Compensated Integrability and Applications to Mathematical Physics

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In the previous lesson, we studied a rather academic model of particle dynamics, where the bodies interact only through collisions. It is hardly realistic. Physics tells us that particles interact through forces. Two of them are typical :

- Coulomb force between charged particles,
- gravity between massive bodies.

We exploit here a remark made in Lesson #7.2, that the mass-momentum S of a point particle, subjected to a force F, satisfies

$$\operatorname{Div} S = \begin{pmatrix} 0 \\ F \, dt|_\gamma \end{pmatrix}, \tag{1}$$

where $\gamma : t \mapsto (t, y(t))$ is the graph of the trajectory.

We intend to build a dynamical DPT for systems of particles interacting through a force deriving from a potential.

Interacting particles

Consider two point particles $P_{1,2}$ with masses m_j, positions $y_j(t)$ and velocities $v_j(t) = \dot{y}_j$. A potential energy $\phi(|y_2 - y_1|)$ is associated with this pair, which generates the forces applied to both particles :

$$\begin{aligned} F_1 &= -\nabla_{y_1}\phi(|y_2 - y_1|) = \phi'(|y_2 - y_1|)\,\frac{y_2 - y_1}{|y_2 - y_1|}\,, \\ F_2 &= -\nabla_{y_2}\phi(|y_2 - y_1|) = -\phi'(|y_2 - y_1|)\,\frac{y_2 - y_1}{|y_2 - y_1|}\,. \end{aligned}$$

The dynamics is governed by

$$m_j\,\frac{d^2 y_j}{dt^2} = F_j\,.$$

The total momentum and energy are conserved :

$$m_1 v_1 + m_2 v_2 \equiv \mathrm{cst}, \qquad \frac{m_1}{2}\,|v_1|^2 + \frac{m_2}{2}\,|v_2|^2 + \phi(|y_2 - y_1|) \equiv \mathrm{cst}.$$

Pairwise DPT

To this pair, we associate a 2-dimensional manifold

$$\mathcal{M} = \left\{(t, y) \in (0, T) \times \mathbb{R}^d \,|\, y \in [y_1(t), y_2(t)]\right\}.$$

The boundary $\partial\mathcal{M}$ is made of the trajectories of P_1 and P_2.

Then we form the tensor [1]

$$\Sigma := -\phi'(|y_2 - y_1|) \begin{pmatrix} 0 & 0 \\ 0 & z \otimes z \end{pmatrix} (d\ell(y)\, dt)|_{\mathcal{M}},$$

where

$$z := \frac{y_2 - y_1}{|y_2 - y_1|}$$

is the unit tangent vector to $[y_1(t), y_2(t)]$, and $d\ell(y)$ is the Lebesgue measure along this segment.

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1. As usual, $n = 1 + d$.

The action over test vector fields $\vec{\psi} = (\psi_0, \dots, \psi_d) \in \mathcal{D}(\mathbb{R}^n)^n$, of the distribution Σ is

$$\langle \Sigma, \vec{\psi} \rangle = -\int dt \int_{y_1(t)}^{y_2(t)} \phi'(|y_2 - y_1|) \sum_1^d (z_j \psi_j)(t, y) \begin{pmatrix} 0 \\ z \end{pmatrix} d\ell(y).$$

Let us compute the Divergence of Σ. For a test function $\theta \in \mathcal{D}(\mathbb{R}^n)$, we have

$$\begin{aligned} \langle \mathrm{Div}\, \Sigma, \theta \rangle &= -\langle \Sigma, \nabla \theta \rangle \\ &= \int dt \int_{y_1(t)}^{y_2(t)} \phi'(|y_2 - y_1|) \frac{d\theta}{d\ell} \begin{pmatrix} 0 \\ z \end{pmatrix} d\ell(y), \end{aligned}$$

where $\frac{d}{d\ell} = \sum_j z_j \partial_j$ is the derivative along the segment $[y_1(t), y_2(t)]$.

Because both z and $\phi'(|y_2 - y_1|)$ are constant along the segment, the last integral is explicit and one finds

$$\begin{aligned}\langle \operatorname{Div}\Sigma, \theta\rangle &= \int \phi'(|y_2 - y_1|)(\theta(t, y_2) - \theta(t, y_1)) \begin{pmatrix} 0 \\ z \end{pmatrix} dt \\ &= -\int \begin{pmatrix} 0 \\ \theta(t, y_1)F_1 + \theta(t, y_2)F_2 \end{pmatrix} dt.\end{aligned}$$

In other words, we have

$$\operatorname{Div}\Sigma = -\begin{pmatrix} 0 \\ F_1\,dt|_{\gamma_1} + F_2\,dt|_{\gamma_2} \end{pmatrix},$$

where γ_j is the graph followed by P_j.

Remark. $\int \operatorname{Div}\Sigma = 0$ (Green's formula) is the principle action/reaction.

Because of (1), we are led to define the mass-momentum tensor of the pair of particles by

$$S_1^2 := m_1 \begin{pmatrix} 1 & v_1^T \\ v_1 & v_1 \otimes v_1 \end{pmatrix} dt|_{\gamma_1} + m_2 \begin{pmatrix} 1 & v_2^T \\ v_2 & v_2 \otimes v_2 \end{pmatrix} dt|_{\gamma_2} + \Sigma.$$

We have therefore

Proposition 1

The pairwise mass-momentum tensor S_1^2 is symmetric and divergence-free.

Is S_1^2 positive semi-definite ?

The question amounts to looking at the sign of the tensor Σ (the analogue of the colliton, an *interacton* ?), whether it is positive or not.

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The answer is yes if, and only if $\phi' \leq 0$, that is ϕ is a non-increasing function. This is equivalent to saying that the force applied to P_1 is oriented in the direction opposite to $y_2 - y_1$, that is opposite to P_2, and symmetrically F_2 is opposite to P_1. In other words,

Proposition 2

The pairwise mass-momentum tensor S_1^2 is a local DPT if, and only if the force is repulsive.

Applications

- For a pair of charged particles, this tensor is a local DPT if, and only if the charges have the **same sign**.
- For a pair of massive body, the force being gravity, the tensor is **not** a local DPT.

1-D configuration

The one-dimensional case ($d = 1$) is interesting, in that we are able to carry out all the calculations. When $n = 2$, a Divergence-free tensor is always special : there exists a potential θ such that

$$S = \widehat{\mathrm{D}^2\theta} = \begin{pmatrix} \partial_y^2\theta & -\partial_y\partial_t\theta \\ -\partial_y\partial_t\theta & \partial_t^2\theta \end{pmatrix}.$$

To identify θ, we start with the region $\mathcal{M}$, defined by $y_1(t) < y < y_2(t)$, where $S \equiv \Sigma$. Since $\Sigma = \begin{pmatrix} 0 & 0 \\ 0 & \bullet \end{pmatrix}$, we have $\partial_y\partial_t\theta = \partial_y^2\theta = 0$, thus $\partial_y\theta$ is constant. We may set (θ is unique up to the addition of an affine function)

$$\partial_y\theta \equiv 0, \qquad (y_1(t) < y < y_2(t)).$$

This tells us that $\theta(t, y) \equiv \beta(t)$ for some function β. Finally, the bottom-right entry of Σ gives us [2]

$$\beta''(t) = -\phi'(y_2(t) - y_1(t)). \tag{2}$$

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2. Beware that $\beta'(t)$ is not equal to $\phi(y_2 - y_1)$.

We then pass to the region $y > y_2(t)$ at the right of $\mathcal{M}$, where $S \equiv 0$. There, $\nabla\theta$ is a constant, but we may not any more choose it. Instead, we use the jump condition across γ_2. On the one hand, we have

$$[\partial_y\theta]\begin{pmatrix} -v_2 \\ 1 \end{pmatrix} \frac{ds}{\sqrt{1+v_2^2}} = [\partial_y\theta]\vec{N}\,ds.$$

On the other hand the singular contribution over γ_2 yields

$$[\partial_y\theta]\vec{N}\,ds = m_2\begin{pmatrix} -v_2 \\ 1 \end{pmatrix} \frac{ds}{\sqrt{1+v_2^2}}.$$

Likewise

$$[\partial_t\theta]\begin{pmatrix} -v_2 \\ 1 \end{pmatrix} \frac{ds}{\sqrt{1+v_2^2}} = [\partial_t\theta]\vec{N}\,ds = m_2\begin{pmatrix} v_2^2 \\ -v_2 \end{pmatrix} \frac{ds}{\sqrt{1+v_2^2}}.$$

This tells us $[\partial_y\theta] = m_2$ and $[\partial_t\theta] = -m_2 v_2$, from which we infer the value

$$\nabla\theta(t,y) = \begin{pmatrix} \beta'(t) - m_2 v_2(t) \\ m_2 \end{pmatrix} \qquad (y > y_2(t)).$$

The constancy of $\nabla\theta$ in this region is equivalent to Newton's law $m_2 \dot{v}_2 = F_2 = -\phi'(y_2 - y_1) = \beta''$.

The same analysis yields

$$\nabla\theta(t,y) = \begin{pmatrix} \beta'(t) + m_1 v_1(t) \\ -m_1 \end{pmatrix} \qquad (y < y_1(t)).$$

The Functional Inequality ($d = 1$)

Let us assume from now on that the force is repulsive, so that S is a DPT in every band $(0, T) \times \mathbb{R}$. We assume that $\phi(\infty) = 0$ (equivalently, the potential energy is bounded below). Thus ϕ is always ≥ 0.

Theorem 3 of L#5 writes

$$\int_0^T dt \int_{\mathbb{R}} (\det S)\, dy \leq k_1 M_0 (\|m(0)\|_{\mathcal{M}} + \|m(T)\|_{\mathcal{M}}),$$

where m stands for the off-diagonal term of S. Here

$$\|m(t)\|_{\mathcal{M}} = m_1|v_1| + m_2|v_2| \leq \sqrt{2M_0 E_{\rm kin}}$$

where the kinetic energy is bounded [3] by the total energy E_0. The bound is therefore $2k_1 M_0 \sqrt{2M_0 E_0}$.

3. Mind that the potential energy is non-negative.

The integral is nothing but the area of the range of $\nabla\theta$.

It turns out that $\nabla\theta$ is constant on each sides of the band : either $t = 0$, or $y > y_2(t)$, or $t = T$, or $y < y_1(t)$. The respective values of the gradient are

$$\begin{pmatrix} \beta'(0) \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \beta'(t) - m_2 v_2 \\ m_2 \end{pmatrix}, \quad \begin{pmatrix} \beta'(T) \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \beta'(t) + m_1 v_1 \\ m_1 \end{pmatrix}.$$

The area must be understood as that of the polygone (a quadrilateral) spanned by these four vertices. It equals

$$\mathrm{Vol}(K) = \frac{m_1 + m_2}{2}\,(\beta'(T) - \beta'(0)) = \frac{m_1 + m_2}{2} \int_0^T \beta''(t)\,dt.$$

Because of $\beta'' = F_2 = -F_1$ and $m_j \dot{v}_j = F_j$, this gives us

$$\mathrm{Vol}(K) = \frac{m_1 m_2}{2} \left(V(T) - V(0) \right)$$

where $V(t) := v_2(t) - v_1(t)$ is an increasing function[4].

Of course, the resulting estimate

$$m_1 m_2 (V(T) - V(0)) \leq c M_0 \sqrt{M_0 E_0}$$

is not a big deal. We could have established it directly from $E_{\mathrm{kin}}(t) \leq E_0$. Its triviality must not surprize us, because Compensated Integrability is a trivial fact when $n = 2$ (*Credo*). Instead the calculation above gives us some hope to obtain interesting estimates in higher space dimension, as well as in the case where there are many more particles.

4. Because the force is repulsive.

Many-particles configurations

The construction above generalizes easily to a system of N particles P_j, which may differ by their masses m_j. The ith particle is subjected to a force $F_i = -\nabla_{y_i} W$, whose potential is defined by

$$W(y_1, \dots, y_N) = \sum_{1 \le i < j \le N} \phi(|y_j - y_i|).$$

We thus have

$$F_i = \sum_{j \neq i} \phi'(|y_j - y_i|) \frac{y_j - y_i}{|y_j - y_i|}.$$

Again, the force is repulsive if $\phi' \le 0$. We still assume that $\phi(\infty) = 0$. Newton's law

$$m_j \frac{d^2 y_j}{dt^2} = F_j$$

implies that the total momentum and energy are conserved :

$$\sum_j m_j v_j \equiv \text{cst}, \qquad \sum_j \frac{m_j}{2} |v_j|^2 + \sum_{i<j} \phi(|y_j - y_i|) \equiv \text{cst}.$$

The mass-momentum tensor S is the sum of the following contributions :

- For each particle P_j, with space-time graph γ_j, the tensor

$$S_j = m_j \begin{pmatrix} 1 & v_j^T \\ v_j & v_j \otimes v_j \end{pmatrix} dt|_{\gamma_j}.$$

- For each pair (P_i, P_j), the tensor

$$\Sigma_i^j = -\begin{pmatrix} 0 & 0 \\ 0 & \phi'(|y_j - y_i|) \frac{(y_j - y_i)\otimes(y_j - y_i)}{|y_j - y_i|^2} \end{pmatrix} (d\ell(y)\, dt)|_{\mathcal{M}_i^j}.$$

  Hereabove, $\mathcal{M}_i^j$ is a 2-dimensional manifold, the union over t of the segments $\{t\} \times [y_i(t), y_j(t)]$, and $d\ell(y)$ is the Lebesgue measure along $[y_i(t), y_j(t)]$.

The same calculation as in the 2-particle configuration gives

$$\mathrm{Div}\, S \equiv 0.$$

When the force is repulsive and $\phi(0) = +\infty$, the conservation of energy ensures on the one hand that the velocities remain bounded, and on the other hand particles do not collide. Therefore the dynamics admits a unique solution, which is defined for all time.

We shall not carry out the application of Compensated Integrability, which can be rather involved. Even the 1-dimensional case becomes complicated when we have more than two particles.

The multi-dimensional case presents the difficulty that $(\det S)^{\frac{1}{n}} \equiv 0$. To extract some information, we need to establish another version of the Functional Inequality, exploiting the remark that in the 1-D case, $\det S$ appears to be a singular measure supported by the graphs γ_j. This is left for future investigations.

Vlasov-type models

In the many-particle configuration, we may form the empirical density, which is a singular measure over $\mathbb{R}_t \times \mathbb{R}_y^d \times \mathbb{R}_v^d$,

$$f_N = \sum_{j=1}^N m_j \, dt|_{\gamma_j} \otimes \delta_{v=v_j}.$$

For every test function $\psi(t, y, v)$,

$$\langle f_N , \psi \rangle = \sum_j m_j \int \psi(t, y_j(t), v_j(t)) \, dt.$$

The conservation of the particles is expressed by the *Liouville equation*,

$$\partial_t f_N + V \cdot \nabla_y f_N + G \cdot \nabla_v f_N = 0,$$

where V and G are vector fields such that

$$V(t, y_j, v_j) = \frac{dy_j}{dt} = v_j(t), \qquad G(t, y_j, v_j) = \frac{dv_j}{dt} = \frac{1}{m_j} F_j(t).$$

Say that the particles are identical, with $m \sim \frac{1}{N}$. When the number of particles tends to infinity, and under suitable assumptions regarding the initial data $f_N(0,\cdot,\cdot)$, we expect that f_N tends to a "continuous" density $f(t,y,v)$, which satisfies in the limit a PDE, the *Vlasov equation*

$$(\partial_t + v\cdot\nabla_y + G\cdot\nabla_v)f = 0. \tag{3}$$

Warning : The vector field $G(t,y)$ is not exactly the force applied to particle, because it has been renormalized ; it is instead a force per unit mass. For instance, in gravitational systems like stars or galaxies, G is just the gravitational field.

Taking the two first moments of (3) yields the following macroscopic PDEs, which reveal the role of the field G :

$$\partial_t \rho + \operatorname{div}_y m = 0, \tag{4}$$

$$\partial_t m + \operatorname{Div}_y T = \rho G, \tag{5}$$

where as usual (see Lesson #6.2)

$$\rho(t,y) = \int_{\mathbb{R}^d} f(t,y,v)\,dv, \qquad m(t,y) = \int_{\mathbb{R}^d} f(t,y,v)\,v\,dv,$$

$$T(t,y) = \int_{\mathbb{R}^d} f(t,y,v)\,v \otimes v\,dv.$$

The Vlasov equation (3) is completed with a definition of the field

$$G = -\nabla\xi.$$

The potential ξ may have an external origin. Here we suppose instead that it is determined by the density ρ (self-induction). It is then a convolution

$$\xi(t,\cdot) = \phi * \rho(t,\cdot), \quad \text{that is} \quad \xi(t,y) = \int_{\mathbb{R}^d} \phi(y-z)\rho(t,z)\,dz.$$

Momentum. We assume that the underlying Physics is rotationally invariant, and therefore $\phi(z)$ is actually $\chi(|z|)$. Then $\nabla\phi$ is odd, and

$$\int_{\mathbb{R}^d} \rho G\,dy = -\int\int \rho(y)\rho(z)\nabla\phi(y-z)\,dz\,dy \equiv 0.$$

This ensures the conservation of global linear momentum :

$$\frac{d}{dt}\int_{\mathbb{R}^d} m(t,y)\,dy \equiv 0.$$

Energy. In the same vein, the energy balance writes

$$\begin{aligned}
\partial_t \int \frac{|v|^2}{2} f\, dv + \operatorname{div}_y \int \frac{|v|^2}{2} fv\, dv &= -\int \operatorname{div}_v(fG) \frac{|v|^2}{2}\, dv \\
&= \int fG \cdot v dv \\
&= m \cdot G = -m \cdot \nabla \xi \\
&= -\operatorname{div}_y(\xi m) + \xi \operatorname{div}_y m \\
&= -\operatorname{div}_y(\xi m) - \xi\, \partial_t \rho.
\end{aligned}$$

Let us integrate in space, killing formally the divergence terms :

$$\partial_t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|v|^2}{2} f\, dv\, dy = -\int_{\mathbb{R}^d} \xi\, \partial_t \rho\, dy.$$

Because of the symmetry of ϕ, the bilinear form

$$B[\rho_1, \rho_2] = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \rho_1(y)\rho_2(z)\phi(y-z)\,dz\,dy$$

is symmetric. Therefore the time derivative of the quadratic form

$$E[\rho] := \frac{1}{2}\,B[\rho,\rho] = \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \rho(y)\rho(z)\chi(|y-z|)\,dy\,dz$$

equals $B[\rho, \partial_t \rho]$. This is nothing but the integral of $\xi\,\partial_t\rho$.

We thus deduce the conservation law of energy :

$$\frac{d}{dt}\left(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|v|^2}{2} f\,dv\,dy + E[\rho(t)]\right) = 0, \qquad (6)$$

where the first integral is the kinetic part. The internal energy $E[\rho]$ is positive for a repulsive force (from $\chi(\infty) = 0$ and $\chi' \leq 0$, we derive $\chi \geq 0$).

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Looking for a DPT

We point out that (5) is not in conservation form. At least,

it is not yet so.

If we wish to let the system (4,5) appear in the form

$$\mathrm{Div}_{t,y} S = 0$$

for some symmetric tensor, then we have to treat the right-hand side ρG, being inspired by what we did in the finitely many-particles situation.

The goal is to write the source term ρG as the Divergence (in space variables) of some symmetric $d \times d$ tensor.

A naive attempt

Consider the so-called *Vlasov–Poisson* equation, which is (3) with $G = -\nabla\xi$, where the potential ξ is given by the Poisson equation [5]

$$\Delta\xi = \alpha\rho.$$

The coefficient α is constant. Its sign reflects whether the force is attractive ($\alpha > 0$, gravity force) or repulsive ($\alpha < 0$, Coulomb force). A simple calculation gives

$$\begin{aligned} -\rho G &= \frac{1}{\alpha}\Delta\xi\nabla\xi = \frac{1}{\alpha}\,\mathrm{Div}\left(\nabla\xi\otimes\nabla\xi - \frac{1}{2}|\nabla\xi|^2 I_d\right) \\ &= \frac{1}{\alpha}\,\mathrm{Div}\left(G\otimes G - \frac{1}{2}|G|^2 I_d\right). \end{aligned}$$

5. Homework : What is the kernel χ ?

We have thus found a Divergence-free tensor, namely

$$\begin{pmatrix} \rho & m^T \\ m & T + \frac{1}{\alpha}\left(G \otimes G - \frac{1}{2}|G|^2 I_d\right) \end{pmatrix}.$$

It has the flaw, however, that we don't control [6] the sign of this tensor, because the contribution

$$G \otimes G - \frac{1}{2}|G|^2 I_d$$

is indefinite : its eigenvalues are $\pm\frac{1}{2}|G|^2$.

This failure is related to the fact that the construction above cannot be adapted to a more general kernel χ.

6. Unless $d = 1$ and $\alpha > 0$.

A general construction

We suppose here that the force is self-induced :

$$G=-\nabla\xi,\qquad \xi(t,y)=\int_{\mathbb{R}^d}\chi(|y-z|)\rho(z)\,dz=\int_{\mathbb{R}^d}\chi(|z|)\rho(y-z)\,dz.$$

Let us drop the time variable and write

$$\begin{aligned}-(\rho G)(y) &= (\rho\nabla\xi)(y)=\rho(y)\int_{\mathbb{R}^n}\nabla\phi(z)\rho(y-z)\,dz\\ &= \rho(y)\int_{\mathbb{R}^n}\chi'(|z|)\rho(y-z)\frac{z}{|z|}\,dz\\ &= \frac{1}{2}\rho(y)\int_{\mathbb{R}^n}\chi'(|z|)(\rho(y-z)-\rho(y+z))\frac{z}{|z|}\,dz\end{aligned}$$

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In order to write $-\rho G$ as the y-Divergence of a symmetric tensor, it suffices to have

$$\rho(y)(\rho(y-z)-\rho(y+z)) = \mathrm{div}_y(A(y,z)z) = z\cdot\nabla_y A(y,z)$$

and then to integrate the corresponding identity against $\frac{1}{2}\,\chi'(|z|)\frac{z}{|z|}\,dz$.

Here is a candidate :

$$A(y,z) = -\int_{-\frac{1}{2}}^{\frac{1}{2}} \rho\left(y+(s-\frac{1}{2})z\right)\,\rho\left(y+(s+\frac{1}{2})z\right)\,ds.$$

Actually

$$\begin{aligned} z\cdot\nabla_y A(y,z) \quad = - \quad & \int_{-\frac{1}{2}}^{\frac{1}{2}}\Big(\rho(y+(s-\frac{1}{2})z)\,z\cdot\nabla\rho(y+(s+\frac{1}{2})z) \\ & +\rho(y+(s+\frac{1}{2})z)\,z\cdot\nabla\rho(y+(s-\frac{1}{2})z)\Big)\,ds \end{aligned}$$

gives

$$z \cdot \nabla_y A(y,z) = -\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{d}{ds}\left(\rho(y+(s-\frac{1}{2})z)\rho(y+(s+\frac{1}{2})z)\right)\,ds$$
$$= -\left[\rho(y+(s-\frac{1}{2})z)\rho(y+(s+\frac{1}{2})z)\right]_{-\frac{1}{2}}^{\frac{1}{2}} = \text{OK}.$$

The system (4,5) can be written as $\mathrm{Div}_{t,y} S \equiv 0$ where

$$S(t,y) = \int_{\mathbb{R}^d} \begin{pmatrix} 1 \\ v \end{pmatrix} \otimes \begin{pmatrix} 1 \\ v \end{pmatrix} f(t,y,v)\,dv + \begin{pmatrix} 0 & 0 \\ 0 & \Sigma \end{pmatrix}$$

and

$$\Sigma(t,y) = \frac{1}{2}\int_{\mathbb{R}^d} \chi'(|z|) A(y,z) \frac{z \otimes z}{|z|}\,dz.$$

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Conclusion

Not only we have been able to construct a divergence-free symmetric tensor for a general kernel χ, but we have the following important property

Proposition 3

If the force is repulsive, that is if $\chi' \leq 0$, then the tensor Σ is positive semi-definite, and S is positive semi-definite as well.

When the force is repulsive, and if the initial mass and energy are finite, then S is a DPT over every band $(0,T)\times\mathbb{R}^d$, and we may apply Theorem 3 of Lesson #5.

The same arguments as for the Boltzmann equation work here. We recall that $\det S$ dominates two quantities, which are thus estimated by the Functional Inequality : respectively

$$\det\int_{\mathbb{R}^d}\begin{pmatrix}1\\v\end{pmatrix}\otimes\begin{pmatrix}1\\v\end{pmatrix}f(t,y,v)\,dv,$$

and

$$\rho\det\Sigma.$$

The first quantity above yields the same estimate as for Boltzmann equation :

$$\int_0^T dt \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} f(v_0) \cdots f(v_d) (\mathrm{Vol}(v_0, \ldots, v_d))^2 \, dv_0 \cdots dv_d \right)^{\frac{1}{d}} dy,$$

is controlled in terms of the total mass and total energy.

Likewise, we have

$$\int_0^T dt \int_{\mathbb{R}^d} (\rho \det \Sigma)^{\frac{1}{d}} \, dy \leq c M_0^{\frac{1}{d}} \sqrt{M_0 E_0} \, . \qquad (7)$$

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One-D case

The calculus becomes especially interesting in one space dimension ($d = 1$). Let us for the moment focus on the space integral of $(\rho \det \Sigma)^{\frac{1}{d}}$, which is simply that of $\rho\Sigma$. It equals

$$\int_{\mathbb{R}} \rho\Sigma\, dy = -\frac{1}{2} \int_{\mathbb{R}^2} \int_{-\frac{1}{2}}^{\frac{1}{2}} |z|\chi'\rho(y)\rho(y+(s-\frac{1}{2})z)\rho(y+(s+\frac{1}{2})z)\, dy\, dz\, ds.$$

Let us make the change of variables $(y, z, s) \mapsto (y, a, b)$ where

$$a := y + (s - \frac{1}{2})z, \qquad b := y + (s + \frac{1}{2})z.$$

A simple calculation yields

$$\int_{\mathbb{R}} \rho\Sigma\, dy = -\iiint_{a<y<b} \chi'(b-a)\rho(a)\rho(b)\rho(y)\, da\, db\, dy. \qquad (8)$$

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To exploit (8), we introduce an auxiliary quantity $M(t,a)$, the total mass at left of $a \in \mathbb{R}$:

$$M(t,a) = \int_{-\infty}^{a} \rho(t,y)\,dy.$$

After two integration by parts, we obtain

$$\int_{\mathbb{R}} \rho\Sigma\,dy = -\frac{1}{6}\int\!\!\int_{a<b} \chi'''(b-a)(M(b)-M(a))^3\,da\,db,$$

where we have used $0 \le M \le M_0 < \infty$, and the extra though natural assumption that $\chi'(\infty) = \chi''(\infty) = 0$.

Our estimate bears now the form

$$-\int_0^{+\infty} dt \int\!\!\int_{a<y<b} \chi'''(b-a)(M(b)-M(a))^3\,da\,db \le cM_0^{\frac{1}{d}}\sqrt{M_0E_0}\,. \tag{9}$$

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There remains to interpret it, as a regularity statement about M, and thus about its derivative $\partial_y M = \rho$.

A naive (again) attempt would be to consider the Coulomb potential $\chi = \frac{1}{|z|}$ *. This is however irrealistic because when* $d = 1$*, the convolution* $\chi * \rho$ *does not make sense for continuous densities : the Vlasov–Poisson model is unphysical in one space dimension.*
Amazingly, the estimate (9) would give

$$\int_0^{+\infty} dt \int\int_{a<b} \frac{(M(b) - M(a))^3}{(b-a)^4} \, da \, db < \infty,$$

a property implying[7] *that* M *is a function of* t *alone, that is* $\rho \equiv 0$ *!*

7. Personal communication from P. Mironescu.

Thus it is more realistic to consider a kernel

$$\chi(|z|) = |z|^{-\theta}$$

for some exponent $\theta \in (0,1)$, even if its physical meaning is unclear.

Then (9) reads

$$\int_0^{+\infty} dt \int\int_{a<b} \frac{(M(b)-M(a))^3}{(b-a)^{\theta+3}}\, da\, db < \infty.$$

The cubic root is the *Sobolev–Slobodeckij* semi-norm over the Sobolev space $W^{s,3}(\mathbb{R})$, where

$$s := 1 - \frac{1-\theta}{3} \in (\frac{2}{3}, 1).$$

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The estimate therefore tells us that $M \in L^3(\mathbb{R}_+; \dot{W}^{s,3}(\mathbb{R}))$.
Equivalently :

$$\rho \in L^3\left(\mathbb{R}_+; W^{\frac{\theta-1}{3},3}(\mathbb{R})\right),$$

where we point out that the order $\frac{\theta-1}{3}$ is negative.

The estimate above completes that of mass and energy, which tell us that

$$\rho \in L^\infty\left(\mathbb{R}_+; L^1(\mathbb{R}) \cap H^{\frac{\theta-1}{2}}(\mathbb{R})\right).$$

Open Problem 1

In several space dimensions ($d \geq 2$), what is the information carried by the estimate (7) ?

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Realistic plasma

We now assume $d \geq 2$, so that the Coulomb force makes sense for continuous densities :

$$\chi(|z|) = \frac{1}{|z|}\,.$$

Warning ! The Vlasov–Poisson model, as described by (3), is not realistic : it is impossible to isolate a large number of identically charged particles (ions). In practice, matter is globally neutral, and the presence of a population of anions is counterbalanced by a similar population of cations.

There are therefore two important kinds of models for plasmas. In the first one (I), both populations move, and their densities $f_{\pm}(t, y, v)$ obey to coupled Vlasov equations.

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In the second model (II), one population is much light (moving) and the other heavy (steady). The light one obeys Vlasov, while the heavy one contribute to the force field G.

Intuitively, both models do not behave the same in our theory. The first one, more complete and exact from a physical point of view, will provide a Divergence-free mass-momentum tensor. However, because anions and cations attract each other, it is not positive semi-definite. Thus Compensated Integrability will not apply.

In model (II), we may consider instead the mass-momentum tensor of the moving population only. It takes in account that part of the Coulomb force between particles that are charged the same way. The force being repulsive, the tensor is positive semi-definite. However, it is not Divergence-free, because we lack the conservation of global momentum.

This illustrate a general

> **Rule**. *Consider a physical model, which admits a DPT. Then the Functional Inequality*
>
> $$\int_0^\infty dt \int_{\mathbb{R}^d} (\det S)^{\frac{1}{d}}\, dy \le c M_0^{\frac{1}{d}} \sqrt{M_0 E_0}$$
>
> *expresses a dispersion property. The mass density tends to decay to* 0 *as* $t \to +\infty$ *in some* L^p*-norm where* $p > 1$*.*
> *Equivalently : If, for some physical reason, we do not expect such a dispersion, then the model does not admit a DPT.*

In Model (II), the light ions will not escape to infinity, because they are attracted by the heavy ones. The configuration should reach instead some equilibrium state as $t \to +\infty$. Hence there is no DPT.

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Going further

We write the Vlasov equation (3) for a light population. The force field splits as

$$G(t,y) = G_{\rm self} + G_{\rm ext}$$

where $G_{\rm self} = -\nabla_y(\chi * \rho)$ is the self-induced component, a repulsive Coulomb force. The external part, generated by the steady population, also derives from a potential, $G_{\rm ext} = -\nabla\psi$, where $\psi = \psi(y)$ is a given function, independent of the unknown f.

The two first moments write

$$\partial_t\rho + {\rm div}_y m = 0, \qquad \partial_t m + {\rm Div}_y T - \rho G_{\rm self} = \rho G_{\rm ext}.$$

The left-hand side can still be written ${\rm Div}_{t,y} S$ for a positive semi-definite tensor, as in the one-population case.

We now have

$$\operatorname{Div} S = \begin{pmatrix} 0 \\ \rho G_{\rm ext} \end{pmatrix}. \tag{10}$$

If $G_{\rm ext}$ is bounded, then S will be Div-controlled and we are allowed to apply a Functional Inequality.

Energy. It is now

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(v)\frac{|v|^2}{2}\,v\,dv + E_{\rm in}[\rho] + \int_{\mathbb{R}^d}\psi(y)\rho(t,y)\,dy.$$

The two first terms are the same as before : kinetic and internal energy of the moving population. The third one is a potential energy in the external field.

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Suppose that Compensated Integrability applies ($G_{\rm ext}$ is bounded). We use Proposition 1 of Lesson #5 :

If

$$S=\begin{pmatrix}\rho & m^T\\ m & A\end{pmatrix}$$

is Div-controlled in $(0,\tau)\times\mathbb{R}^d$, then we have

$$\begin{aligned}\int_0^\tau dt\int_{\mathbb{R}^d}(\det S)^{\frac1d}\,dy \quad &\leq\quad K_d\left(\|\rho(0)\|_{\mathcal M}+\|\rho(\tau)\|_{\mathcal M}+\|\partial_t\rho+{\rm div}\,m\|_{\mathcal M}\right)^{\frac1d}\\ &\quad\cdot\left(\|m(0)\|_{\mathcal M}+\|m(\tau)\|_{\mathcal M}+\|\partial_t m+{\rm Div}\,A\|_{\mathcal M}\right)\end{aligned}$$

The first parenthesis in the rhs is $2M_0$. In the last one, $\|m(t)\|_{\mathcal M}$ is always bounded by $\sqrt{2M_0E_0}$, and the last term is the space-time integral of $\rho|G_{\rm ext}|$.

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Because $\|\rho G_{\rm ext}\|_{\mathcal M} \le \tau \| G_{\rm ext}\|_\infty M_0$, we derive

$$\int_0^\tau dt \int_{\mathbb{R}^d} (\det S)^{\frac{1}{d}} \, dy \le K'_d M_0^{\frac{1}{d}} \left(\sqrt{8 M_0 E_0} + \tau \| G_{\rm ext}\|_\infty M_0 \right). \qquad (11)$$

Remark that asymptotically, this estimate depends only upon the total mass and not on the energy.

Thank you for your attention ! !