Compensated Integrability and Applications to Mathematical Physics

Denis SERRE

UMPA, UMR 5669 CNRS
École Normale Supérieure de Lyon
France

University of L'Aquila, spring 2020

L. #7 - Singular tensors (1). Determinantal masses

Compensated Integrability says that for DPTs, or Div-controlled tensors, $(\det S)^{\frac{1}{n}} \in L^{\frac{n}{n-1}}$. This would be a trivial statement if S itself belonged to $L^{\frac{n}{n-1}}$. We investigate below some of these tensors, which are not $L^{\frac{n}{n-1}}_{\rm loc}$.

There are mainly two situations.

Part 1. DPTs that are homogeneous in x. The higher negative degree $1-n$ displays a phenomenon of rigidity. This class is central in Minkowski's problem for convex bodies. We find that $(\det S)^{\frac{1}{n-1}}$ exhibits a Dirac singularity at the origin, which we call determinantal mass.

Part 2. Other tensors are supported by submanifolds, or by unions such as graphs. They are used in the analysis of particle dynamics, and also serve in the characterization of minimal surface.

Notations : B the unit ball of $\mathbb{R}^n$. We denote $r = |x|$ and $e = \frac{x}{r}$.

Homogeneous tensors

Let us begin with a simple but fundamental example.

Special DPTs. Recall that if θ is convex, then $S := \widehat{\mathrm{D}^2\theta}$ is a DPT. In the extreme case, a convex function is positively homogeneous of degree 1. Typical example : a norm. Since a derivative lets the degree of homogeneity drop by one unit, the Hessian is homogeneous of degree -1. Then S is homogeneous of degree $1-n$.

Notice that (recall $dx = r^{n-1}dr\,ds(e)$)

$$\frac{1}{r^{n-1}} \in L^1_{\mathrm{loc}} \qquad \text{but } \notin L^{\frac{n}{n-1}}_{\mathrm{loc}},$$

marginally.

In addition, the Euler identity $\mathrm{D}^2\theta e \equiv 0$ (θ is linear along rays) implies that S is proportional to $e \otimes e$, because of the following (homework)

Lemma 1

If $M \in \mathbf{M}_n(k)$ has rank $n-1$, then $\widehat{M}$ is rank-one, with $\ker \widehat{M} = (\ker M)^\perp$.

Thus

$$S = \frac{\lambda(e)}{r^{n-1}}\, e \otimes e \tag{1}$$

for some non-negative measure λ over the unit sphere S_{n-1}. For instance $\theta^0(x) \equiv |x|$ yields

$$S^0 = \frac{1}{r^{n-1}}\, e \otimes e.$$

Two natural questions come immediately :

> *Conversely, given a tensor S of the form (1) for some measure $\lambda \geq 0$, is it divergence-free ?*
> *Is it special ?*

The first answer is

Proposition 1

Let λ be a non-negative measure over S_{n-1}. Then the tensor

$$S = \frac{\lambda(e)}{r^{n-1}}\, e \otimes e$$

is Div-controlled, with

$$\operatorname{Div} S = V_\lambda \delta_{x=0}, \qquad V_\lambda = \int_{S_{n-1}} e\, d\lambda(e).$$

In particular, S is a DPT if, and only if $V_\lambda = 0$.

Remark that if λ is even, then $V_\lambda = 0$; S is always a DPT in this case.

iNδAM

Proof

Notice $S = \lambda S^0$ where $S^0 = \frac{1}{r^{n-1}}\, e \otimes e$. Away from the origin, S^0 is smooth, divergence-free, thus

$$\mathrm{Div}\, S = \lambda \mathrm{Div}\, S^0 + S^0 \nabla\lambda = S^0 \nabla\lambda = (\nabla\lambda \cdot e)\, \frac{e}{r^{n-1}}\,.$$

Since λ is homogeneous of degree 0, we have $\nabla\lambda \cdot e \equiv 0$ and we conclude that S is divergence-free away from the origin. Eventually, we calculate

$$\langle \mathrm{Div}\, S, \phi \rangle = -\langle S, \nabla\phi \rangle = -\int_B S : \nabla\phi = -\int_{B_\epsilon} - \int_{B \setminus B_\epsilon},$$

where $\epsilon \in (0,1)$ and $\phi \in \mathcal{D}(B)^n$.

Consider the first integral :

iNδAM

$$
\begin{aligned}
\int_{B_\epsilon} S : \nabla\phi &= \int_0^\epsilon r^{n-1}\,dr \int_{S_{n-1}} (e\cdot\nabla\phi(re))\,e\,\frac{d\lambda(e)}{r^{n-1}} \\
&= \int_0^\epsilon dr \int_{S_{n-1}} (e\cdot\nabla\phi(re))\,e\,d\lambda(e) \\
&\sim \epsilon\left(\int_{S_{n-1}} e\otimes e\,d\lambda(e)\right) : \nabla\phi(0) \stackrel{\epsilon\to 0+}{\longrightarrow} 0.
\end{aligned}
$$

Because S is divergence-free away from the origin, we may apply Green's Formula to the second integral (the outward unit normal is $-e$) :

$$
\begin{aligned}
-\int_{B\setminus B_\epsilon} S : \nabla\phi &= -\int_{B\setminus B_\epsilon} \mathrm{div}\,(S\phi) = \int_{S_\epsilon} (Se)\cdot\phi \\
&= \int_{S_{n-1}} \phi(\epsilon e)\cdot e\,d\lambda(e) \stackrel{\epsilon\to 0+}{\longrightarrow} \phi(0)\cdot V_\lambda.
\end{aligned}
$$

Hence $\langle \mathrm{Div}\,S, \phi\rangle = \phi(0)\cdot V_\lambda$, that is $\mathrm{Div}\,S = V_\lambda \delta_{x=0}$.

Geometrical interpretation

Denote $\partial\theta$ the sub-differential. Its image is exactly the convex body $K = \partial\theta(0)$, that is

$$K = \{x \in \mathbb{R}^n \,|\, x \cdot e \leq \theta(e),\ \forall e \in S_{n-1}\}.$$

The boundary $\Sigma = \partial K$ is the image $\partial\theta(\mathbb{R}^n \setminus \{0\})$. The Gauß map $\nu : \Sigma \to S_{n-1}$ associates to a point $x \in \Sigma$ the unit outward normal $\vec{N}$ to Σ at x.

Lemma 2

The measure λ in $S = r^{1-n}\lambda e \otimes e$ is the push forward of the $(n-1)$-dimensional Hausdorff measure over Σ, by the Gauß map.

The identity

$$\int_{S_{n-1}} e \, d\lambda(e) = 0 \qquad (2)$$

expresses that the centroid of λ is at the origin.

Minkowski's problem

It asks, given $\lambda \in \mathcal{M}_+(S_{n-1})$ satisfying (2), whether there exists a convex body K such that λ is the measure described above. This amounts to finding a positively homogeneous potential θ, such that S given by (1) equals $\widehat{\mathrm{D}^2\theta}$.

In order that the solution correspond to a genuine convex body, it is necessary that the support of λ is not contained in a sphere of smaller dimension (non-degenerate case). But if we are interested in the analytic part only (finding the potential θ), this hypothesis is not needed.

Minkowski's problem amounts to solving a Monge–Ampère equation on the unit sphere, whose data is λ and the unknown is θ.

Eugenio Calabi said

> *From the geometric view point, the Minkowski problem is the Rosetta Stone, from which several related problems can be solved.*

The solution of Minkowski's problem is a long story, beginning with Hermann Minkowski himself, when K is a polytope. It boosted the modern theory of elliptic nonlinear PDEs, with major contributions by A. Aleksandrov, L. Nirenberg and others. The final word was uttered by Pogorelov [1].

1. A. V. Pogorelov. *The Minkowski multidimensional problem*. John Wiley & sons, NY (1978).

Translating the complete result in terms of DPTs, we have

Theorem 1

Let λ be a non-negative finite measure over S_{n-1} such that $V_\lambda = 0$. Then there exists a convex potential θ, positively homogeneous of degree one, such that the special DPT $S = \widehat{\mathrm{D}^2\theta}$ equals

$$\frac{\lambda(e)}{r^{n-1}}\, e \otimes e.$$

This potential is unique in the non-degenerate case, up to the addition of an affine function.

What does homogeneity of degree $1-n$ mean ?

A paradox : Consider a homogeneous potential as above, but one that depends only upon $\widehat{x}_1 = (x_2, \dots, x_n)$. For example $\theta = \|\widehat{x}_1\|$. Then not only $\mathrm{D}^2\theta e = 0$, but also $\mathrm{D}^2\theta \vec{e}_1 = 0$. For $\widehat{x}_1 \neq 0$, the rank of $\mathrm{D}^2\theta$ is at most $n-2$, implying

$$S := \widehat{\mathrm{D}^2\theta} = 0_n.$$

Therefore S is supported by the first axis $L_1 := \mathbb{R}\vec{e}_1$. But since θ is translation-invariant, the same is true for S and we can write $S = T\mathcal{L}_{L_1}$, where $\mathcal{L}_L$ denotes the Lebesgue measure over a line L, and $T \in \mathbf{Sym}_n^+$ is a constant matrix. In the example above, we simply have

$$T \equiv \vec{e}_1 \otimes \vec{e}_1.$$

How can S be homogeneous of degree $1-n$, and simultaneously constant in the x_1-direction ?

iNδAM

Explanation. For measures, and more generally for distributions, the notion of homogeneity is defined by duality, *via* the same integral formula that is valid for functions. For instance, one finds that δ_0 is homogeneous of degree $-n$ (!).

More generally

Proposition 2

The Lebesgue measure $\mathcal{L}_E$ over a linear subspace E of codimension k is positively homogeneous of degree $-k$.

Coming back to the case of special DPTs supported by a line, we see that $\mathcal{L}_{L_1}$ is homogeneous of degree $1-n$. This reconciles the homogeneity of S with the translation invariance in the x_1-direction.

To conclude with homogeneity, one may ask whether there exist DPTs over B that are positively homogeneous with an arbitrary degree α.

- $\alpha > 1 - n$. **Yes**. Just take the special DPT associated with a convex homogeneous potential of degree $2 + \frac{\alpha}{n-1} > 1$. There are actually plenty of others example that are nor "special".
- $\alpha < 1 - n$. **No** : the only positively homogeneous DPT of degree $\alpha < 1 - n$ is $S \equiv 0_n$.
- $\alpha = 1 - n$. **Rigidity** : the only positively homogeneous DPTs of degree $1 - n$ are the special ones described above.

Determinantal masses

An annoying fact. Let θ be a convex function, positively homogeneous of degree 1, and S be the associated special DPT. Because $S=\lambda S^0$ is rank-one away from the origin, and $(\det S)^{\frac1n}$ is absolutely continuous (Comp. Int.), it is $\equiv 0$. A direct application of Thm 6 (Lesson #3) is useless, as the Functional Inequality is trivial :

$$\int_\Omega (\det S)^{\frac{1}{n-1}}\,dx=\int_\Omega 0\,dx=0\le c_n\|S\vec N\|_{\mathcal M}^{\frac{n}{n-1}}\quad !!$$

If instead θ is smooth (hence not homogeneous), then

$$(\det S)^{\frac{1}{n-1}}=\det \mathrm{D}^2\theta \tag{3}$$

is the Jacobian of the one-to-one vector field $\nabla\theta$. The quantity

$$\int_B (\det S)^{\frac{1}{n-1}}\,dx=\mathrm{Vol}(\nabla\theta(B)), \tag{4}$$

estimated by Compensated Integrability, is nothing the volume of the image of B under $\nabla\theta$.

Equality (4) sheds light on the nature of $(\det S)^{\frac{1}{n-1}}$ when θ, instead of being smooth, is not differentiable at the origin. Then $\nabla\theta$ has to be interpreted as the sub-gradient $\partial\theta$. The Jacobian of $\nabla\theta$ will be understood as the pullback of the Lebesgue measure by $\partial\theta$.

This means

$$\int_{B_\epsilon} d\det \mathrm{D}^2\theta := \mathrm{Vol}(\partial\theta(B_\epsilon)) = \mathrm{Vol}(\partial\theta(0)).$$

This suggests to define $\det \mathrm{D}^2\theta$, hence $(\det S)^{\frac{1}{n-1}}$ as the Dirac mass

$$\mathrm{Vol}(\partial\theta(0))\delta_{x=0}.$$

Warning : This singular part of $(\det S)^{\frac{1}{n-1}}$ does not come from the singular part of S (the latter can be $\equiv 0$, if λ is absolutely continuous over S_{n-1}).

We are led to the

Definition 1

Let S be a Div-controlled tensor over an open domain $\Omega \subset \mathbb{R}^n$.
For $a \in \Omega$, let $Z(a)$ be the set of convex functions θ, positively homogeneous of degree one about a, such that $\widehat{\mathrm{D}^2\theta} \le S$ in a neighbourhood of a.
Then the determinantal mass of S at a is the supremum of $\mathrm{Vol}(\partial\theta(a))$ among those $\theta \in Z(a)$.
It is denoted

$$\mathrm{Dm}(S; a).$$

Improved Functional Inequality

A natural question is whether FI is still valid when we incorporate the determinantal masses to $(\det S)^{\frac{1}{n-1}}$.

Theorem 2

Let S be a Div-controlled tensor in an n-dimensional bounded open domain Ω. Let $(a_j)_{j=0,\dots}$ be a discrete (either finite or countable) subset of Ω. Then the following generalized form of the Functional Inequality holds true :

$$\int_\Omega (\det S)^{\frac{1}{n-1}}\,dx + \sum_j \mathrm{Dm}(S;a_j) \leq c_n \left(\|\mathrm{Div}\, S\|_{\mathcal{M}(\Omega)} + \|S\vec{N}\|_{\mathcal{M}(\partial\Omega)} \right)^{\frac{n}{n-1}}. \tag{5}$$

Corollary 1

A Div-controlled tensor S carries at most countably many determinantal masses.

Proof (of the Theorem)

It is enough to consider the case of finitely many singularities. But since the calculations below are local, we may assume only one singularity at $a \in \Omega$.

Let $\theta \in Z(a)$ and $B_r \subset \Omega$ be a ball in which $S \geq \mathrm{D}^2\theta$. We decompose

$$S = T + \mathrm{D}^2\theta \quad \text{in } B_r.$$

By assumption, T is a Div-controlled tensor in B_r, where it satisfies $\mathrm{Div}\, T = \mathrm{Div}\, S$.

Lemma 3

There exists a convex function θ', which coincides with θ in $B_r \setminus B_{r/2}$ but is smooth at the origin.

iNδAM

The modified potential satisfies

$$\int_{B_r} \det \mathrm{D}^2\theta' \, dx = \mathrm{Vol}(\nabla\theta'(B_r)) = \mathrm{Vol}(\partial\theta(0)). \tag{6}$$

We form the Div-controlled tensor $S' = T + \mathrm{D}^2\theta'$. The formula defines S' in B_r, but since $S' \equiv S$ in $B_r \setminus B_{r/2}$, we may concatenate with S in $\Omega \setminus B_{r/2}$. Thus S' is defined in Ω.

Apply FI to S', using three facts : first of all, $\mathrm{Div}\, S' \equiv \mathrm{Div}\, S$, then $\det S' \geq (\det \mathrm{D}^2\theta')^{n-1}$ in $B_{r/2}$ and finally $S' \equiv S$ away from $B_{r/2}$:

$$\begin{aligned} \int_{\Omega\setminus B_{r/2}} (\det S)^{\frac{1}{n-1}}\, dx \quad &+ \int_{B_{r/2}} \det \mathrm{D}^2\theta' \\ &\leq c_d \left(\|\mathrm{Div}\, S\|_{\mathcal{M}(\Omega)} + \|S\vec{N}\|_{\mathcal{M}(\partial\Omega)} \right)^{\frac{n}{n-1}} . \end{aligned}$$

Then use (6), and take the supremum over $\theta \in Z(a)$.

A curious example

Product of distributions. Recall that if Γ_1, Γ_2 are distributions, their product $\Gamma_1 \Gamma_2$ does not make sense in general, but it does whenever their wave front sets are transversal to each other.
Transversality arises in the planar $(n = 2)$ example where Γ_j is the Lebesgue measure along a line L_j, and L_1, L_2 are generic. For instance

$$\mathcal{L}_{x_1=0} \cdot \mathcal{L}_{x_2=0} = \delta_{x=0} \qquad (n = 2).$$

This generalizes to the n-dimensional case as follows. Consider the axes $L_1, \dots, L_n$ of the canonical basis and their Lebesgue measures $\mathcal{L}_1, \dots, \mathcal{L}_n$. The tensor $S = \mathrm{diag}(\mathcal{L}_1, \dots, \mathcal{L}_n)$ is a DPT ($\partial_j \mathcal{L}_j \equiv 0$). It is homogeneous of degree $1 - n$, hence a special one (rigidity). Its potential $\theta(x) = \frac{1}{2} \sum_{j=1}^n |x_j|$ yields the identity, in the light of Definition 1,

$$(\mathcal{L}_1 \cdots \mathcal{L}_n)^{\frac{1}{n-1}} = \delta_{x=0} \quad (!).$$

Practical aspects

To exploit at best the improved FI, we need a precise knowledge of $\mathrm{Dm}(S;a)$.

Say that $a=0$ and we know a lower barrier $\frac{\lambda(e)}{r^{n-1}}\,e\otimes e=\widehat{\mathrm{D}^2\theta}$ of S.

How do we determine $\mathrm{Vol}(\partial\theta(0))$ *?*

There is no explicit formula in general because the knowledge of θ requires the resolution of a Monge–Ampère equation over the unit sphere.

The next slides give tools in this directions.

iNδAM

The two-dimensional case

When $n=2$, Minkowski's problem is solved explicitly, because the correspondance $S=\widehat{\mathrm{D}^2\theta}\longleftrightarrow\theta$ is linear.

Parametrize S_1 by the angle $\alpha\in\mathbb{R}/2\pi\mathbb{Z}$. The compatibility condition $V_\lambda=0$ reads

$$\int_0^{2\pi}\cos\alpha\,d\lambda(\alpha)=0,\qquad\int_0^{2\pi}\sin\alpha\,d\lambda(\alpha)=0.$$

This is precisely the condition of solvability for the ODE

$$p+\frac{d^2p}{d\alpha^2}=\lambda,\qquad p(\alpha+2\pi)\equiv p(\alpha).$$

The potential is given by (homework)

$$\theta\left(re^{i\alpha}\right)=rp(\alpha).$$

The sub-differential at $x = 0$ is the domain enclosed by the curve

$$\alpha \mapsto \nabla\theta\left(e^{i\alpha}\right) = \begin{pmatrix} p(\alpha)\cos\alpha - p'(\alpha)\sin\alpha \\ p(\alpha)\sin\alpha + p'(\alpha)\cos\alpha \end{pmatrix}.$$

The determinantal mass is its area,

$$\begin{aligned} \mathrm{Dm}(S;0) &= \frac{1}{2}\int_0^{2\pi} \theta_{,1}\, d\theta_{,2} \\ &= \frac{1}{2}\int_0^{2\pi} (p(\alpha)\cos\alpha - p'(\alpha)\sin\alpha)(p+p'')\cos\alpha\, d\alpha \\ &= \frac{1}{2}\int_0^{2\pi} (p(\alpha)\cos\alpha - p'(\alpha)\sin\alpha)\cos\alpha\, d\lambda(\alpha). \end{aligned}$$

Let us denote

$$\mu(\alpha) := \int_0^{\alpha} \cos s\, d\lambda(s).$$

Integrating by parts,

$$\begin{aligned}
\mathrm{Dm}(S;0) &= \frac{1}{2}\int_0^{2\pi}(p(\alpha)\cos\alpha - p'(\alpha)\sin\alpha)\mu'(\alpha)\,d\alpha \\
&= -\frac{1}{2}\int_0^{2\pi}(p(\alpha)\cos\alpha - p'(\alpha)\sin\alpha)'\mu(\alpha)\,d\alpha \\
&= \frac{1}{2}\int_0^{2\pi}\mu(\alpha)\sin\alpha\,d\lambda(\alpha) \\
&= \frac{1}{2}\int_0^{2\pi}\sin\alpha\,d\lambda(\alpha)\int_0^{\alpha}\cos s\,d\lambda(s).
\end{aligned}$$

A symmetrization of the formula above yields

Proposition 3 ($n = 2$.)

$$\mathrm{Dm}(\lambda S^0;0) = \frac{1}{8}\int_0^{2\pi}\int_0^{2\pi}\sin|\beta-\alpha|\,d\lambda(\beta)\,d\lambda(\alpha). \tag{7}$$

Simplest non-trivial case ($n = 2$)

The following will be useful in applications. Suppose λ is the sum of three Dirac masses

$$\lambda = a\delta_X + b\delta_Y + c\delta_Z, \qquad a, b, c \geq 0 \text{ and } X, Y, Z \in S_1.$$

The condition that $S = \lambda S^0$ be a DPT writes

$$aX + bY + cZ = 0.$$

Then Formula (7) gives

$$\mathrm{Dm}(S; 0) = \frac{1}{4} |\det(aX, bY)|. \tag{8}$$

Remark that this expression is symmetric in the data, since for instance

$$\det(cZ, aX) = \det(aX, aX + bY) = \det(aX, bY).$$

Block diagonal tensors

Since our applications occur in higher space dimension $n \geq 3$, we need an other tool.

Let us decompose $\mathbb{R}^n = \mathbb{R}^p \oplus \mathbb{R}^q$, with $x = (x_-, x_+)$. If θ_- (resp. θ_+) is a convex homogeneous potential over $\mathbb{R}^p$ (resp. $\mathbb{R}^q$), we form

$$\theta(x) = \theta_-(x_-) + \theta_+(x_+). \tag{9}$$

The corresponding special DPT $\widehat{\mathrm{D}^2\theta}$ is block-diagonal :

$$S = \begin{pmatrix} S_- \otimes \delta_{x_+=0} & 0 \\ 0 & \delta_{x_-=0} \otimes S_+ \end{pmatrix}. \tag{10}$$

Conversely every homogeneous DPT of the from (10) comes from a potential such as in (9).

iNδAM

Remark that S_- and S_+ are themselves DPTs in $\mathbb{R}^p$ and $\mathbb{R}^q$, homogeneous of degree $1-p$ and $1-q$, respectively.

The calculation trick is

Proposition 4

For a block-diagonal DPT, homogeneous of degree $1-n$, we have

$$\mathrm{Dm}(S;0) = \left[\mathrm{Dm}(S_-;0)^{p-1}\mathrm{Dm}(S_+;0)^{q-1}\right]^{\frac{1}{n-1}}.$$

This interprets, in a rigorous form, the equality

$$\det\begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix} = \det M \cdot \det N.$$

The formula remains valid when $\mathbb{R}^n$ splits as $E \oplus^\perp F$ where $\dim E = p$ and $\dim F = q$, and $S_\pm$ are replaced by homogeneous special DPTs over E and F, respectively. When the factors E and F are not orthogonal, the formula remains the same, up to a constant multiplicative factor.

Proof

Using $\mathrm{D}^2\theta = \mathrm{diag}(\mathrm{D}^2\theta_-(x_-), \mathrm{D}^2\theta_+(x_+))$ and the formula

$$\det \mathrm{D}^2\theta_- = \mathrm{Vol}(\partial\theta_-(0))\delta_{x_-=0},$$

we have

$$S_- = \mathrm{Vol}(\partial\theta_+(0))\widehat{\mathrm{D}^2\theta_-}\,, \qquad S_+ = \mathrm{Vol}(\partial\theta_-(0))\widehat{\mathrm{D}^2\theta_+}\,.$$

From these expressions, we have

$$\begin{aligned} \mathrm{Dm}(S_-;0) &= \mathrm{Vol}(\partial\theta_+(0))^{\frac{p}{p-1}} \cdot \mathrm{Vol}(\partial\theta_-(0)) \\ \mathrm{Dm}(S_+;0) &= \mathrm{Vol}(\partial\theta_-(0))^{\frac{q}{q-1}} \cdot \mathrm{Vol}(\partial\theta_+(0)). \end{aligned}$$

Eliminating between both formula, we obtain

$$
\begin{aligned}
\mathrm{Vol}(\partial\theta_+(0))^{n-1} &= \mathrm{Dm}(S_+;0)^{-(p-1)(q-1)} \cdot \mathrm{Dm}(S_-;0)^{p(q-1)} \\
\mathrm{Vol}(\partial\theta_-(0))^{n-1} &= \mathrm{Dm}(S_-;0)^{-(p-1)(q-1)} \cdot \mathrm{Dm}(S_+;0)^{q(p-1)}
\end{aligned}
$$

Using the fact that $\partial\theta(0) = \partial\theta_-(0) \times \partial\theta_+(0)$, thus

$$\mathrm{Vol}(\partial\theta(0)) = \mathrm{Vol}(\partial\theta_-(0)) \cdot \mathrm{Vol}(\partial\theta_+(0)),$$

we obtain the announced formula. ∎

iNδAM