Compensated Integrability and Applications to Mathematical Physics

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L#7 - Singular tensors (2). DPTs over graphs ; Applications

In Part 1, we explored a few examples of singular DPTs, especially those that are homogeneous of degree $1-n$. They are of the form

$$S = \frac{\lambda(e)}{r^{n-1}}\, e \otimes e, \qquad r = |x|,\ e = \frac{x}{r}\,,$$

where λ is a non-negative measure over S_{n-1}, satisfying

$$\int_{S_{n-1}} e\, d\lambda(e) = 0.$$

Example : If $\lambda = \delta_u + \delta_{-u}$ for some unit vector u, then $S = \pi\mathcal{L}_u$ is concentrated along the line $\mathbb{R}u$, and its constant density (a symmetric positive semi-definite matrix) $\pi = u \otimes u$ is the orthogonal projection over $\mathbb{R}u$.

How far can this example be generalized ?

Tensors supported by a curve

Let $s \mapsto X(s)$ be the arc-length parametrization of a smooth curve γ. The parameter runs over a bounded interval $I = (0, \ell)$ where ℓ is the length. If the curve is closed, then s runs over $\mathbb{R}/\ell\mathbb{Z}$ instead. Denote $\tau(s) = \dot{X}(s)$ the unit tangent vector.

For a smooth map $s \mapsto \Sigma(s) \in \mathbf{Sym}_n^+$, we may define a singular, symmetric positive semi-definite tensor

$$S = \Sigma \, \mathcal{L}_\gamma,$$

where $\mathcal{L}_\gamma$ denotes the 1-dimensional Lebesgue measure along γ. For every vector-valued $\phi \in \mathcal{D}(\mathbb{R}^n)$, we have

$$\langle S, \phi \rangle = \int_I \Sigma(s) \phi(X(s)) \, ds.$$

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For which pairs (γ,Σ) is the tensor S Div-controlled, or divergence-free ?

Let us compute $\mathrm{Div}\,S$.

$$\langle \mathrm{Div}\,S,\phi\rangle = -\langle S,\nabla\phi\rangle = -\int_I \mathsf{Tr}\,(\Sigma(s)\nabla\phi(X(s)))\,ds.$$

$\mathrm{Div}\,S$ is supported by a subset of $\mathrm{Supp}\,S$, hence by the closure of γ (the curve together its ends). S is Div-controlled if

$$|\langle S,\nabla\phi\rangle| \le C\sup_\gamma |\phi|$$

for some finite constant C. This implies that

$$\int_I \mathsf{Tr}\,(\Sigma(s)\nabla\phi(X(s)))\,ds$$

does not involve the normal part of $\nabla\phi$.

In other words, $\Sigma(s)$ must be rank-one, proportional to the orthogonal projector $\tau(s) \otimes \tau(s)$ onto the tangent line. Thus let

$$\Sigma(s) = \sigma(s)\tau(s) \otimes \tau(s)$$

for some smooth function $\sigma \geq 0$. We have

$$\langle \mathrm{Div}\, S, \phi \rangle = -\int_I \sigma(s) \left(\frac{d}{ds} \phi(X(s)) \right) \tau(s)\, ds.$$

Integrating by parts, (for the sake of generality, we consider the open case $I = (0, \ell)$)

$$\begin{aligned} \langle \mathrm{Div}\, S, \phi \rangle &= \int_I \phi(X(s)) \frac{d}{ds} (\sigma(s)\tau(s))\, ds \\ &\quad + \phi(X(0))\sigma(0)\tau(0) - \phi(X(\ell))\sigma(\ell)\tau(\ell). \end{aligned}$$

The calculations above yield the statement

Proposition 1

Let $\gamma : s \mapsto X(s)$ be a smooth curve in $\mathbb{R}^n$ and $s \mapsto \Sigma(s) \in \mathbf{Sym}_n^+$ be a density along this curve.
The tensor $S = \Sigma\,\mathcal{L}_\gamma$ is Div-controlled if, and only if $\Sigma = \sigma\tau \otimes \tau$ for some absolutely continuous function $s \mapsto \sigma(s) \geq 0$. We have

$$\operatorname{Div} S = \frac{d}{ds}\left(\sigma(s)\tau(s)\right)\,\mathcal{L}_\gamma + \sigma(0)\tau(0)\delta_{X(0)} - \sigma(\ell)\tau(\ell)\delta_{X(\ell)}.$$

If γ is a closed curve, then the Dirac masses are absent from the formula.

Corollary 1

The tensor S defined in Proposition 1 is a DPT over an open domain Ω if, and only if the curve γ is a segment with vertices on $\partial\Omega$, and the density σ is a constant.

Example : moving particle

Let P be a particle of mass m, whose position at time t is $t \mapsto Y(t) \in \mathbb{R}^d$. Denote its velocity $v(t) = \dot{Y}(t)$.

As usual, $n = 1 + d$ and $x = (t, y)$ is the time-space coordinate. The curve γ will be the graph of P. Arc-length parametrization and tangent vector :

$$ds = \sqrt{1 + |v|^2}\, dt, \qquad \tau = \frac{1}{\sqrt{1 + |v|^2}} \begin{pmatrix} 1 \\ v \end{pmatrix} \in S_d.$$

Define the mass-momentum tensor of the particle :

$$S_P := m\sqrt{1 + |v|^2}\, \tau \otimes \tau\, \mathcal{L}_\gamma = \frac{1}{\sqrt{1 + |v|^2}} \begin{pmatrix} m & mv^T \\ mv & mv \otimes v \end{pmatrix} \mathcal{L}_\gamma,$$

where

$$\frac{1}{\sqrt{1 + |v|^2}}\, \mathcal{L}_\gamma = dt|_\gamma.$$

We compute as before

$$
\begin{aligned}
\langle \mathrm{Div}\, S, \phi \rangle &= -\langle S, \nabla\phi \rangle \\
&= -\int \frac{m}{\sqrt{1+|v|^2}} (\phi_t + v\cdot\nabla\phi) \begin{pmatrix} 1 \\ v \end{pmatrix} ds \\
&= -\int m \frac{d}{dt}\phi(Y(t)) \begin{pmatrix} 1 \\ v \end{pmatrix} dt
\end{aligned}
$$

Integrating by parts, we find (up to Dirac masses at the ends of the interval)

$$
\mathrm{Div}\, S = \frac{m}{\sqrt{1+|v|^2}} \begin{pmatrix} 0 \\ \frac{dv}{dt} \end{pmatrix} \mathcal{L}_\gamma .
$$

.

If the particle obeys to Newton's law

$$m\,\frac{dv}{dt}=F,$$

then

$$\mathrm{Div}\,S=\frac{1}{\sqrt{1+|v|^2}}\begin{pmatrix}0\\F\end{pmatrix}\mathcal{L}_\gamma.$$

Proposition 2

Let P be a particle moving in a force field F. Then the mass-momentum tensor satisfies

$$\mathrm{Div}\,S=\begin{pmatrix}0\\F\,dt|_\gamma\end{pmatrix}.$$

Junctions

The situation becomes richer when considering a tensor supported by a graph $G \subset \Omega$.

Let V be the set of vertices and E that of edges (segments). According to Corollary 1, if S is a DPT supported by G, then its restriction to an edge e of direction τ_e must have a constant density of the form $\sigma_e \tau_e \otimes \tau_e$ with $\sigma_e \geq 0$. Thus besides the sets (V, E), our data consists in a third set $\sigma = (\sigma_e)_{e \in E}$.

Proposition 1 gives

$$\operatorname{Div} S = \sum_{e \in E} \sigma_e \tau_e (\delta_{\text{in}} - \delta_{\text{fin}}) = \sum_{v \in V} \left(\sum_{v \vdash e} \sigma_e \tau_{e,v} \right) \delta_v, \tag{1}$$

where $v \vdash e$ means that v is a vertex of e, and $\tau_{e,v}$ is the outward unit tangent vector. In particular, τ_e comes in the last sum with two opposite signs at the opposite vertices of e.

We deduce

Proposition 3

Let $G = (V, E)$ be a graph in Ω, made of segments. Let $\sigma = (\sigma_e)_{e\in E}$ be a list of constant ≥ 0 densities. Then the tensor

$$S = \sum_{e\in E} \sigma_e \, \tau_e \otimes \tau_e \, \mathcal{L}_e$$

is a DPT if, and only if (Kirchhoff's law)

$$\forall v \in V, \qquad \sum_{v\vdash e} \sigma_e \tau_{e,v} = 0. \tag{2}$$

Example : colliding particles

Consider two point particles $P_{1,2}$ with masses $m_{1,2}$. No force : they move freely with a constant velocity. However it happens that they collide at time t^*, position $y^* \in \mathbb{R}^d$. The velocities experience jumps

$$v_1 \mapsto v_1', \qquad v_2 \mapsto v_2'.$$

The conservation of momentum writes

$$m_1 v_1' + m_2 v_2' = m_1 v_1 + m_2 v_2.$$

(Conservation of the kinetic energy is not use at this stage.)

The trajectory of P_j is the union of two semi-lines γ_{j-} (before the collision) and γ_{j+} (after), both ending at $x^* = (t^*, y^*)$. Thus the graph of the whole system looks like an $\mathbf{X}$, with center at x^* (see Figure 1.a next slide).

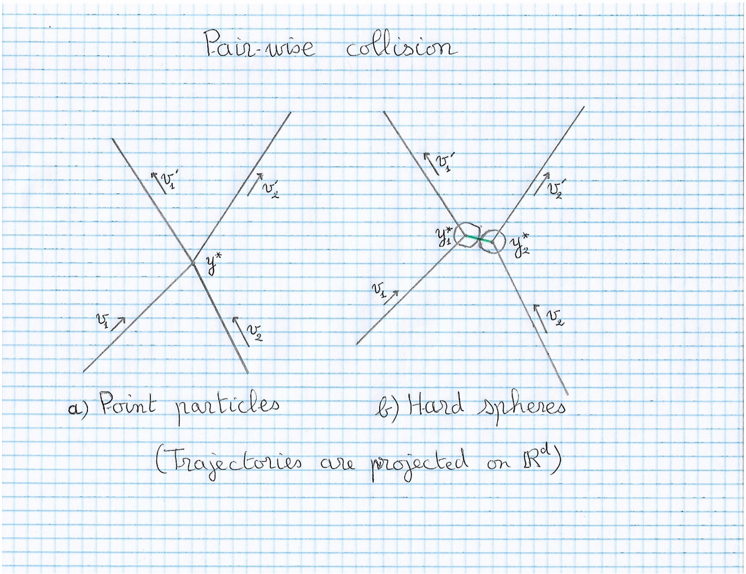

Fig. 1. Graph supporting the mass-momentum tensor of a colliding pair.

The particle P_j has its own mass-momentum tensor S^j, which is supported by its trajectory :

$$S^j = \frac{1}{\sqrt{1+|v_j|^2}} \begin{pmatrix} m_j & m_j v_j^T \\ m_j v_j & m_j v_j \otimes v_j \end{pmatrix} \mathcal{L}_{j-} + \frac{1}{\sqrt{1+|v_j'|^2}} \begin{pmatrix} m_j & m_j v_j'^T \\ m_j v_j' & m_j v_j' \otimes v_j' \end{pmatrix} \mathcal{L}_{j+},$$

with an obvious notation for the Lebesgue measures.

Remark that S^j is **not** divergence-free. Instead (1) gives

$$\mathrm{Div}\, S^j = m_j \begin{pmatrix} 0 \\ v_j' - v_j \end{pmatrix} \delta_{x^*}.$$

We deduce however from the conservation of linear momentum that

$$\mathrm{Div}\,(S^1 + S^2) = 0.$$

This leads us defining the mass-momentum tensor of the system of particles by

$$S := S^1 + S^2.$$

We have proved

Proposition 4

For a system of two un-accelerated point particles, the mass-momentum tensor is a DPT.

Of course, this is valid for an arbitrary finite system $P_1, \dots, P_N$ of colliding point particles. Just form

$$S := S^1 + \cdots + S^N.$$

The model of colliding point particles is not realistic in space dimension $d \geq 2$.

Actually, given an initial configuration of N particles (positions $y_j(0)$, velocities $v_j(0)$), there is no reason why two of them would collide at some time : Generically, the vectors $y_j(0) - y_i(0)$ and $v_j(0) - v_i(0)$ are not colinear and thus the particles P_i and P_j ignore each other.

This is why we must turn to a slightly more elaborate model, where particles have the shape of a ball.

Hard spheres dynamics

Consider N identical particles of mass m, whose shapes are balls of radius a. Their "positions" $y_j(t)$ is actually that of their centers.

A collision between P_i and P_j happens when (see Figure 1.b)

$$|y_j(t^*) - y_i(t^*)| = 2a.$$

Denoting again the velocities before/after the shock by v_i, v_j, v_i', v_j', we have

$$(v_j - v_i) \cdot (y_j(t^*) - y_i(t^*)) \leq 0, \qquad (v_j' - v_i') \cdot (y_j(t^*) - y_i(t^*)) \geq 0. \quad (3)$$

The conservation of momentum is still

$$v_i' + v_j' = v_i + v_j. \quad (4)$$

With the "jump" notation $[v] = v' - v$, this is $[v_j] = -[v_i]$.

Assume that the collisions are friction-less :

$$[v_i] \parallel y_j(t^*) - y_i(t^*). \tag{5}$$

Because of (4) and (3), we have

$$2[v_i] \cdot (y_j(t^*) - y_i(t^*)) = [v_i - v_j] \cdot (y_j(t^*) - y_i(t^*)) \leq 0,$$

and therefore $[v_i]$ is colinear with $y_i(t^*) - y_j(t^*)$, and points in the same direction.

At last, the collisions are elastic : they conserve the kinetic energy

$$|v_i'|^2 + |v_j'|^2 = |v_i|^2 + |v_j|^2. \tag{6}$$

As in the simplified model, the conservation of energy does not serve in the construction of the mass-momentum tensor, nor to verify its divergence-freeness. But it is useful in proving that its entries are finite measures.

For the moment, we content ourselves with the following consequence of (6) : the total energy

$$E(t) := \frac{m}{2} \sum_{j=1}^{N} |v_j|^2 \equiv E_0$$

is a constant.

The mass-momentum tensor

The trajectory of one particle P_j is a broken line, with vertices at $x_{j,k}$ for $k = 0, 1 \dots$. By convention, $x_{j,0} = (0, y_j(0))$ is at initial time. Along the segment $\gamma_{j,k}$ between $x_{j,k}$ and $x_{j,k+1}$, the constant velocity is $v_{j,k}$. In particular $v_{j,0} = v_j(0)$.

The mass-momentum tensor of P_j is defined as before, by

$$S^j = m \sum_{k \geq 0} \frac{1}{\sqrt{1 + |v_{j,k}|^2}} \begin{pmatrix} 1 & v_{j,k}^T \\ v_{j,k} & v_{j,k} \otimes v_{j,k} \end{pmatrix} \mathcal{L}_{j,k},$$

with an obvious notation for the 1-dimensional Lebesgue measure.

The naive attempt to define a mass-momentum tensor of the system is to sum all the S^j's.

However a flaw happens : this sum is not divergence-free.

The reason why Proposition 4 does not apply is that at a collision the points $y_i(t^*)$ and $y_j(t^*)$ do not coincide ; they are instead separated by a distance $2a$. Therefore the contributions

$$\begin{pmatrix} 0 \\ m[v_i] \end{pmatrix} \delta_{t^*, y_i(t^*)} \quad \text{and} \quad \begin{pmatrix} 0 \\ m[v_j] \end{pmatrix} \delta_{t^*, y_j(t^*)} \tag{7}$$

in the Divergence do not cancel each other.

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To recover the divergence-freeness, we add a corrector.

Consider the tensor

$$C = \frac{1}{|[v_i]|} \begin{pmatrix} 0 & 0 \\ 0 & m[v_i] \otimes [v_i] \end{pmatrix} \mathcal{L}_{[x_i^*, x_j^*]}, \tag{8}$$

supported by the segment joining the centers of P_i and P_j at the time of the collision.

Because of (5), and by Proposition 1, we have

$$\mathrm{Div}\, C = \begin{pmatrix} 0 \\ m[v_i] \end{pmatrix} (\delta_{x_j^*} - \delta_{x_i^*}),$$

where we have used the fact that $[v_i]$ points towards y_i^*. This compensates exactly the contributions (7).

In conclusion, we **define** the mass-momentum tensor of the whole hard spheres system, as the sum of the following contributions :

- for every particle P_j, its dedicated tensor S^j,
- for every pairwise collision, the tensor defined in (8).

See Figure 1.b.

We thus have

Proposition 5

The mass-momentum tensor of a hard sphere system experiencing only pairwise collisions is divergence-free.

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Interpretation

If we think that a mass-momentum tensor describes only the dynamics of physical particles, then every contribution C as (8) must be interpreted as some kind of particle, of a new type.

On the one hand, it is mass-less because its upper-left entry is 0. It travels at infinite speed because it exists only at the time t^* of the collision.

On the other hand, its role is to carry the exchange $m[v_i]$ of momentum between colliding particles.

We suggest to call this virtual particle a

Colliton.

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Higher-dimensional supports and mean curvature

More generally, we may consider tensors of the form

$$S = \Sigma\,\mathcal{L}_M$$

supported by a sub-manifold M of dimension p. Here $\mathcal{L}_M$ denotes the p-dimensional Lebesgue measure induced over M, and $\Sigma : M \to \mathbf{Sym}_n^+$ is a reasonnably smooth density.

For a test vector-field $\phi \in \mathcal{D}(\mathbb{R}^n)$, we have

$$\langle S, \phi \rangle = \int_M \Sigma(x)\phi(x)\, d\mathcal{L}_M.$$

An argument similar to that employed for curves (that is when $p = 1$) tells us that S is Div-controlled if, and only if for every $x \in M$, the range of $\Sigma(x)$ is contained in the tangent space $\vec{TM}_x$.

The calculation of $\operatorname{Div} S$, whose support is contained in the closure of M, might be complicated. We have however the following remarkable fact

Theorem 1

Let M be a smooth immersed submanifold of dimension p. For $x \in M$, let $\pi(x)$ be the orthogonal projector onto $\vec{TM}_x$.
The tensor

$$S_M := \pi \, \mathcal{L}_M$$

is divergence-free if, and only if M is a minimal surface.

Remark that this $\pi(x)$ is symmetric, positive semi-definite with rank p.

Proof (For hypersurfaces.)

If $p = n - 1$, then $\pi(x) = I_n - \vec{N}(x) \otimes \vec{N}(x)$ where $\vec{N}(x)$ is the unit normal to M at x.

Since the Divergence is a local operator, it suffices to consider a small piece of M, written as a graph over its tangent space. Up to a rotation, we may suppose that M is locally represented as

$$x_n = w(\hat{x}_n)$$

where w is smooth and $w(0) = 0$ and $\nabla w(0) = 0$.

The projector is given by

$$\pi(x) = \frac{1}{1 + |\nabla w|^2} \begin{pmatrix} (1 + |\nabla w|^2) I_{n-1} - \nabla w \otimes \nabla w & \nabla w \\ \nabla w & |\nabla w|^2 \end{pmatrix}.$$

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For a test function $\phi \in \mathcal{D}(B)$, we denote $\psi(\hat{x}_n) := \phi(\hat{x}_n, w(\hat{x}_n))$ the restriction of ϕ to M. Since the area element over M is

$$d\mathcal{L}_M = \sqrt{1 + |\nabla w|^2}\, d\hat{x}_n,$$

we write

$$\langle \mathrm{Div}\, S_M, \phi \rangle = -\langle S_M, \nabla\phi \rangle = -\int_{B_{n-1}} \pi(x)\nabla\phi(x)\, \sqrt{1 + |\nabla w|^2}\, d\hat{x}_n.$$

With $\nabla\phi = (\hat{\nabla}\phi, \partial_n\phi)$, the chain rule gives $\nabla\psi = \hat{\nabla}\phi + \partial_n\phi\nabla w$, whence

$$\begin{aligned}
&\langle \mathrm{Div}\, S_M, \phi \rangle = -\langle S, \nabla\phi \rangle = \\
&-\int_{B_{n-1}} \frac{d\hat{x}_n}{\sqrt{1 + |\nabla w|^2}} \begin{pmatrix} (1 + |\nabla w|^2)\hat{\nabla}\phi - (\nabla w \cdot \hat{\nabla}\phi)\nabla w + \partial_n\phi\nabla w \\ \nabla w \cdot \hat{\nabla}\phi + |\nabla w|^2\partial_n\phi \end{pmatrix} \\
&= -\int_{B_{n-1}} \frac{d\hat{x}_n}{\sqrt{1 + |\nabla w|^2}} \begin{pmatrix} (1 + |\nabla w|^2)\nabla\psi - (\nabla w \cdot \nabla\psi)\nabla w \\ \nabla w \cdot \nabla\psi \end{pmatrix}.
\end{aligned}$$

Integrating by parts, we get

$$\langle \mathrm{Div}\, S_M , \phi \rangle = \int_{B_{n-1}} \psi \vec{K}\, d\hat{x}_n ,$$

where for instance

$$K_n = \mathrm{div} \frac{\nabla w}{\sqrt{1+|\nabla w|^2}},$$

is $n-1$ times the mean curvature H.

Likewise, an elementary calculation gives

$$\hat{K}_n = \nabla \sqrt{1+|\nabla w|^2} - \mathrm{Div}\, \frac{\nabla w \otimes \nabla w}{\sqrt{1+|\nabla w|^2}} = -K_n \nabla w.$$

Finally we have

$$\langle \mathrm{Div}\, S_M , \phi \rangle = (n-1) \int_{B_{n-1}} \psi H \begin{pmatrix} -\nabla w \\ 1 \end{pmatrix} d\hat{x}_n = (n-1) \int_M \phi H \vec{N}\, d\mathcal{L}_M .$$

The proof above gives us a more general statement, which covers arbitrary hypersurfaces. Taking care of the boundary terms in the integration by parts, we can even write

Theorem 2

Let M be an immersed hypersurface in $\mathbb{R}^n$, with smooth boundary γ. Then the tensor S_M defined in Theorem 1 satisfies

$$\mathrm{Div}\, S_M = (n-1)H\vec{N}\,\mathcal{L}_M - \vec{\nu}\,\mathcal{L}_\gamma. \tag{9}$$

Here above, H is the mean curvature, $\vec{N}$ is the unit normal to M, $\mathcal{L}_\gamma$ is the $(n-2)$-dimensional Lebesgue measure induced over γ and $\vec{\nu}(x)$ is the outward unit vector tangent to M, normal to γ.

In soap bubble experiences, we observe minimal surfaces, though not only that. There are often more (and possibly many more) than one such surface. When they meet along lines, we see three of them, forming angles of 120 degrees pairwise (Plateau's laws).

This phenomenon is interpreted in terms of a DPT in the following way : say that

$$M = M_1 \cup M_2 \cup M_3$$

is the union of three hypersurfaces, whose boundaries have a common component, an $(n-2)$-dimensional submanifold γ.

Let us define the natural tensor associated with M,

$$S_M = S_{M_1} + S_{M_2} + S_{M_3}\,.$$

From Theorem 2, we have

$$\operatorname{Div} S_M = (n-1)\left(H_1\vec{N}_1\,\mathcal{L}_1 + H_2\vec{N}_2\,\mathcal{L}_2 + H_3\vec{N}_3\,\mathcal{L}_3\right) - (\vec{\nu}_1+\vec{\nu}_2+\vec{\nu}_3)\,\mathcal{L}_\gamma,$$

with obvious notations.
The tensor S_M is therefore a local DPT if, and only if on the one hand each M_j is a minimal surface ($H_j \equiv 0$), and on the other hand

$$\vec{\nu}_1+\vec{\nu}_2+\vec{\nu}_3 \equiv 0.$$

Since each $\vec{\nu}_j$ is a unit vector, this exactly means that the hypersurfaces form pairwise an angle of 120 degrees. See an illustration of Figure 2.

Mind that the calculation above tells nothing about the self-intersections ; if the relative interiors of two sheets intersect, S is just locally the sum $S_{M_1}+S_{M_2}$ of two local DPTs, hence is local DPT. See an example in Figure 3.

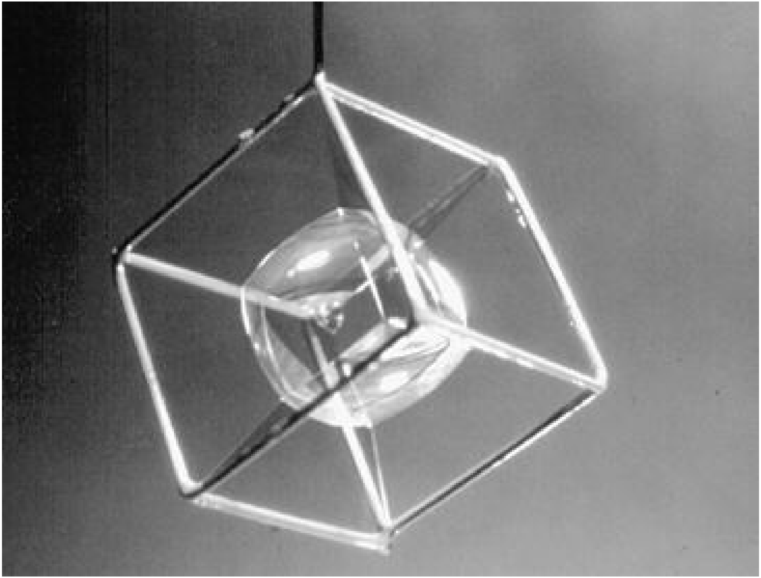

Fig. 2. At least 20 ternary lines bounding minimal sheets are visible here.

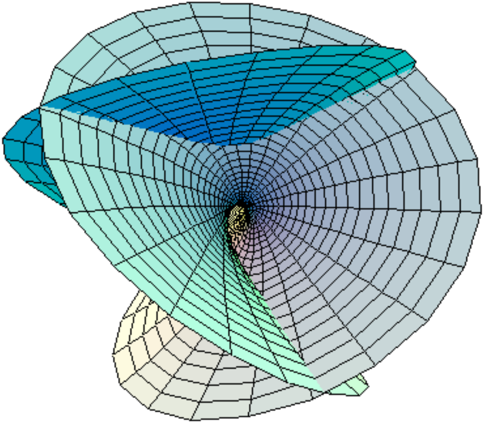

Fig. 3. The Enneper minimal surface is self-intersecting, thus not immersed.

Conclusion

In this lesson, we have seen that the theory of divergence-free positive symmetric tensors relates to various domains. Besides gas dynamics, which was considered in Lesson 6, we encounter two important topics in differential geometry :

- Minkowski's problem,
- minimal surfaces.

Finally it let us discovering a virtual particle at work in the collision between hard spheres.

In the next lesson, we shall apply the improved Functional Inequality, which involves the determinantal masses at central singularities, to the hard spheres model. We shall derive new estimates about the collision set for a finite collection of N particles in $\mathbb{R}^d$.