Compensated Integrability and Applications to Mathematical Physics

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In a dilute gas, the velocity $v \in \mathbb{R}^3$ is an independent variable. The molecules are described statistically by a function $f(t, y, v) \geq 0$. The macroscopic mass density and linear momentum are given by

$$\rho(t, y) = \int_{\mathbb{R}^3} f(t, y, v)\, dv, \qquad m = \int_{\mathbb{R}^3} f(t, y, v) v\, dv,$$

from which we can define a mean velocity $u := \frac{m}{\rho}$.

The motion is governed by a kinetic equation, named after L. Boltzmann,

$$\underbrace{(\partial_t + v \cdot \nabla_y) f}_{\text{transport}} = Q[f]. \tag{1}$$

The right-hand side accounts for particle interactions, for instance collisions. The expression $Q[f](t,y,v)$ depends only upon $f(t,y,\cdot)$, in a quadratic manner because one takes in account only binary interactions.

When $v \mapsto g(v) \geq 0$ is a particle distribution, the interaction term is given by a formula

$$Q[g](v) = \int_{\mathbb{R}^3} \int_{S_2} B(z, |v - v_*|)(\underbrace{g(v')g(v'_*)}_{\text{gain}} - \underbrace{g(v)g(v_*)}_{\text{loss}})\, ds(z)\, dv_*,$$

in which z is a unit vector, a collision parameter.

The kernel B is non-negative and its behaviour depends upon the assumptions that are made at the microscopic level.

Loss term : $g(v)g(v_*)$ is the probability that the particle with velocity v interacts with a random one of velocity v_* ;

Gain term : Then both particles exit the interaction (collision) with modified velocities v', v'_* that are compatible with the conservation of momentum and energy :

$$v' + v'_* = v + v_*, \qquad |v'|^2 + |v'_*|^2 = |v|^2 + |v_*|^2.$$

Whence the term $g(v')g(v'_*)$.

The outgoing velocities are parametrized by $z \in S_{d-1}$

$$v' = v + (z \cdot (v_* - v))z, \quad v'_* = v_* + (z \cdot (v - v_*))z.$$

For instance, $z = \frac{v - v*}{|v - v_*|}$ yields $(v', v'_*) = (v_*, v)$, while $z \perp (v_* - v)$ gives $(v', v'_*) = (v, v_*)$.

The kernel B tells us how the redistribution $(v, v_*) \mapsto (v', v'_*)$ is made. For instance, the hard spheres model yields $B = |v - v_*|$.

Minimum principle. Remark that the loss term writes (after integration) $-(Lg)g$ for some linear operator L. Hence $Q[g] \geq -(Lg)g$, and the differential inequality

$$(\partial_t + v \cdot \nabla_y)f + (Lf)f \geq 0$$

ensures that the particle density stays ≥ 0 as time increases.

Symmetry of Q

Let $v \mapsto \phi(v)$ be a given, reasonable function. The symmetry between v and v^* gives us (denote $g = g(v), g' = g(v')$ and so on)

$$\begin{aligned}\int_{\mathbb{R}^3} Q[g](v)\phi(v)\,dv &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S_2} B(g'g'_* - gg_*)\phi(v)\,dv_*dv\,dz \\ &= \frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S_2} B(g'g'_* - gg_*)(\phi(v)+\phi(v_*))\,dv_*dv\,dz.\end{aligned}$$

Observing that

- $(v, v_*, z) \mapsto (v', v'_*, z)$ is a volume-preserving change of variable [1]
- and $|v'_* - v'| = |v_* - v|$,

1. This is microscopic reversibility.

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we see that the expression above equals

$$\frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S_2} B(gg_* - g'g'_*)(\phi(v') + \phi(v'_*))\,dv_*\,dvdz.$$

Taking the sum of both, we conclude that

$$\int_{\mathbb{R}^3} Q[g](v)\phi(v)\,dv = \tag{2}$$

$$\frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S_2} B(g'g'_* - gg_*)(\phi(v) + \phi(v_*) - \phi(v') - \phi(v'_*))\,dv_*\,dvdz.$$

As noted by C. Cercignani, the identity (2) can actually be used in order to define $Q[g]$ by duality.

Collision invariants

These are the functions such that

$$\phi(v) + \phi(v_*) - \phi(v') - \phi(v'_*) \equiv 0.$$

Proposition 1

The collision invariants are linear combinations of the functions $\mathbf{1}$, v_i *and* $|v|^2$.

The proof is not easy !

Applying (2) to the basic collision invariants, we deduce that for every density $g \in L^1((1+|v|^2)\,dv)$,

Local conservation of mass. $\int_{\mathbb{R}^3} Q[g](v)\,dv = 0.$

Local C. of momentum. $\int_{\mathbb{R}^3} Q[g](v)v\,dv = 0.$

Local C. of energy. $\int_{\mathbb{R}^3} Q[g](v)|v|^2\,dv = 0.$

The DPT (formal)

At least formally, the two first properties above imply the macroscopic conservation laws of mass and momentum :

$$\begin{aligned} \partial_t \rho + \mathrm{div}_y m &= 0, & (3) \\ \partial_t m + \mathrm{Div}_y T &= 0, & (4) \end{aligned}$$

where

$$T(t,y) := \int_{\mathbb{R}^3} f(t,y,v) v \otimes v \, dv.$$

This yields the Divergence-free symmetric positive tensor

$$S := \begin{pmatrix} \rho & m^T \\ m & T \end{pmatrix} = \int_{\mathbb{R}^3} f(t,y,v) \, V \otimes V \, dv, \qquad V := \begin{pmatrix} 1 \\ v \end{pmatrix}.$$

The third property yields formally the conservation law

$$\partial_t \varepsilon + \operatorname{div} \vec{q} = 0, \tag{5}$$

where

$$\varepsilon := \int_{\mathbb{R}^3} \frac{|v|^2}{2} f(t,y,v)\,dv, \quad \vec{q} := \int_{\mathbb{R}^3} \frac{|v|^2}{2} f(t,y,v) v\,dv,$$

are the macroscopic energy density, which accounts for the kinetic energy of the molecules, and its flux.

Consider a flow in the physical domain $\mathbb{R}^3$, for $t \in (0, \tau)$.

When the total mass and energy at initial time

$$M_0 = \int_{\mathbb{R}^3} dy \int_{\mathbb{R}^3} f_0(y, v)\, dv, \qquad E_0 = \int_{\mathbb{R}^3} dy \int_{\mathbb{R}^3} \frac{|v|^2}{2} f_0(y, v)\, dv$$

are finite, then (3) and (5) give formally the conservation of mass and energy,

$$\int_{\mathbb{R}^3} dy \int_{\mathbb{R}^3} f(t, y, v)\, dv \equiv M_0, \qquad \int_{\mathbb{R}^3} dy \int_{\mathbb{R}^3} \frac{|v|^2}{2} f(t, y, v)\, dv \equiv E_0, \tag{6}$$

telling us that $f \in L^\infty(0, \tau; L^1((1 + |v|^2) dv\, dy))$.

This implies that S is of class $L^\infty(0, \tau; L^1(\mathbb{R}^3))$, hence is a DPT over $(0, \tau) \times \mathbb{R}^3$.

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Two other balance laws

On the one hand, observing that the expression $|y-tv|^2$ belongs to the kernel of the transport operator, and is a linear combination of the collision invariants $\mathbf{1}, v, |v|^2$, we find formally

$$\partial_t I + \operatorname{div} \vec{J} = 0,$$

where

$$I(t,y) := \int_{\mathbb{R}^3} \frac{|y-tv|^2}{2} f(t,y,v)\,dv, \quad \vec{J}(t,y) := \int_{\mathbb{R}^3} \frac{|y-tv|^2}{2} f(t,y,v) v\,dv.$$

We infer

$$\int_{\mathbb{R}^3} dy \int_{\mathbb{R}^3} \frac{|y-tv|^2}{2} f(t,y,v)\,dv \equiv \int_{\mathbb{R}^3} dy \int_{\mathbb{R}^3} \frac{|y|^2}{2} f_0(y,v)\,dv =: I_0. \tag{7}$$

On the other hand, taking $\phi = \log g$ in (2) yields an interesting result. The quantity

$$\log g(v') + \log g(v'_*) - \log g(v) - \log g(v_*) = \log(g(v')g(v'_*)) - \log(g(v)g(v_*))$$

has the same sign as the factor $g' g'_* - g g_*$.

The dissipation rate

$$R[g] := -\int_{\mathbb{R}^3} Q[g] \log g \, dv$$

is thus a non-negative quantity.

Multiplying the Boltzman equation (1) by $\log f$ and integrating with respect to the velocity, we obtain the balance law, called Boltzman's H-Theorem,

$$\partial_t \int_{\mathbb{R}^3} h(f)\, dv + \operatorname{div} \int_{\mathbb{R}^3} h(f) v \, dv + R[f(t,y,\cdot)] = 0, \qquad h(s) := s \log s. \tag{8}$$

Entropy

The integral of $f \log f$ is viewed as the opposite of the thermodynamical entropy.

Integrating (8) in the space variable, we end up (again formally) with our last estimate

$$\sup_{t>0}\left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} h(f(t,y,v))\,dv\,dy + \int_0^t ds\int_{\mathbb{R}^3} R[f(s,y,\cdot)]\,dy\right) \leq \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} h(f_0)\,dv\,dy =: H_0. \tag{9}$$

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Equilibria

The dissipation rate $R[g]$ vanishes if and only if $g'g'_* - gg_* \equiv 0$.

This condition,

$$\log g(v') + \log g(v'_*) = \log g(v) + \log g(v_*)$$

tells us that $\log g$ is a collision invariants.
Hence $R[g] = 0$ is equivalent to g being a Gaussian distribution, called a *Maxwellian* in this context :

$$g(v) = \frac{\rho}{(2\pi\theta)^{3/2}} \exp\left(-\frac{|v-u|^2}{2\theta}\right).$$

Here ρ and $m = \rho u$ are as usual the moments of g of order 0 and 1, respectively. The temperature θ is a second order moment

$$\theta := \frac{1}{3\rho} \int_{\mathbb{R}^3} |v-u|^2 g(v)\, dv.$$

As it is customary in the realm of nonlinear PDEs, the existence of a solution is proven through an approximation procedure. We have to pass to the limit in a sequence of approximate solutions, and the only information at our disposal is given by *a priori* estimates[2].

Assembling (6,7) and (9), we have therefore a control

$$\sup_{t>0}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}(1+|v|^2+|y|^2+\log_+ f)f(t,y,v)\,dv\,dy < C(M_0,E_0,I_0,H_0) \tag{10}$$

where the bound depends only upon the initial data, and we have denoted $\log_+$ the positive part of the logarithm.

2. Those presented above

On the other hand, (9) gives

$$\int_0^\tau \int_{\mathbb{R}^3} R[f(t,y,\cdot)]\,dy\,dt < C. \tag{11}$$

Let us insist on the word *formal(ly)* :

- The current theory of the Cauchy Problem for the Boltzmann equation is not powerful enough to ensure all of the conservation laws described above (mass, momentum, energy).
- It is not even good enough to prove the existence of a solution of the Boltzmann equation at all.

The reason of this weakness is that the *a priori* estimates (10) and (11) do not allow us to give a meaning to the interaction term $Q[f]$ in the sense of distributions.

Renormalized solutions

What is know instead is the global existence [3] of a density $f(t,y,v)$ satisfying the identity

$$(\partial_t + v\cdot\nabla_y)\beta(f) = \beta'(f)Q[f] \tag{12}$$

for smooth increasing functions β such that $\beta'(s) = O(\frac{1}{s})$.

Within DiPerna & Lions' theory, one may take the first moment (of order zero) of the equation, to obtain the conservation of mass (3). But when one takes the next moment (of order one), it is unclear whether that of momentum (4) is correct.

3. R. J. DiPerna & P.-L. Lions. On the Cauchy problem for Boltzmann equations : global existence and weak stability. *Ann. of Math.*, **130** (1989), 321–366.

Modified DPT

We are only able to obtain a modified conservation law

$$\partial_t m + \mathrm{Div}\,(T + \Sigma) = 0, \tag{13}$$

where the so-called defect measure Σ is a positive semi-definite symmetric tensor.

At least, this tells us that the modified *mass-momentum* tensor

$$\widehat{S} := \begin{pmatrix} \rho & m^T \\ m & T + \Sigma \end{pmatrix} = S + \begin{pmatrix} 0 & 0 \\ 0 & \Sigma \end{pmatrix}$$

is Divergence-free over $(0, \tau) \times \mathbb{R}^3$.

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It is known in addition that the modified energy

$$\begin{aligned}\hat{E}(t) &= \frac{1}{2}\int_{\mathbb{R}^3} \mathsf{Tr}\,(T+\Sigma) \\ &= \frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f(t,y,v)|v|^2\,dv\,dy + \frac{1}{2}\int_{\mathbb{R}^3} \mathsf{Tr}\,\Sigma\end{aligned}$$

is non-increasing in time, bounded by E_0.

Hence the entries of $\widehat{S}$ are finite measures provided that

$$M_0, E_0 < \infty.$$

Applying Compensated Integrability

We are now in position to apply Theorem 3 of Lesson #5 :

$$\int_0^\tau \int_{\mathbb{R}^3} (\det \widehat{S})^{\frac{1}{3}}\, dy \le k_3 M_0^{\frac{1}{3}} (\|m(0)\|_{\mathcal{M}} + \|m(\tau)\|_{\mathcal{M}}).$$

We bound the right-hand side as usual with the Cauchy–Schwarz inequality :

$$\begin{aligned}
\|m(t)\|_{\mathcal{M}} &\le \left(\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f(t,y,v)\, dv\, dy \cdot \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f(t,y,v) |v|^2 dv\, dy \right)^{\frac{1}{2}} \\
&\le \sqrt{2M_0 E_0}\, .
\end{aligned}$$

The left-hand side is treated with the following observations. On the one hand, we have $S \le \widehat{S}$ (because of $\Sigma \ge 0_3$) and therefore

$$\det S \le \det \widehat{S}.$$

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On the other hand, the Hadamard inequality tells us that

$$\rho \det \Sigma \leq \det \widehat{S}.$$

We obtain therefore two informations, namely

Theorem 1

Let the initial density $f_0 \geq 0$ satisfy the requirements that M_0, E_0, I_0 and H_0 are finite. Then a renormalized solution of the Cauchy problem for (1) satisfies

$$\int_0^\tau \int_{\mathbb{R}^3} (\det S)^{\frac{1}{3}} \, dy \quad \text{and} \quad \int_0^\tau \int_{\mathbb{R}^3} (\rho \det \Sigma)^{\frac{1}{3}} \, dy \leq k_3 M_0^{\frac{1}{3}} \sqrt{8 M_0 E_0}\,. \tag{14}$$

Comments.

- Since Σ is a 3×3 tensor whose entries are finite measures, $(\det \Sigma)^{\frac{1}{3}}$ is itself a finite measure. The second estimate tells us therefore that $\rho^{\frac{1}{3}}$ is integrable with respect to this measure.
- In the estimate of
$$\int_0^\tau \int_{\mathbb{R}^3} (\det S)^{\frac{1}{3}} \, dy \, dt,$$
the integrand is homogeneous of degree $\frac{4}{3}$ in the density f. It is significantly better than the basic estimates, which are either linear in f (mass and energy), or $\log$-linear (entropy).
- The same argument as for the Euler system (Part 1 of this Lesson, minimization of $M_0 E_0$ with respect to the choice of an inertial frame) yields the improved upper bound $k_3 M_0^{\frac{1}{3}} \sqrt{8D_0}$ where now
$$D_0 := \frac{1}{4} \int_{\mathbb{R}^3} \cdots \int_{\mathbb{R}^3} f_0(y,v) f_0(y',v') |v'-v|^2 \, dv \, dv' \, dy \, dy'.$$

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- In the first estimate, we can develop $\det S$ by applying Andreev's formula. It involves the 4×4 determinant

$$\begin{vmatrix} 1 & \cdots & 1 \\ v^0 & \dots & v^3 \end{vmatrix},$$

which equals, up to a sign, the volume of the tetrahedron spanned in $\mathbb{R}^3$ by the vertices $v^0,\dots,v^3$. Denoting this volume $\mathrm{vol}(v^0,\dots,v^3)$, we have

$$(\det S)(t,y) =$$

$$\frac{1}{24}\int_{\mathbb{R}^3}\cdots\int_{\mathbb{R}^3} f(t,y,v_0)\cdots f(t,y,v_3)(\mathrm{vol}(v^0,\dots,v^3))^2\,dv^0\cdots dv^3.$$

Remark that it estimates the product $f(v^0)\cdots f(v^3)$ as much as the vertices $v^0,\dots,v^3$ are not coplanar.

How to use the new estimate ?

A natural question is whether the estimate (14.1), which is super-linear in the density, can be used to improve the theory of the Cauchy problem. Ultimate target : Prove that the renormalized solutions are true solutions of the Boltzmann equation. This means proving that the interaction term $Q[f]$ makes sense as a distribution. At least, we'd like to prove that the renormalized equation (12) is satisfied for a larger class of functions β.

Observe that (14.1) is somewhat complementary to the estimate (11) of the dissipation rate. The second one forces $f(t, y, \cdot)$ to ressemble a Maxwellian distribution, while the first one tends to force $f(t, y, \cdot)$ to concentrate over a hyperplane. Since these behaviours are not compatible, it seems that some stronger control of $f(t, y, \cdot)$ could be available. Whether this allows us to improve the notion of solution remains an *open problem*.

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A less optimistic remark is that (14.1) is sub-quadratic. It is doubtful that we could use it to define a distribution $Q[f]$, since Q is a quadratic operator.

So far, the only situation where the program has been carried out is the Cauchy problem for (1) when one considers an initial data that is uniform in the directions y_2 and y_3 :

$$f_0(y,v) = F_0(y_1,v). \tag{15}$$

One looks therefore for a solution of the form

$$f(t,y,v) = F(t,y_1,v).$$

We speak of the quasi-1D Cauchy problem [4]

4. The genuinely 1D Boltzmann equation is trivial, reducing to free transport.

In the quasi-1D framework, the macroscopic system (3,4) is posed on a 2-dimensional slab $(0,\tau)\times\mathbb{R}$. The DPT reduces to a 2×2 tensor

$$S^1 := \begin{pmatrix} \rho & m_1 \\ m_1 & t_{11}+\sigma_{11} \end{pmatrix},$$

where we recall that σ_{11} is a non-negative finite defect measure.

The same application of Theorem 3 of Lesson #5 yields now

$$\int_0^\tau dt \int_{\mathbb{R}} dy_1 \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} (v_1-w_1)^2 f(t,y_1,v) f(t,y_1,w)\, dv\, dw \le \mathrm{cst}\cdot M_1\sqrt{M_1 E_1} \tag{16}$$

in terms of the *fake* mass and energy

$$M_1 := \int_{\mathbb{R}} dy_1 \int_{\mathbb{R}^3} F_0(y_1,v)\, dv, \qquad E_1 := \frac{1}{2}\int_{\mathbb{R}} dy_1 \int_{\mathbb{R}^3} F_0(y_1,v) v_1^2\, dv.$$

Because estimate (16) is fully quadratic in the density, and using the formulation (2), C. Cercignani [5] could prove

Theorem 2 (Cercignani.)

Let $F_0 \in L^1(\mathbb{R} \times \mathbb{R}^3)$ be such that

$$\int F_0(y_1, v)(1 + y_1^2 + |v|^2 + |\log F_0(y_1, v)|)\, dv\, dy_1 < \infty.$$

Also, assume a technical condition on the collision kernel B (see the reference paper). Then there exists a weak solution of the initial value problem $f \in C(\mathbb{R}_+, L^1(\mathbb{R} \times \mathbb{R}^3))$ and $f(0, \cdot) = F_0$. This solution conserves the energy globally.

5. Global weak solutions of the Boltzmann equation. *J. Stat. Phys.*, **118** (2005), 333–342.

Comments.

- The conservation of energy implies

$$\int_0^\tau \int_{\mathbb{R}} d\,\mathsf{Tr}\,\Sigma \equiv 0$$

  and therefore $\Sigma \equiv 0$. This is the only case where we do know that the defect measure is not present.

- Cercignani's Theorem precedes our theory of Compensated Integrability. Yet, he did know the estimate (16), because the case $n = 2$ is always "trivial". The same estimate had been established earlier by J.-M. Bony [6] in the context of discrete velocity models in one space dimension.

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6. J.-M. Bony. Existence globale et diffusion en théorie cinétique discrète, in *Advances in kinetic theory and continuum mechanics*,ÊR. Gatignol & Soubbaramayer eds. (Springer-Verlag, Berlin 1991), 81–90.

- A similar result was proved independently by C. Villani in his PhD thesis (page 416). It has not been published elsewhere.
- Whether our new estimate can be used to prove a true multi-dimensional version of Theorem 2 is left as an

Open Problem.