

A representation formula for large deviations rate functionals of invariant measures on the one dimensional torus

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Abstract. We consider a generic diffusion on the 1D torus and give a simple representation formula for the large deviation rate functional of its invariant probability measure, in the limit of vanishing noise. Previously, this rate functional had been characterized by M. I. Freidlin and A. D. Wentzell as solution of a rather complex optimization problem. We discuss this last problem in full generality and show that it leads to our formula. We express the rate functional by means of a geometric transformation that, with a Maxwell-like construction, creates flat regions.

We then consider piecewise deterministic Markov processes on the 1D torus and show that the corresponding large deviation rate functional for the stationary distribution is obtained by applying the same transformation. Inspired by this, we prove a universality result showing that the transformation generates viscosity solution of stationary Hamilton–Jacobi equation associated to any Hamiltonian H satisfying suitable weak conditions.

Résumé. Dans cet article, nous considérons une diffusion sur le tore de dimension 1 pour laquelle nous établissons une formule simple pour la fonction de taux d'un principe de grandes déviations pour sa mesure invariante, lorsque le bruit tend vers 0. Cette fonction de taux a été précédemment caractérisée par M. I. Freidlin et A. D. Wentzell comme la solution d'un problème d'optimisation plutôt compliqué. En réduisant ce problème, nous établissons notre formule à travers une transformation géométrique qui, à l'aide d'une construction de type Maxwell, crée des régions plates.

Nous considérons également des processus de Markov déterministes par morceaux, sur le tore de dimension 1. Nous montrons que la fonction de taux, dans un principe de grandes déviations pour la mesure invariante, peut être obtenue en considérant de nouveau la même transformation géométrique.

Inspirés par ces similarités, nous prouvons un résultat d'universalité de cette transformation en montrant qu'elle engendre la solution de viscosité d'une équation d'Hamilton–Jacobi stationnaire, associée à un Hamiltonien H qui satisfait à certaines hypothèses minimales.

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1. Introduction

We consider two different random dynamical systems on the one dimensional torus \mathbb{T} that, in suitable regimes, can be thought of as random perturbations of deterministic dynamical systems. The first one is a diffusion on \mathbb{T} , with small noise of intensity ε . This system has an invariant distribution μ^ε , whose large deviation (LD) functional has been expressed by M. I. Freidlin and A. D. Wentzell as solution of an optimization problem [8]. As discussed in [8], already for a very simple example of diffusion on \mathbb{T} with velocity field having only three attractor points, the solution of this optimization problem requires a rather long procedure.

The second system we consider is given by a piecewise deterministic Markov process (PDMP) on \mathbb{T} : the state is described by a pair $(x, \sigma) \in \mathbb{T} \times \{0, 1\}$, the continuous variable x follows a piecewise deterministic dynamics with nonvanishing σ -dependent velocity field, the discrete variable σ evolves by an x -dependent stochastic jump dynamics and the two resulting evolutions are fully-coupled. When the jump rates of the discrete variable are multiplied by a factor λ , in the limit $\lambda \rightarrow +\infty$ the evolution of the continuous variable x is well approximated by a deterministic dynamical system (cf. [6,10]). In [7], an expression of the probability distribution μ^λ of the continuous variable x in the steady state is computed up to a normalization constant. In addition, from this expression the LD functional of μ^λ is computed in the limit of diverging frequency jumps (i.e. $\lambda \rightarrow \infty$). The resulting formula is simple and concise.

Although the two models are not similar, we show here that the LD rate functionals for the measures μ^ε and μ^λ share a common structure. In particular they admit a very simple expression, that we further investigate. We then show that the optimization problem of M. I. Freidlin and A. D. Wentzell leads indeed to the same expression, by solving this optimization problem in the general case.

For both models the rate functional is given by a geometric transformation applied to a specific non periodic function. The result is a periodic function, whose graph differs from the original one due to new flat regions. The function to be transformed is model dependent while the transformation is always the same. In this sense our result is universal. We discuss this issue in terms of Hamilton–Jacobi equations. More precisely we discuss the regularity properties of the functions obtained by this procedure showing that they are viscosity solution of a suitable class of Hamilton–Jacobi equations.

When the models are reversible the transformation reduces to the identity. In all the other cases intervals on which the rate functional is constant appear. This reveals the presence of a phase transition. This kind of stationary non equilibrium states have a physical relevance and have been created and studied experimentally (see for example [9]).

2. Models and results

Without loss of generality, we think of \mathbb{T} as the interval $[0, 1]$ with identification of the extreme points 0 and 1.

2.1. Models

The first model we consider is a generic diffusion $(X_t^\varepsilon)_{t \geq 0}$ described by the equation

$$\dot{X}_t^\varepsilon = b(X_t^\varepsilon) + \varepsilon \dot{w}_t, \tag{2.1}$$

where $b: \mathbb{T} \rightarrow \mathbb{R}$ is a Lipschitz continuous vector field, w_t is a Wiener process and ε is a positive parameter. A detailed analysis of the above diffusion, as well as of diffusions on generic manifolds \mathcal{M} , in the limit $\varepsilon \downarrow 0$ is given in [8]. In the case $\mathcal{M} = \mathbb{T}$, Theorem 4.3 in Section 6.4 of [8] under the assumption that the closed set $\{x \in \mathbb{T} : b(x) = 0\}$ has a finite number of connected components gives

$$\lim_{\varepsilon \downarrow 0} -\varepsilon^2 \log \mu^\varepsilon(x) = W(x) - \min_{y \in \mathbb{T}} W(y), \quad x \in \mathbb{T}, \tag{2.2}$$

where $\mu^\varepsilon(x) dx$ denotes the invariant probability measure of the diffusion (2.1) and where the continuous function W is described in [8] by a rather complex variational characterization that we recall in Section 3. The r.h.s. of (2.2) is the LD rate functional for μ^ε . Here and in all the paper we state our large deviations results in the simple and direct formulation used in (2.2). Of course we mean that $\mu^\varepsilon(x)$ is a continuous version of the density of the invariant measure.

A very simple expression both of the invariant measure and of the LD rate functional can be given. To this aim, in the formulas below we will think of the field $b(\cdot)$ also as a periodic function on \mathbb{R} , with periodicity 1. With this convention and without requiring that the set $\{b = 0\}$ has a finite number of connected components, we get:

Proposition 2.1. *Define the function $S: \mathbb{R} \rightarrow \mathbb{R}$ as $S(x) = -2 \int_0^x b(s) ds$. Then,*

$$\mu^\varepsilon(x) = \frac{1}{c(\varepsilon)} \int_x^{x+1} e^{\varepsilon^{-2}(S(y)-S(x))} dy, \tag{2.3}$$

where $c(\varepsilon)$ is the normalization constant

$$c(\varepsilon) = \int_0^1 dx \int_x^{x+1} e^{\varepsilon^{-2}(S(y)-S(x))} dy. \quad (2.4)$$

In particular, it holds

$$\begin{aligned} \lim_{\varepsilon \downarrow 0} -\varepsilon^2 \log \mu^\varepsilon(x) &= \max_{x' \in [0,1]} \max_{y' \in [x', x'+1]} (S(y') - S(x')) - \max_{y \in [x, x+1]} (S(y) - S(x)) \\ &= \min_{y \in [x, x+1]} (S(x) - S(y)) - \min_{x' \in [0,1]} \min_{y' \in [x', x'+1]} (S(x') - S(y')). \end{aligned} \quad (2.5)$$

Note that $S(a+1) - S(a) = -\int_0^1 2b(s) ds$, and in particular the difference does not depend on $a \in \mathbb{R}$. Hence, given $x \in \mathbb{T}$, the expression $\max_{y \in [x, x+1]} (S(y) - S(x))$ does not generate any confusion, both if we think $x \in \mathbb{T} \hookrightarrow \mathbb{R}$ by identifying \mathbb{T} with $[0, 1)$, and if we think $\max_{y \in [x, x+1]} (S(y) - S(x))$ as $\max_{y \in [\bar{x}, \bar{x}+1]} (S(y) - S(\bar{x}))$ with $\bar{x} \in \mathbb{R}$ such that $\pi(\bar{x}) = x$, where $\pi: \mathbb{R} \rightarrow \mathbb{T}$ denotes the canonical projection. In what follows, when writing $\max_{y \in [x, x+1]} (S(y) - S(x))$ with $x \in \mathbb{T}$ we will mean any of the above interpretations. The same considerations hold if we consider the minimum instead of the maximum.

Proof of Proposition 2.1. The proof is elementary. The fact that (2.3) is the density of the invariant measure follows by a direct computation. See for example [11] where a similar expression has been obtained. Then (2.5) follows from (2.3) and (2.4) by a direct application of the Laplace theorem [8]. \square

As a consequence of Proposition 2.1, the r.h.s. of (2.5) coincides with the r.h.s. of (2.2), where we recall that the function W is characterized as the solution of the Freidlin–Wentzell variational problem. This fact is not evident. The general solution of the variational problem determining W on the 1D torus is described in detail in Theorem 4.1, which is stated only in Section 4 after introducing some preliminaries. Its proof is given in Section 5 and is independent from Proposition 2.1. The identification of W with the r.h.s. of (2.5) is stated in Theorem 2.3.

The second model we discuss is a PDMP on the 1D torus \mathbb{T} . Let $F_0, F_1: \mathbb{T} \rightarrow \mathbb{R}$ be Lipschitz continuous fields. In addition, let $r(0, 1|\cdot), r(1, 0|\cdot)$ be positive continuous functions on \mathbb{T} . Given the parameter $\lambda > 0$, we denote by $\{(X_t^\lambda, \sigma_t^\lambda): t \geq 0\}$ the stochastic process with states in $\mathbb{T} \times \{0, 1\}$ whose generator is given by

$$\mathbb{L}_\lambda f(x, \sigma) = F_\sigma(x) \cdot \nabla f(x, \sigma) + \lambda r(\sigma, 1 - \sigma|x) (f(x, 1 - \sigma) - f(x, \sigma)), \quad (2.6)$$

for all $(x, \sigma) \in \mathbb{T} \times \{0, 1\}$. The above process is a generic PDMP on the torus \mathbb{T} (cf. [4] for a detailed discussion on PDMPs). Following [6,7], we call x and σ the mechanical and the chemical state of the system, respectively. The dynamics can be roughly described as follows. Given the initial state $(x_0, \sigma_0) \in \Omega \times \Gamma$, consider the positive random variable τ_1 with distribution

$$\mathbb{P}(\tau_1 > t) = e^{-\lambda \int_0^t r(\sigma_0, 1 - \sigma_0|x_0(s)) ds}, \quad t \geq 0,$$

where $x_0(s)$ is the solution of the Cauchy problem

$$\begin{cases} \dot{x} = F_{\sigma_0}(x), \\ x(0) = x_0. \end{cases} \quad (2.7)$$

The evolution of the system in the time interval $[0, \tau_1)$ is given by $(x_0(s), \sigma_0)$. At time τ_1 the chemical state changes, i.e. $\sigma_{\tau_1}^\lambda = 1 - \sigma_0$, and the dynamics starts afresh from the state $(x(\tau_1), \sigma_{\tau_1}^\lambda)$. Note that the mechanical trajectory is continuous and piecewise C^1 . Moreover, if the jump rates $r(0, 1|x)$ and $r(1, 0|x)$ do not depend on x , then the process $(\sigma_t^\lambda: t \geq 0)$ reduces to a continuous-time Markov chain with jump rates $r(0, 1), r(1, 0)$, independent from $(X_t^\lambda: t \geq 0)$. In general, the chemical and the mechanical evolutions are fully-coupled.

In [6] (see also [7]) an averaging principle has been proved. In the limit of high frequency of the chemical jumps (i.e. $\lambda \rightarrow \infty$), the mechanical variable x behaves deterministically according to an ODE with a suitable averaged

vector field \bar{F} . In this sense a PDMP can be thought of as a stochastic perturbation of the deterministic dynamical system $\dot{x} = \bar{F}(x)$.

For simplicity, we restrict to the case of non vanishing force fields F_0, F_1 . Then for each $\lambda > 0$, there exists a unique invariant measure $\tilde{\mu}^\lambda$ for the PDMP $(X_t^\lambda, \sigma_t^\lambda)$ and it has the form $\tilde{\mu}^\lambda(x, \sigma) = \tilde{\mu}_0^\lambda(x) dx \delta_{\sigma,0} + \tilde{\mu}_1^\lambda(x) dx \delta_{\sigma,1}$, where δ is the Kronecker delta (cf. Theorem (34.19), p. 118 and Theorem 3.10, p. 130 in [4] [Section 34] together with [7]). Let us observe now the evolution of the mechanical state alone in the steady state. We set

$$\mu^\lambda(x) = \tilde{\mu}_0^\lambda(x) + \tilde{\mu}_1^\lambda(x)$$

for the probability density at x of the mechanical variable in the steady state. The following result holds [7]:

Proposition 2.2. *Suppose that F_0 and F_1 are Lipschitz continuous nonvanishing fields and define the function $S : \mathbb{R} \rightarrow \mathbb{R}$ as*

$$S(x) = \int_0^x \left(\frac{r(0, 1|y)}{F_0(y)} + \frac{r(1, 0|y)}{F_1(y)} \right) dy. \tag{2.8}$$

Then

$$\mu^\lambda(x) = \frac{1}{Z^\lambda} \int_x^{x+1} \left[\frac{r(1, 0|y)}{F_0(x)F_1(y)} + \frac{r(0, 1|y)}{F_1(x)F_0(y)} \right] e^{\lambda(S(y)-S(x))} dy, \tag{2.9}$$

where Z^λ denotes the normalization constant. In particular, it holds

$$\begin{aligned} \lim_{\lambda \uparrow \infty} -\frac{1}{\lambda} \ln \mu^\lambda(x) &= \max_{x' \in [0, 1]} \max_{y' \in [x', x'+1]} (S(y') - S(x')) - \max_{y \in [x, x+1]} (S(y) - S(x)) \\ &= \min_{y \in [x, x+1]} (S(x) - S(y)) - \min_{x' \in [0, 1]} \min_{y' \in [x', x'+1]} (S(x') - S(y')). \end{aligned} \tag{2.10}$$

The analogy with Proposition 2.1 is evident. We refer the reader to [7] for a proof of the above results.

2.2. Common geometric structure of the LD functionals.

We can finally describe the common structure behind the LD functionals of μ^ε and μ^λ :

Theorem 2.3. *Given a continuous function $F : \mathbb{T} \rightarrow \mathbb{R}$, define for all $x \in \mathbb{R}$*

$$S(x) = \int_0^x F(s) ds, \tag{2.11}$$

$$\Phi(x) = - \max_{y \in [x, x+1]} (S(y) - S(x)) = \min_{y \in [x, x+1]} (S(x) - S(y)). \tag{2.12}$$

Then the following holds:

- (i) Φ is Lipschitz continuous and periodic with unit period. If S is monotone, then Φ is constant and equals $\min\{0, -S(1)\}$. If $S(1) = 0$, then S is periodic and $\Phi = S$ up to an additive constant. If $S(1) \neq 0$ and S is not monotone, then the set

$$U = \{x \in \mathbb{R} : \Phi(x) \neq \min\{0, -S(1)\}\} \tag{2.13}$$

is an open subset $U \subset \mathbb{R}$ such that $\mathbb{R} \setminus U$ has nonempty interior part. On each connected component of U it holds $\Phi = S$ up to an additive constant, i.e.

$$\Phi(x) = S(x) - S(a) + \Phi(a) \quad \forall x \in (a, b) \subset U. \tag{2.14}$$

On $\mathbb{R} \setminus U$ the function Φ is constant and satisfies

$$\Phi(x) = \min\{0, -S(1)\} \quad \forall x \in \mathbb{R} \setminus U. \quad (2.15)$$

Moreover, Φ reaches its maximum on $\mathbb{R} \setminus U$:

$$\max_{x \in [0, \infty)} \Phi(x) = \max_{x \in [0, 1]} \Phi(x) = \min\{0, -S(1)\}. \quad (2.16)$$

(ii) Suppose that the set $\{x \in \mathbb{T} : F(x) = 0\}$ has a finite number of connected components. Then,

$$\Phi(x) - \min_{x' \in [0, 1]} \Phi(x') = W(x) - \min_{y \in \mathbb{T}} W(y), \quad (2.17)$$

where W is the function entering in (2.2) and defined by an optimization problem in [8], taking $b(x) = -F(x)/2$.

The map from S to Φ is the geometric transformation mentioned in the Introduction.

Remark 1. It is interesting to observe that if we define

$$\tilde{\Phi}(x) := \inf_{y \in [x, x+1]} \left(\int_x^y F_-(s) ds + \int_y^{x+1} F_+(s) ds \right), \quad (2.18)$$

then $\tilde{\Phi}$ and Φ differ by a constant. In fact, we have

$$\begin{aligned} \Phi(x) &= \min_{y \in [x, x+1]} (S(x) - S(y)) = \min_{y \in [x, x+1]} \left(\int_x^y F_-(s) ds - \int_x^y F_+(s) ds \right) \\ &= \min_{y \in [x, x+1]} \left(\int_x^y F_-(s) ds + \int_y^{x+1} F_+(s) ds \right) - \int_0^1 F_+(s) ds \\ &= \tilde{\Phi}(x) - \int_0^1 F_+(s) ds. \end{aligned}$$

Indeed, comparing with (5.1) in Proposition 5.1, one gets $\tilde{\Phi} = W$ under the assumptions of Theorem 2.3(ii).

Theorem 2.3 suggests a simple algorithm to determine the graph of the function Φ . To avoid trivial cases, we assume that $S(1) \neq 0$ and that S is not monotone. For simplicity of notation we suppose that the set $\{F = 0\}$ has finite cardinality.

• *Case $S(1) > 0$.* We can always find a point of local maximum b such that $S(b+1) = \max_{y \in [b, b+1]} S(y)$. This point b can be found as follows: let $a \in \mathbb{R}$ be any point of local maximum for S , then let $b \in [a, a+1]$ be such that $S(b) = \max_{y \in [a, a+1]} S(y)$. This trivially implies that

$$S(b+1) = S(1) + S(b) \geq \begin{cases} S(1) + \max_{y \in [b, a+1]} S(y), \\ S(1) + \max_{y \in [a, b]} S(y) = \max_{y \in [a+1, b+1]} S(y). \end{cases}$$

The above inequalities imply that $S(b+1) = \max_{y \in [b, b+1]} S(y)$. If $b \in (a, a+1)$ then trivially b is a point of local maximum for S . Otherwise, it must be $b = a+1$ (since $S(a+1) = S(a) + S(1) > S(a)$) and in particular b is again a point of local maximum for S since a and therefore $a+1$ satisfy this property.

The following algorithm shows how to construct the function Φ on the interval $[b, b+1]$. Due to the periodicity of Φ this construction extends to all \mathbb{R} .

Let $\Gamma = \{x_i : 1 \leq i \leq m\}$ be the set of points of local maximum for S in $[b, b+1]$ (note that $b, b+1 \in \Gamma$). Let $L_1 = S(b+1)$, $z_1 = \min\{x \in \Gamma : S(x) = L_1\}$, and define inductively

$$L_k = \max\{S(x) : x \in \Gamma, x < z_{k-1}\}, \quad z_k = \min\{x \in \Gamma : S(x) = L_k\}$$

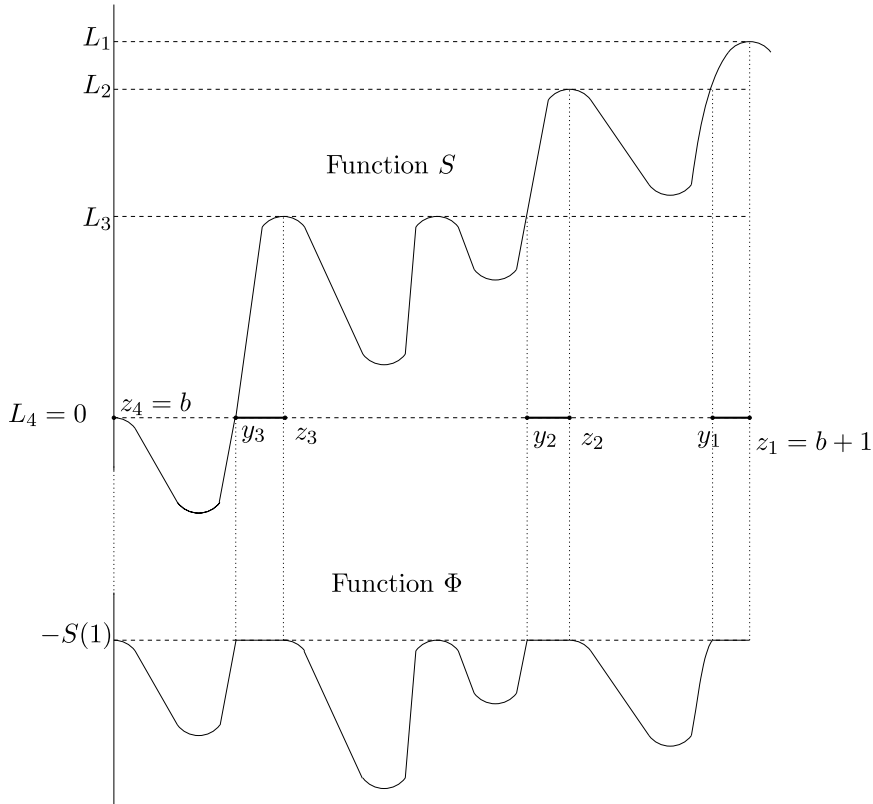


Fig. 1. Algorithm to determine the flat regions and the function Φ : case $S(1) > 0$.

for all $k \geq 2$ such that the set $\{S(x) : x \in \Gamma, x < z_{k-1}\}$ is nonempty. At the end, we get n levels L_1, \dots, L_n and points $z_n < z_{n-1} < \dots < z_1$, which are all points of local maximum for S on $[b, b + 1]$. In addition, it must be $z_n = b$. For each i : $1 \leq i < n$ let y_i be the point

$$y_i = \max\{x \in [b, z_i] : S(x) = L_{i+1}\}.$$

Then the continuous function Φ on $[b, b + 1]$ is obtained by the following rules. Fix the value $\Phi(b) = -S(1)$. Set $V = \bigcup_{i=1}^{n-1} [y_i, z_i]$, then the function Φ must be equal to the constant $-S(1)$ on V and must satisfy $\nabla\Phi = \nabla S$ on $[b, b + 1] \setminus V$. See Fig. 1.

We do not give a formal proof of the above algorithm, it can be easily obtained for example from the following alternative representation of the function Φ when x belongs to the special unitary period $[b, b + 1]$

$$\begin{aligned} \Phi(x) &= \min_{y \in [x, x+1]} (S(x) - S(y)) = S(x) - \max_{y \in [x, x+1]} S(y) \\ &= S(x) - \max_{y \in [b+1, x+1]} S(y) = S(x) - S(1) - \max_{y \in [b, x]} S(y). \end{aligned} \tag{2.19}$$

Note that in the last term we are maximizing only over the interval $[b, x]$. The algorithm follows easily.

The algorithm can be summarized by the following simple and intuitive procedure. Think of the graph of S in Fig. 1 as a mountain profile and imagine also that light is arriving from the left with rays parallel to the horizontal axis. On a point x such that the corresponding point on the mountain profile is in the shadow we have $\nabla\Phi = \nabla S$. On a point x such that the corresponding point on the mountain profile is enlightened, we have $\nabla\Phi = 0$. Note in particular that flat intervals are always to the left of some local maxima.

• *Case $S(1) < 0$.* The algorithm in this case is similar to the case $S(1) > 0$, it is enough to invert left with right. More precisely, first one determines a point of local maximum b such that $S(b) = \max_{y \in [b, b+1]} S(y)$. One defines Γ as above, let $L_1 = S(b)$, $z_1 = \max\{x \in \Gamma : S(x) = L_1\}$, and define inductively

$$L_k = \max\{S(x) : x \in \Gamma, x > z_{k-1}\}, \quad z_k = \max\{x \in \Gamma : S(x) = L_k\}$$

for all $k \geq 2$ such that the set $\{S(x) : x \in \Gamma, x > z_{k-1}\}$ is nonempty. It must be $z_n = b + 1$. For each $i : 1 \leq i < n$ let y_i be the point

$$y_i = \min\{x \in [z_i, b + 1] : S(x) = L_{i+1}\}.$$

Setting $\Phi(b) = -S(1)$ and $V = \bigcup_{i=1}^{n-1} [z_i, y_i]$, the conclusion is the same as in the case $S(1) > 0$.

Also in this case an interpretation of the algorithm in terms of mountains and light can be given. The difference is that light is arriving now from the right. Note also that in this case flats intervals are always located at the right of some local maxima.

2.3. Hamilton–Jacobi equations and universality

Finally, we illustrate the relation of the function Φ with Hamilton–Jacobi equations:

Theorem 2.4. *Given a continuous function $F : \mathbb{T} \rightarrow \mathbb{R}$, let $H(x, p)$, with $(x, p) \in \mathbb{T} \times \mathbb{R}$, be a Hamiltonian such that:*

- (A) $H(x, \cdot)$ is a convex function for any $x \in \mathbb{T}$,
- (B) $H(x, 0) = H(x, F(x)) = 0$ for any $x \in \mathbb{T}$.

Then the function $\Phi(x)$ defined in Theorem 2.3 is a viscosity solution of the Hamilton–Jacobi equation

$$H(x, \nabla \Phi(x)) = 0, \quad x \in \mathbb{T}. \quad (2.20)$$

Hypothesis (A) is a rather weak assumption since usually the Hamiltonian is obtained as Legendre transform of the convex Lagrangian function. Theorem 2.4 states a universality property: independently from the exact expression of the Hamiltonian, as soon as conditions (A) and (B) are satisfied, Φ is a viscosity solution of the Hamilton–Jacobi equation (2.20).

We recall (see [1,3,5]) that a function φ is a viscosity solution for the Hamilton–Jacobi equation $H(x, \nabla \varphi(x)) = 0$ iff $\varphi \in C(\mathbb{T})$ and

$$H(x, p) \geq 0 \quad \forall x \in \mathbb{T}, p \in D^- \varphi(x), \quad (2.21)$$

$$H(x, p) \leq 0 \quad \forall x \in \mathbb{T}, p \in D^+ \varphi(x), \quad (2.22)$$

where the superdifferential $D^+ \varphi(x)$ and the subdifferential $D^- \varphi(x)$ are defined as

$$D^+ \varphi(x) = \left\{ p \in \mathbb{R} : \limsup_{y \rightarrow x} \frac{\varphi(y) - \varphi(x) - p(y-x)}{|y-x|} \leq 0 \right\}, \quad (2.23)$$

$$D^- \varphi(x) = \left\{ p \in \mathbb{R} : \liminf_{y \rightarrow x} \frac{\varphi(y) - \varphi(x) - p(y-x)}{|y-x|} \geq 0 \right\}. \quad (2.24)$$

As explained in [8] for the diffusion and in [7] for the PDMP, Hamilton–Jacobi equations appear in a natural way in problems related to the computation of the quasipotential. In the case of the diffusive model the associated Hamiltonian is

$$H(x, p) := p(p - F(x)), \quad (x, p) \in \mathbb{T} \times \mathbb{R}. \quad (2.25)$$

It is easy to check that the Hamiltonian in (2.25) satisfies hypotheses (A) and (B) of Theorem 2.4. For PDMPs the corresponding Hamiltonian has a more complex structure and is written in [7], p. 298. Also in this case it is possible to check that hypotheses (A) and (B) are satisfied.

2.4. Outline of the paper

In Section 3 we recall the definition of the function W given in [8]. This definition consists of two optimization problems, whose detailed solution is described in Theorem 4.1 in Section 4. In Section 5 we prove Theorem 4.1 and part (ii) of Theorem 2.3. In Section 6 we prove part (i) of Theorem 2.3, while in Section 7 we prove Theorem 2.4.

3. Definition of the function W given in [8]

For the reader's convenience and in order to set the notation for further developments, in this section we briefly recall the definition of the function W given in Chapter 6 of [8]. It is convenient to set $F(x) = -2b(x)$ and to work with F instead of the field b entering in the definition of the diffusion (2.1). We use the same notation of Theorem 2.3. Moreover, we write F_- and F_+ for the positive and negative part of F , respectively. This means that $F_-(x) = |F(x) \wedge 0|$ and $F_+(x) = F(x) \vee 0$. In what follows we identify the torus \mathbb{T} with the 1D circle and we write π for the canonical projection $\pi : \mathbb{R} \rightarrow \mathbb{R}/\mathbb{Z} = \mathbb{T}$. In particular, the clockwise orientation of the 1D circle corresponds to the orientation of the path $\pi(x)$ as x goes from 0 to 1. In order to avoid confusion in the rest of the paper we will not identify \mathbb{T} with the interval $[0, 1)$. Finally, we recall that we think of F both as function on the torus and as periodic function on \mathbb{R} with unit period and that, given $x \in \mathbb{T}$, we write $\max/\min_{y \in [x, x+1]}(S(y) - S(x))$ for the quantity $\max/\min_{\bar{y} \in [\bar{x}, \bar{x}+1]}(S(\bar{y}) - S(\bar{x}))$ where \bar{x} is any point in \mathbb{R} such that $\pi(\bar{x}) = x$.

Since the function S is constant on each connected component C of $\{x \in \mathbb{R} : F(x) = 0\}$, we denote by $S(C)$ the constant value $S(x)$, $x \in C$. Recall the assumption for (2.2) that the closed set $\{x \in \mathbb{T} : F(x) = 0\}$ has a finite number of connected components. We say that a connected component $[a, b]$ is stable if for some $\varepsilon > 0$ it holds $F(s) < 0$ for all $s \in [a - \varepsilon, a)$ and $F(s) > 0$ for all $s \in (b, b + \varepsilon]$. We say that $[a, b]$ is totally unstable if for some $\varepsilon > 0$ it holds $F(s) > 0$ for all $s \in [a - \varepsilon, a)$ and $F(s) < 0$ for all $s \in (b, b + \varepsilon]$. If the connected components of $\{x \in \mathbb{T} : F(x) = 0\}$ are given by isolated points, then the stable and unstable ones are obtained by the canonical projection $\pi : \mathbb{R} \rightarrow \mathbb{T}$ of the points of local minimum and local maximum for S , respectively.

We call K_1, K_2, \dots, K_ℓ the stable connected components of $\{x \in \mathbb{T} : F(x) = 0\}$ labeled in clockwise way, i.e. K_i and K_{i+1} are nearest-neighbors and K_{i+1} follows K_i clockwise. Moreover, for generic $i \in \mathbb{Z}$, we denote by K_i the component K_j , such that $1 \leq j \leq \ell$ and $i \equiv j$ in $\mathbb{Z}/\ell\mathbb{Z}$. We observe that between K_i and K_{i+1} (with respect to clockwise order) there exists only one connected component of $\{x \in \mathbb{T} : F(x) = 0\}$ totally unstable. We call A_i such connected component.

Given two distinct points $x, y \in \mathbb{T}$ we write $\gamma_{x,y}^+$ and $\gamma_{x,y}^-$ for the unoriented paths in \mathbb{T} connecting x to y clockwise and anticlockwise, respectively. If $x = y$ we set $\gamma_{x,x}^+ = \gamma_{x,x}^- = \{x\}$. Note that $\gamma_{x,y}^- = \gamma_{y,x}^+$. Given a generic function $h : \mathbb{T} \rightarrow \mathbb{R}$ the integrals $\int_{\gamma_{x,y}^+} h(s) ds$, $\int_{\gamma_{x,y}^-} h(s) ds$ are defined as

$$\int_{\gamma_{x,y}^+} h(s) ds = \begin{cases} \int_{\bar{x}}^{\bar{y}} h(s) ds & \text{if } \bar{x} < \bar{y}, \\ \int_{\bar{x}}^{\bar{y}+1} h(s) ds & \text{if } \bar{y} < \bar{x}, \end{cases} \quad (3.1)$$

and

$$\int_{\gamma_{x,y}^-} h(s) ds = \begin{cases} \int_{\bar{y}}^{\bar{x}} h(s) ds & \text{if } \bar{y} < \bar{x}, \\ \int_{\bar{y}}^{\bar{x}+1} h(s) ds & \text{if } \bar{x} < \bar{y}, \end{cases} \quad (3.2)$$

where \bar{x}, \bar{y} are the only elements in $[0, 1)$ such that $\pi(\bar{x}) = x$ and $\pi(\bar{y}) = y$.

Given $x, y \in \mathbb{T}$ we define

$$V_+(x, y) = \int_{\gamma_{x,y}^+} F_+(s) ds, \quad (3.3)$$

$$V_-(x, y) = \int_{\gamma_{x,y}^-} F_-(s) ds, \quad (3.4)$$

$$V(x, y) = V_+(x, y) \wedge V_-(x, y). \quad (3.5)$$

Due to Lemma 3.1 in Section 4.3 in [8] and the discussion in Section 6.4 in [8], the above function $V(x, y)$ coincides with the function $V(x, y)$ defined in [8] at page 161 as well as our definition of stable component coincides with the one given on p. 188 in [8].

Given two distinct connected components C_1, C_2 in $\{F = 0\} \subset \mathbb{T}$, the quantities $V(x, y)$, $V_{\pm}(x, y)$ with $x \in C_1$ and $y \in C_2$ do not depend on the particular choice of the points x, y and are denoted respectively by $V(C_1, C_2)$, $V_{\pm}(C_1, C_2)$. We set $V(C_1, C_1) = V_{\pm}(C_1, C_1) = 0$ (note that $V(x, y) = 0$ for all $x, y \in C_1$, while due to our definition $V_{\pm}(x, y)$ typically depend from $x, y \in C_1$).

Let us now pass to define the function W . If $\{F = 0\} = \emptyset$, then set $W \equiv 0$. When $\{F = 0\} \neq \emptyset$ then necessarily there is at least one stable connected component. If there is only one stable connected component, then set $W(K_1) = 0$. Let us now assume $\ell \geq 2$.

Given an index i in $\{1, \dots, \ell\}$, an oriented graph g is said to belong to the family $G\{i\}$ if it is a directed tree having vertices $\{1, \dots, \ell\}$, rooted at i and pointing towards the root. This means that:

- (1) $1, \dots, \ell$ are the vertices of g ,
- (2) every point j in $\{1, \dots, \ell\} \setminus \{i\}$ is the initial point of exactly one arrow,
- (3) for any point j in $\{1, \dots, \ell\} \setminus \{i\}$ there exists a (unique) sequence of arrows leading from it to the point i ,
- (4) no arrow exits from i .

Given a graph $g \in G\{i\}$ the arrow from m to n is denoted by $m \rightarrow n$. Then in [8] the authors define

$$W(K_i) = \min_{g \in G\{i\}} \sum_{(m \rightarrow n) \in g} V(K_m, K_n). \quad (3.6)$$

Definition 3.1. The function W on \mathbb{T} is defined in [8] as follows: if $\ell = 0$ then $W(x) = 0$ for all $x \in \mathbb{T}$, otherwise

$$W(x) = \min_{i \in \{1, \dots, \ell\}} [W(K_i) + V(K_i, x)]. \quad (3.7)$$

We point out that it holds $W(x) = W(K_i)$ for all $x \in K_i$ as discussed in [8].

4. Solution of the variational problems entering in the definition of W

Given x, y in \mathbb{T} we define $\Delta S(x, y)$ as

$$\Delta S(x, y) := \int_{\gamma_{x,y}^+} F(s) ds. \quad (4.1)$$

We note that $S(v) - S(u) = \Delta S(x, y)$ whenever $v \in [u, u + 1)$ and $x = \pi(u)$, $y = \pi(v)$, π being the canonical projection $\pi : \mathbb{R} \rightarrow \mathbb{T}$. Given C, C' two distinct connected components of $\{F = 0\} \subset \mathbb{T}$, we set $\Delta S(C, C') := S(x, y)$ for any $x \in C$, $y \in C'$. Trivially the definition does not depend on the choice of x, y .

Given disjoint connected subsets $A, B \subset \mathbb{T}$ and a point $x \in \mathbb{T}$ we write $A \leq x \leq B$ if $x \in \gamma_{a,b}^+$ for some $a \in A$ and $b \in B$. We write $A < x \leq B$ if $A \leq x \leq B$ and $x \notin A$. Similarly, we define $A \leq x < B$ and $A < x < B$. Then, it holds

$$\Delta S(K_i, A_i) = \max\{\Delta S(x, y) : K_i \leq x \leq y \leq K_{i+1}\}.$$

(Recall that A_i is the totally unstable connected component of $\{F = 0\} \subset \mathbb{T}$ between K_i and K_{i+1} .)

We introduce some special graphs $g_{i,j} \in G\{i\}$ with $i, j \in \{1, \dots, \ell\}$. The simplest definition is given by Fig. 2.

In order to give a formal definition, at cost of a rotation we can assume that $i = 1 \leq j \leq \ell$. Then, $g_{1,j}$ has anticlockwise arrows $j \rightarrow j - 1$, $j - 1 \rightarrow j - 2$, \dots , $2 \rightarrow 1$ and clockwise arrows $j + 1 \rightarrow j + 2$, $j + 2 \rightarrow j + 3$, \dots , $\ell \rightarrow \ell + 1 \equiv 1$. Note that if $j = 1$ the graph g has only clockwise arrows ($2 \rightarrow 3$, $3 \rightarrow 4$, \dots , $\ell \rightarrow \ell + 1 \equiv 1$), while if $j = \ell$ the graph g has only anticlockwise arrows ($\ell \rightarrow \ell - 1$, $\ell - 1 \rightarrow \ell - 2$, \dots , $2 \rightarrow 1$). As we will show, in the minimization problem (3.6) only the graphs of the type $g_{i,j}$ are relevant.

We can finally state our result:

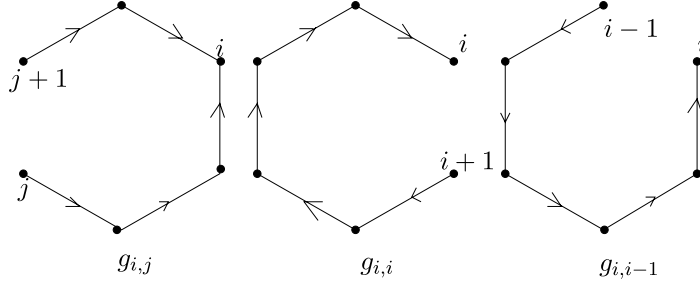


Fig. 2. The graphs $g_{i,j}$. Vertices are labeled clockwise from 1 to ℓ .

Theorem 4.1. *The minimizers in (3.6) and (3.7) can be described as follows.*

(i) Take $\ell \geq 2$. Fixed $i \in \{1, \dots, \ell\}$, take $J \in \{1, \dots, \ell\}$ such that

$$\Delta S(K_i, A_J) = \max_{y \in [x, x+1]} (S(y) - S(x))$$

for some (and therefore for all) $x \in K_i$. Then

$$W(K_i) = \sum_{(m \rightarrow n) \in g_{i,J}} V(K_m, K_n), \tag{4.2}$$

and for each arrow $m \rightarrow n$ in $g_{i,J}$ it holds

$$V(K_m, K_n) = \begin{cases} V_-(K_m, K_{m-1}) & \text{if } n = m - 1, \\ V_+(K_m, K_{m+1}) & \text{if } n = m + 1. \end{cases} \tag{4.3}$$

(ii) Take $\ell \geq 1$. Take $x \in \mathbb{T} \setminus (\bigcup_{r=1}^{\ell} K_r)$. Take $i \in \{1, \dots, \ell\}$ such that $K_i < x < K_{i+1}$. Fix $x_i \in K_i$, $x_{i+1} \in K_{i+1}$. Then, fix $\bar{x}_i \in \mathbb{R}$ such that $x_i = \pi(\bar{x}_i)$, and afterwards fix \bar{x} , $\bar{x}_{i+1} \in (\bar{x}_i, \bar{x}_i + 1]$ such that $x = \pi(\bar{x})$, $x_{i+1} = \pi(\bar{x}_{i+1})$.

Then exactly one of the following cases holds:

- (1) $x_i < x \leq A_i$ and $\max_{y \in [\bar{x}_i, \bar{x}_i+1]} S(y) = \max_{y \in [\bar{x}, \bar{x}+1]} S(y)$;
- (2) $x_i < x \leq A_i$ and $\max_{y \in [\bar{x}_i, \bar{x}_i+1]} S(y) < \max_{y \in [\bar{x}, \bar{x}+1]} S(y)$;
- (3) $A_i \leq x < x_{i+1}$ and $\max_{y \in [\bar{x}_{i+1}, \bar{x}_{i+1}+1]} S(y) = \max_{y \in [\bar{x}, \bar{x}+1]} S(y)$;
- (4) $A_i \leq x < x_{i+1}$ and $\max_{y \in [\bar{x}_{i+1}, \bar{x}_{i+1}+1]} S(y) < \max_{y \in [\bar{x}, \bar{x}+1]} S(y)$.

Then in cases (1) and (4) it holds

$$W(x) = W(K_i) + V(K_i, x), \quad V(K_i, x) = V_+(K_i, x), \tag{4.4}$$

while in cases (2) and (3) it holds

$$W(x) = W(K_{i+1}) + V(K_{i+1}, x), \quad V(K_{i+1}, x) = V_-(K_{i+1}, x). \tag{4.5}$$

5. Proof of Theorem 2.3(ii) and Theorem 4.1

Part (ii) of Theorem 2.3 follows easily from the following result:

Proposition 5.1. *For each $x \in \mathbb{T}$ it holds*

$$W(x) - \Phi(x) = W(x) + \max_{y \in [x, x+1]} (S(y) - S(x)) = \int_0^1 F_+(s) ds, \tag{5.1}$$

In particular, the l.h.s. does not depend on x .

We divide the proof of the above proposition in several steps.

Lemma 5.2. *Given $\ell \geq 2$ and $i, j \in \{1, \dots, \ell\}$, consider*

$$t(i, j) := V_-(K_j, K_i) + V_+(K_{j+1}, K_i).$$

Then,

$$W(K_i) = \min_{1 \leq j \leq \ell} t(i, j). \quad (5.2)$$

Moreover, $t(i, J) = \min_{1 \leq j \leq \ell} t(i, j)$ if and only if

$$\Delta S(K_i, A_J) = \max_{y \in [x, x+1]} (S(y) - S(x)) \quad \forall x \in K_i. \quad (5.3)$$

Proof. In order to simplify the notation, without loss of generality we take $i = 1$. Let us first show that

$$W(K_1) = \min_{g \in G\{1\}} \sum_{(m \rightarrow n) \in g} V(K_m, K_n) \leq \min_{1 \leq j \leq \ell} t(1, j). \quad (5.4)$$

For each $j \in \{1, \dots, \ell\}$, consider the graph $g_{1,j} \in G\{1\}$ defined in Section 3. If $j \neq 1, \ell$ we have

$$t(1, j) \geq \sum_{r=2}^j V(K_r, K_{r-1}) + \sum_{r=j+1}^{\ell-1} V(K_r, K_{r+1}) = \sum_{(m \rightarrow n) \in g_{1,j}} V(K_m, K_n), \quad (5.5)$$

since

$$V_-(K_j, K_1) = \sum_{r=2}^j V_-(K_r, K_{r-1}) \geq \sum_{r=2}^j V(K_r, K_{r-1}), \quad (5.6)$$

$$V_+(K_{j+1}, K_1) = \sum_{r=j+1}^{\ell} V_+(K_r, K_{r+1}) \geq \sum_{r=j+1}^{\ell} V(K_r, K_{r+1}). \quad (5.7)$$

We point out that (5.6) holds also for $j = \ell$, while (5.7) holds also for $j = 1$. As a consequence, (5.5) is valid also for $j = 1, \ell$ and this readily implies (5.4).

Let us now prove that (5.4) remains valid with opposite inequality. To this aim, we take a generic graph $g \in G\{1\}$ and fix $x_n \in K_n$ for each stable connected component K_n . Given an arrow $m \rightarrow n$, we define $\sigma(m \rightarrow n) \in \{-, +\}$ as

$$\sigma(m \rightarrow n) = \begin{cases} + & \text{if } V_-(K_m, K_n) \geq V_+(K_m, K_n), \\ - & \text{if } V_-(K_m, K_n) < V_+(K_m, K_n). \end{cases}$$

Note that

$$V(K_m, K_n) = V_{\sigma(m \rightarrow n)}(K_m, K_n) = \int_{\mathcal{Y}_{x_m, x_n}^{\sigma(m \rightarrow n)}} F_{\sigma(m \rightarrow n)}(s) ds. \quad (5.8)$$

Given $g \in G\{1\}$, let us call

$$\Delta(g) := \sum_{(m \rightarrow n) \in g} \sum_{j \in \{1, \dots, \ell\} \setminus \{m, n\}} \chi(x_j \in \mathcal{Y}_{x_m, x_n}^{\sigma(m \rightarrow n)}), \quad (5.9)$$

where χ denotes the characteristic function ($\chi(A)$ equals 1 if the condition A is satisfied and zero otherwise). We denote by $G^0\{1\}$ the subset of $G\{1\}$ containing the elements g satisfying $\Delta(g) = 0$, i.e. such that for any $(m \rightarrow n) \in g$

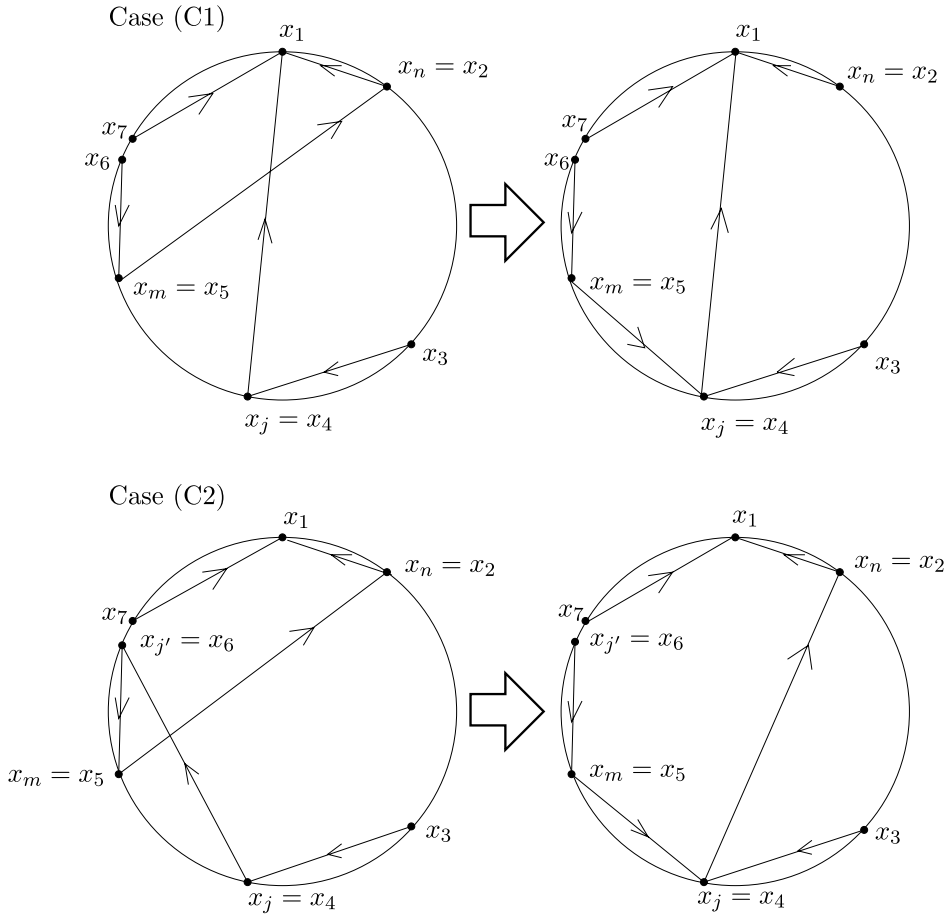


Fig. 3. Transformation $g \rightarrow g'$ in cases (C1) and (C2), with $m = 5, n = 2, j = 4$ and $\sigma(m \rightarrow n) = -$. In the above picture, we have identified graphs on $\{1, \dots, \ell\}$ with graphs on $\{x_1, \dots, x_\ell\}$.

and for any $j \neq m, n$, it holds $x_j \notin \gamma_{x_m, x_n}^{\sigma(m \rightarrow n)}$. We next show that $G^0\{1\}$ is not empty and that we can restrict to $G^0\{1\}$ the first minimum appearing in Eq. (5.4) (we assume $\ell > 2$ otherwise this statement is obviously true). To this aim, if $\Delta(g) \geq 1$ we fix m, n, j such that $x_j \in \gamma_{x_m, x_n}^{\sigma(m \rightarrow n)}$ and construct a new graph g' satisfying the following properties:

- (i) $g' \in G\{1\}$,
- (ii) $\Delta(g') \leq \Delta(g) - 1$,
- (iii) $\sum_{(r \rightarrow s) \in g'} V(K_r, K_s) \leq \sum_{(r \rightarrow s) \in g} V(K_r, K_s)$.

We need to distinguish two cases:

- (C1) the unique path in g from j to 1 does not contain the arrow $m \rightarrow n$,
- (C2) the unique path in g from j to 1 contains the arrow $m \rightarrow n$.

In case (C1) the graph g' is obtained from g by removing the edge $m \rightarrow n$ and adding the edge $m \rightarrow j$. In case (C2) we call j' the arriving point of the unique arrow in g exiting from j (note that in this case necessarily $j \neq 1$), then we remove from g the arrows $m \rightarrow n$ and $j \rightarrow j'$ and add the arrows $m \rightarrow j$ and $j \rightarrow n$. See Fig. 3.

Let us now show that g' satisfies the properties (i), (ii) and (iii).

- Proof of (i). Trivially, in all cases properties (1), (2) and (4) in the definition of $G\{i\}$ given in Section 3 are satisfied. We focus on property (3).

Consider first case (C1). Removing the arrow $m \rightarrow n$ the graph g splits into two connected components. One component coincides with the vertices whose unique path towards the root 1 in g contains the arrow $m \rightarrow n$ (in particular

m belongs to this component), the other component coincides with the vertices whose unique path towards the root 1 in g does not use the arrow $m \rightarrow n$ (in particular both j and n belong to this component). Both components are trees. Moreover, by definition of g , the component containing m is oriented towards m . Likewise the other component is a directed tree oriented towards its root 1. If we add the arrow $m \rightarrow j$ the graph that we obtain is connected and it is a tree. Using the orientations of the two merged components we easily obtain that it is oriented towards its root 1.

Consider now case (C2). By arguments similar to the previous case we have the following. Removing the arrows $m \rightarrow n$ and $j \rightarrow j'$ we obtain three connected components. One is a directed tree oriented towards its root j , one is a directed tree oriented towards its root m and one is a directed tree containing n and oriented towards its root 1. Adding the arrows $m \rightarrow j$ and $j \rightarrow n$ we obtain a directed tree oriented towards the root 1. See [2] for the basic characterizing properties of trees that we implicitly used in the proof.

- Proof of (ii). We first consider case (C1). We have

$$\Delta(g') = \Delta(g) - \sum_{r \neq m, n} \chi(x_r \in \gamma_{x_m, x_n}^{\sigma(m \rightarrow n)}) + \sum_{r \neq m, j} \chi(x_r \in \gamma_{x_m, x_j}^{\sigma(m \rightarrow j)}).$$

Since it must be $\sigma(m \rightarrow j) = \sigma(m \rightarrow n)$ and $x_n \notin \gamma_{x_m, x_j}^{\sigma(m \rightarrow j)}$, we conclude that

$$\sum_{r \neq m, j} \chi(x_r \in \gamma_{x_m, x_j}^{\sigma(m \rightarrow j)}) \leq \sum_{r \neq m, n} \chi(x_r \in \gamma_{x_m, x_n}^{\sigma(m \rightarrow n)}) - 1, \tag{5.10}$$

thus implying (ii). Let us now consider case (C2). We have

$$\begin{aligned} \Delta(g') &= \Delta(g) - \sum_{r \neq m, n} \chi(x_r \in \gamma_{x_m, x_n}^{\sigma(m \rightarrow n)}) - \sum_{r \neq j, j'} \chi(x_r \in \gamma_{x_j, x_{j'}}^{\sigma(j \rightarrow j')}) \\ &\quad + \sum_{r \neq m, j} \chi(x_r \in \gamma_{x_m, x_j}^{\sigma(m \rightarrow j)}) + \sum_{r \neq j, n} \chi(x_r \in \gamma_{x_j, x_n}^{\sigma(j \rightarrow n)}). \end{aligned} \tag{5.11}$$

Again it must be $\sigma(m \rightarrow j) = \sigma(m \rightarrow n) = \sigma(j \rightarrow n)$. Therefore the last two terms above equal

$$\sum_{r \neq m, j, n} \chi(x_r \in \gamma_{x_m, x_n}^{\sigma(m \rightarrow j)}) = \sum_{r \neq m, n} \chi(x_r \in \gamma_{x_m, x_n}^{\sigma(m \rightarrow n)}) - 1.$$

This identity together with (5.11) leads to (ii).

- Proof of (iii). We first consider case (C1). Then we have

$$\begin{aligned} &\sum_{(r \rightarrow s) \in g} V(K_r, K_s) - \sum_{(r \rightarrow s) \in g'} V(K_r, K_s) \\ &= V(K_m, K_n) - V(K_m, K_j) = \int_{\gamma_{x_m, x_n}^{\sigma(m \rightarrow n)}} F_{\sigma(m \rightarrow n)}(s) ds - \int_{\gamma_{x_m, x_j}^{\sigma(m \rightarrow j)}} F_{\sigma(m \rightarrow j)}(s) ds. \end{aligned} \tag{5.12}$$

Since $\sigma(m \rightarrow n) = \sigma(m \rightarrow j)$ the last difference must be nonnegative.

In case (C2) we have

$$\begin{aligned} &\sum_{(r \rightarrow s) \in g} V(K_r, K_s) - \sum_{(r \rightarrow s) \in g'} V(K_r, K_s) \\ &= V(K_m, K_n) + V(K_j, K_{j'}) - V(K_m, K_j) - V(K_j, K_n). \end{aligned}$$

The last difference must be positive since $\sigma(m \rightarrow j) = \sigma(m \rightarrow n) = \sigma(j \rightarrow n)$ and therefore $V(K_m, K_j) + V(K_j, K_n) = V(K_m, K_n)$.

We have now proved our claim concerning the new graph g' . This claim trivially implies that, starting from any initial graph belonging to $G\{1\}$, with a finite number of iterations of the above procedure we end with a graph belonging to $G^0\{1\}$. In particular $G^0\{1\}$ is not empty. Moreover from property (iii) we have that

$$\min_{g \in G^0\{1\}} \sum_{(r \rightarrow s) \in g} V(K_r, K_s) = \min_{g \in G\{1\}} \sum_{(r \rightarrow s) \in g} V(K_r, K_s).$$

Clearly if $g \in G^0\{1\}$ then it can contain only arrows of the type $r \rightarrow r + 1$ or $r \rightarrow r - 1$. Moreover if it contains $r \rightarrow r + 1$ then necessarily $V(K_r, K_{r+1}) = V_+(K_r, K_{r+1})$; if it contains $(r \rightarrow r - 1)$ then necessarily $V(K_r, K_{r-1}) = V_-(K_r, K_{r-1})$. This implies that if $g \in G^0\{1\}$ then $g = g_{1,j}$ for some j and moreover

$$\sum_{(r \rightarrow s) \in g} V(K_r, K_s) = t(1, j).$$

Summarizing we have

$$\begin{aligned} \min_{g \in G\{1\}} \sum_{(r \rightarrow s) \in g} V(K_r, K_s) &= \min_{g \in G^0\{1\}} \sum_{(r \rightarrow s) \in g} V(K_r, K_s) \\ &= \min_{\{j: g_{1,j} \in G^0\{1\}\}} \sum_{(r \rightarrow s) \in g} V(K_r, K_s) = \min_{\{j: g_{1,j} \in G^0\{1\}\}} t(1, j) \geq \min_{1 \leq j \leq \ell} t(1, j), \end{aligned}$$

that is (5.4) with the reversed inequality.

It remains to prove the last statement in Lemma 5.2. Again, we take $i = 1$ for simplicity of notation. We fix two distinct indices j, J in $\{1, 2, \dots, \ell\}$ and we fix $a_j \in A_j$ and $a_J \in A_J$. If $1 \leq j < J$ then it holds

$$V_-(K_J, K_j) = \int_{\gamma_{a_j^+, a_J}^+} F_-(s) ds, \quad V_+(K_{j+1}, K_{J+1}) = \int_{\gamma_{a_j^+, a_J}^+} F_+(s) ds.$$

Therefore we can write

$$\begin{aligned} t(1, J) - t(1, j) &= V_-(K_J, K_j) - V_+(K_{j+1}, K_{J+1}) \\ &= - \int_{\gamma_{a_j^+, a_J}^+} F(s) ds = -\Delta S(a_j, a_J) = -\Delta S(A_j, A_J). \end{aligned} \tag{5.13}$$

Similarly, if $J < j \leq \ell$, it holds $t(1, J) - t(1, j) = \Delta S(A_J, A_j)$. In particular, an index J realizes the $\min_{1 \leq j \leq \ell} t(1, j)$ if and only if $\Delta S(A_j, A_J) \geq 0$ for all j such that $1 \leq j < J$ and $\Delta S(A_J, A_j) \leq 0$ for all j such that $J < j \leq \ell$. These inequalities read as follows: considering S on the interval $[x_0, x_0 + 1]$ such that $\pi(x_0) \in K_1$, the highest local maximum (and therefore the maximum) of S is attained at all points $y \in [x_0, x_0 + 1]$ such that $\pi(y) \in A_J$. This coincides with the characterization (5.3). \square

Remark 2. Fixed $i \in \{1, \dots, \ell\}$, set

$$\begin{aligned} N_i^+ &= \{j \in \{1, \dots, \ell\}: V_+(K_j, K_i) \leq V_-(K_j, K_i)\}, \\ N_i^- &= \{j \in \{1, \dots, \ell\}: V_+(K_j, K_i) > V_-(K_j, K_i)\}. \end{aligned}$$

It is simple to check that there exists $a \in \{0, \dots, l - 1\}$ such that

$$\begin{aligned} N_i^+ &= \{i - m: 0 \leq m \leq a\}, \\ N_i^- &= \{i + m: 1 \leq m \leq l - a - 1\}. \end{aligned}$$

One could ask if the index J of Lemma 5.2 can be characterized as $J = i + l - a - 1$ or $J = i + l - a$. It is easy to check that this simple characterization cannot hold by drawing suitable functions S .

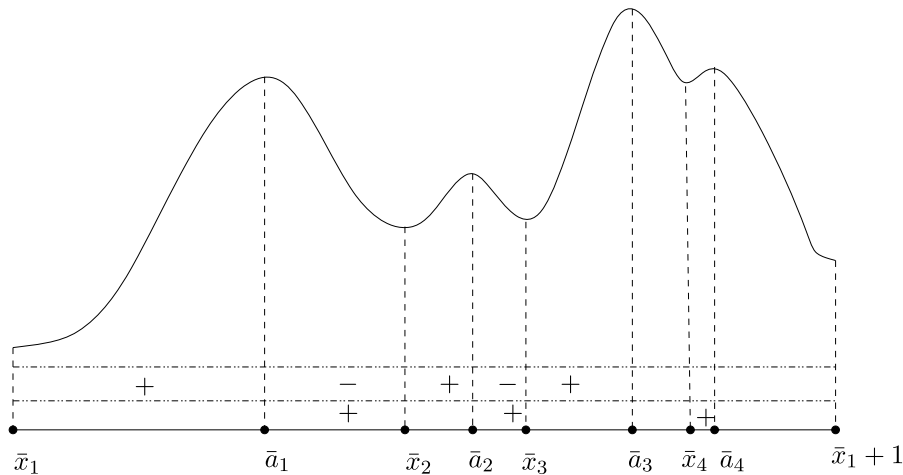


Fig. 4. Proof of Lemma 5.3 with $i = 1$, $J = 3$, $\ell = 4$. The first line of signs illustrates how to sum slope heights to get $\max_{y \in [\bar{x}_1, \bar{x}_1+1]}(S(y) - S(\bar{x}_1))$. The second line of signs illustrates how to sum slope heights to get $W(K_1)$.

In what follows, in order to make the discussion more intuitive, it is convenient to use geometric arguments. To this aim we fix some language. We call *slope* of the function S its graph restricted to intervals I of the form $I = [\bar{x}_i, \bar{a}_i]$ or $I = [\bar{a}_i, \bar{x}_{i+1}]$, where $\bar{x}_i, \bar{x}_{i+1}, \bar{a}_i$ are points in \mathbb{R} such that $\pi(\bar{x}_i) \in K_i, \pi(\bar{x}_{i+1}) \in K_{i+1}, \pi(\bar{a}_i) \in A_i$ and $\bar{a}_i - \bar{x}_i < 1, \bar{x}_{i+1} - \bar{a}_i < 1$. If $I = [\bar{x}_i, \bar{a}_i]$, then the slope is increasing and its height is set equal to $S(\bar{a}_i) - S(\bar{x}_i)$; if $I = [\bar{a}_i, \bar{x}_{i+1}]$, then the slope is decreasing and its height is set equals to $S(\bar{a}_i) - S(\bar{x}_{i+1})$.

Lemma 5.3. *Identity (5.1) holds for all $x \in K_i, 1 \leq i \leq \ell, \ell \geq 1$.*

Proof. If $\ell = 1$, then the thesis is trivial. Suppose that $\ell \geq 2$ and see Fig. 4. Without loss assume that $i = 1$. Fix points $\bar{x}_1 < \bar{a}_1 < \bar{x}_2 < \dots < \bar{x}_\ell < \bar{a}_\ell < \bar{x}_1 + 1$ such that $\pi(\bar{x}_j) \in K_j$ and $\pi(\bar{a}_j) \in A_j$. Take J as in Lemma 5.2 such that $W(K_1) = t(1, J)$.

Then $\max_{y \in [x_1, x_1+1]}(S(y) - S(x_1))$, by (5.3) in Lemma 5.2, equals the sum of the heights of the slopes associated to $[\bar{x}_1, \bar{a}_1], [\bar{a}_1, \bar{x}_2], \dots, [\bar{x}_2, \bar{a}_2], \dots, [\bar{x}_J, \bar{a}_J]$ with alternating signs $+, -, +, \dots, +$. On the other hand, by Lemma 5.2 again, $W(K_1)$ equals the sum of the heights of the slopes associated to $[\bar{a}_1, \bar{x}_2], \dots, [\bar{a}_{J-1}, \bar{x}_J]$ and $[\bar{x}_{J+1}, \bar{a}_{J+1}], \dots, [\bar{x}_\ell, \bar{a}_\ell]$. Hence, $W(K_1) + \max_{y \in [x_1, x_1+1]}(S(y) - S(x_1))$ simply equals the sum of the heights of the increasing slopes in $[\bar{x}_1, \bar{x}_1 + 1]$, i.e. $\int_0^1 F_+(s) ds$. □

Lemma 5.4. *Suppose $\ell \geq 1$. Then, given $x \in \mathbb{T}$ and $i \in \{1, \dots, \ell\}$, it holds*

$$W(K_i) + V_{\pm}(K_i, x) \geq \int_0^1 F_+(s) ds - \max_{y \in [x, x+1]} (S(y) - S(x)), \tag{5.14}$$

where $V_{\pm}(K_i, x) := 0$ if $x \in K_i$ and $V_{\pm}(K_i, x) := V_{\pm}(x_i, x)$ for any $x_i \in K_i$ if $x \notin K_i$.

Proof. If $x \in K_i$ we have nothing to prove due to Lemma 5.3. We assume $x \notin K_i$ and we fix $x_i \in K_i$, and also $\bar{x}, \bar{x}_i \in \mathbb{R}$ such that $\pi(\bar{x}) = x, \pi(\bar{x}_i) = x_i$ and $|\bar{x} - \bar{x}_i| < 1$. Then, substituting $W(K_i)$ by means of Lemma 5.3, we get that (5.14) reads

$$V_{\pm}(x_i, x) \geq \max_{y \in [\bar{x}_i, \bar{x}_i+1]} (S(y) - S(\bar{x}_i)) - \max_{y \in [\bar{x}, \bar{x}+1]} (S(y) - S(\bar{x})). \tag{5.15}$$

We give the proof for $V_+(x_i, x)$. The other case is completely similar (indeed, specular). We can always choose \bar{x} and \bar{x}_i such that $\bar{x}_i < \bar{x} < \bar{x}_i + 1$. Then we can bound

$$V_+(x_i, x) = \int_{\bar{x}_i}^{\bar{x}} F_+(s) ds \geq \max_{y \in [\bar{x}_i, \bar{x}]} \int_{\bar{x}_i}^y F(s) ds = \max_{y \in [\bar{x}_i, \bar{x}]} (S(y) - S(\bar{x}_i)). \tag{5.16}$$

Hence, to conclude it is enough to show that the last member in (5.16) bounds from above the r.h.s. of (5.15). This is equivalent to the inequality

$$\max_{y \in [\bar{x}_i, \bar{x}]} S(y) \geq \max_{y \in [\bar{x}_i, \bar{x}_i + 1]} S(y) - \max_{y \in [\bar{x}, \bar{x} + 1]} S(y) + S(\bar{x}).$$

If the l.h.s. equals the first term in the r.h.s., then the inequality is obviously verified. Otherwise, it must be $\max_{y \in [\bar{x}_i, \bar{x}_i + 1]} S(y) < \max_{y \in [\bar{x}, \bar{x} + 1]} S(y)$ and the conclusion becomes trivial. \square

Lemma 5.5. *Suppose $\ell \geq 1$. Take $x \in \mathbb{T} \setminus (\bigcup_{r=1}^{\ell} K_r)$. Take $i \in \{1, \dots, \ell\}$ such that $K_i < x < K_{i+1}$. Fix $x_i \in K_i$, $x_{i+1} \in K_{i+1}$ (if $\ell = 1$ take $x_i = x_{i+1}$). Then, fix $\bar{x}_i \in \mathbb{R}$ such that $x_i = \pi(\bar{x}_i)$, and afterwards fix $\bar{x}, \bar{x}_{i+1} \in (\bar{x}_i, \bar{x}_i + 1]$ such that $x = \pi(\bar{x})$, $x_{i+1} = \pi(\bar{x}_{i+1})$.*

Then exactly one of the four cases (1), ..., (4) mentioned in Theorem 4.1(ii) holds. Moreover, in cases (1) and (4) it holds

$$V_+(K_i, x) + \max_{y \in [\bar{x}, \bar{x} + 1]} S(y) - S(\bar{x}) = \max_{y \in [\bar{x}_i, \bar{x}_i + 1]} S(y) - S(\bar{x}_i), \tag{5.17}$$

while in cases (2) and (3) it holds

$$V_-(K_{i+1}, x) + \max_{y \in [\bar{x}, \bar{x} + 1]} S(y) - S(\bar{x}) = \max_{y \in [\bar{x}_{i+1}, \bar{x}_{i+1} + 1]} S(y) - S(\bar{x}_{i+1}). \tag{5.18}$$

Proof. In case (1) it holds $V_+(K_i, x) = S(\bar{x}) - S(\bar{x}_i)$ and the check of (5.17) is immediate (see Fig. 5). Case (3) is similar. We only need to treat case (2), since case (4) is specular (take a reflection at the origin). To this aim, we fix $a_i \in (\bar{x}_i, \bar{x}_i + 1]$ such that $\pi(a_i) \in A_i$. Then, due to the definition of case (2) (see Fig. 5), it must be $\max_{y \in [\bar{x}, \bar{x} + 1]} S(y) = S(\bar{x} + 1)$ and $\max_{y \in [\bar{x}_{i+1}, \bar{x}_{i+1} + 1]} S(y) = S(\bar{a}_i + 1)$. In particular, (5.18) is equivalent to

$$V_-(K_{i+1}, x) + S(\bar{x} + 1) - S(\bar{x}) = S(\bar{a}_i + 1) - S(\bar{x}_{i+1}).$$

Since $S(\bar{x} + 1) - S(\bar{x}) = S(\bar{a}_i + 1) - S(\bar{a}_i) = S(1)$, the above identity is equivalent to $V_-(K_{i+1}, x) = S(\bar{a}_i) - S(\bar{x}_{i+1})$ which is trivially true (see Fig. 5).

It remains now to prove that the above four cases are exhaustive. Suppose for example that $x_i \leq x < A_i$. Then it is trivial to check that it cannot be $\max_{y \in [\bar{x}_i, \bar{x}_i + 1]} S(y) > \max_{y \in [\bar{x}, \bar{x} + 1]} S(y)$. \square

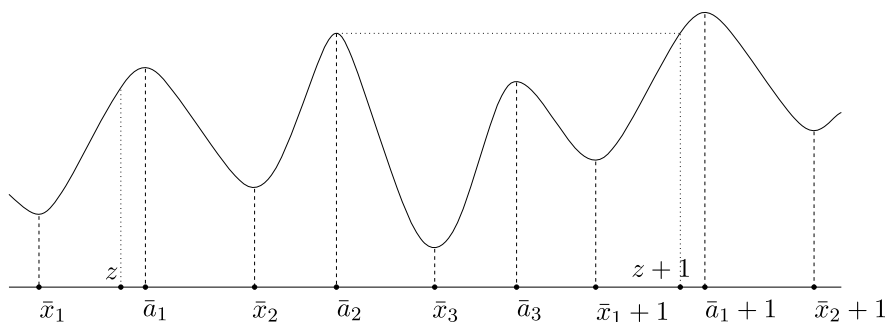


Fig. 5. Proof of Lemma 5.5 with $i = 1$, $\ell = 3$. If $\bar{x}_1 < \bar{x} \leq z$, then case (1) takes place. If $z < \bar{x} < \bar{a}_1$, then case (2) takes place.

We can finally conclude:

Proof of Proposition 5.1 and Theorem 4.1(ii). If $\ell = 0$, then by definition $W \equiv 0$ and we only need to prove Proposition 5.1. We take $\bar{x} \in \mathbb{R}$ such that $\pi(\bar{x}) = x$ and note that the function S is monotone. If it is weakly increasing, then

$$\max_{y \in [\bar{x}, \bar{x}+1]} (S(y) - S(\bar{x})) = S(\bar{x} + 1) - S(\bar{x}) = \int_0^1 F(s) ds = \int_0^1 F_+(s) ds$$

which coincides with (5.1). Similarly, one gets (5.1) if S is weakly decreasing.

Let us restrict now to $\ell \geq 1$. If $x \in \bigcup_{i=1}^{\ell} K_i$, we only need to prove Proposition 5.1 and in this case the thesis coincides with Lemma 5.3. Suppose that $x \notin \bigcup_{i=1}^{\ell} K_i$. Due to the definition of the function W and inequality (5.14) in Lemma 5.4,

$$W(x) \geq \int_0^1 F_+(s) ds - \max_{y \in [x, x+1]} (S(y) - S(x)).$$

Moreover, if for some $j \in \{1, \dots, \ell\}$ and some sign $\sigma \in \{-, +\}$ it holds

$$W(K_j) + V_{\sigma}(K_j, x) = \int_0^1 F_+(s) ds - \max_{y \in [x, x+1]} (S(y) - S(x)), \quad (5.19)$$

then it must be

$$W(x) = W(K_j) + V_{\sigma}(K_j, x) = \int_0^1 F_+(s) ds - \max_{y \in [x, x+1]} (S(y) - S(x)) \quad (5.20)$$

and

$$V(K_j, x) = V_{\sigma}(K_j, x). \quad (5.21)$$

By Lemma 5.3 we can rewrite $W(K_j)$ as

$$W(K_j) = \int_0^1 F_+(s) ds - \max_{y \in [x_j, x_j+1]} (S(y) - S(x_j)),$$

where $x_j \in K_j$. Hence (5.19) reads

$$V_{\sigma}(K_j, x) - \max_{y \in [x_j, x_j+1]} (S(y) - S(x_j)) = - \max_{y \in [x, x+1]} (S(y) - S(x)). \quad (5.22)$$

By Lemma 5.5, in order to fulfill (5.22) it is enough to take $j = i$ and $\sigma = +$ in cases (1) and (4) (see (5.17)), while it is enough to take $j = i + 1$ and $\sigma = -$ in cases (2) and (3) (see (5.18)). Since as already observed (5.19) implies both (5.20) and (5.21), this concludes the proof of both Proposition 5.1 and Theorem 4.1(ii). \square

Proof of Theorem 2.3(ii). As already observed, part (ii) of Theorem 2.3 is an immediate consequence of Proposition 5.1. \square

Proof of Theorem 4.1(i). If we take J as in Theorem 4.1, by Lemma 5.2 it must be

$$\begin{aligned} W(K_i) &= t(i, J) = \sum_{(m \rightarrow n) \in g_{i,J}} [V_-(K_m, K_n)\chi(n = m - 1) + V_+(K_m, K_n)\chi(n = m + 1)] \\ &\geq \sum_{(m \rightarrow n) \in g_{i,J}} V(K_m, K_n) \geq \min_{g \in G\{i\}} \sum_{(m \rightarrow n) \in g} V(K_m, K_n) = W(K_i). \end{aligned} \quad (5.23)$$

In particular all the inequalities in the above expression must be equalities and this proves the first part of Theorem 4.1. \square

6. Proof of Theorem 2.3 part (i)

The periodicity of Φ follows by

$$\begin{aligned} \Phi(x+1) &= \min_{y \in [x+1, x+2]} (S(x+1) - S(y)) = \min_{y \in [x+1, x+2]} - \int_{x+1}^y F(s) ds \\ &= \min_{y \in [x+1, x+2]} - \int_x^{y-1} F(t) dt = \min_{y \in [x, x+1]} - \int_x^y F(t) dt = \Phi(x). \end{aligned} \quad (6.1)$$

In the third equality we used the periodicity of F .

Next we show that Φ is Lipschitz. Due to the periodicity of Φ it is enough to show it in $[0, 1]$. The function S is Lipschitz with Lipschitz constant $K := \max_{x \in \mathbb{T}} |F(x)|$. Fix $x < y \in [0, 1]$. Then we have

$$\begin{aligned} |\Phi(x) - \Phi(y)| &\leq |S(x) - S(y)| + \left| \max_{z \in [x, x+1]} S(z) - \max_{w \in [y, y+1]} S(w) \right| \\ &\leq K|x - y| + \left| \max_{z \in [x, x+1]} S(z) - \max_{w \in [y, y+1]} S(w) \right|. \end{aligned} \quad (6.2)$$

We now estimate the second term in (6.2). If the maxima are achieved respectively in z^* and w^* both belonging to $[x, x+1] \cap [y, y+1] = [y, x+1]$, then necessarily $S(w^*) = S(z^*)$ and the second term in (6.2) is zero. Let us suppose that $S(w^*) > S(z^*)$. Then necessarily $w^* \in (x+1, y+1]$ and

$$|S(w^*) - S(z^*)| = S(x+1) + S(w^*) - S(z^*) - S(x+1) \leq S(w^*) - S(x+1) \leq K|x - y|.$$

The remaining case can be treated similarly. Summarizing we have

$$|\Phi(x) - \Phi(y)| \leq 2K|x - y|,$$

that is Φ is Lipschitz with Lipschitz constant $2K$.

If S is monotone, then trivially it holds $\Phi(x) = \min\{0, -S(1)\}$ for all $x \in \mathbb{R}$. When S is periodic, $\max_{y \in [x, x+1]} S(y)$ does not depend on x and therefore it holds $\Phi(x) = S(x) - \max_{y \in [0, 1]} S(y)$. Let us suppose now that S is not monotone and it has not period one, i.e. that $S(1) = \int_0^1 F(s) ds \neq 0$. Similarly to our definition on the torus \mathbb{T} , we say that $[a, b] \subset \mathbb{R}$ is a totally unstable connected component of $\{\nabla S = 0\}$ if S is constant on $[a, b]$ and there exists $\varepsilon > 0$ such that $S(x) < S(a)$ for all $x \in [a - \varepsilon, a) \cup (b, b + \varepsilon]$. Due to the continuity of F , the totally unstable connected components are countable and we enumerate them as A_j , $j \in J$. Due to the assumption that S is not monotone, the index set J is nonempty. Below, we write $S(A_j)$ for the value $S(x)$ with $x \in A_j$.

We write $\Phi(x) = S(x) - \max_{y \in [x, x+1]} S(y)$. Since $S(1) = S(x+1) - S(x) \neq 0$, then the maximum of S on $[x, x+1]$ is achieved on

$$\begin{cases} \{y \in [x, x+1]: y \in A_j \text{ for some } j\} \cup \{x+1\} & \text{if } S(1) > 0, \\ \{y \in [x, x+1]: y \in A_j \text{ for some } j\} \cup \{x\} & \text{if } S(1) < 0. \end{cases}$$

Since $0 = S(x) - S(x)$ and $-S(1) = S(x) - S(x+1)$, if we define as in (2.13)

$$U = \left\{ x \in \mathbb{R}: \Phi(x) = S(x) - \max_{y \in [x, x+1]} S(y) \neq \min\{0, -S(1)\} \right\}, \quad (6.3)$$

then for all $x \in U$ the maximum of S on $[x, x+1]$ is not achieved on x or $x+1$ (one has to distinguish the cases $S(1) > 0$ and $S(1) < 0$). Since Φ is continuous, we get that U is an open subset of \mathbb{R} .

We define the function Ψ as $\Psi(x) = \max_{y \in [x, x+1]} S(y)$. Then, $\Phi(x) = S(x) - \Psi(x)$. Due to the above observations, when $x \in U$ we have

$$\Psi(x) = \max_{j: A_j \cap [x, x+1] \neq \emptyset} S(A_j).$$

Since $\Psi(x) = S(x) - \Phi(x)$ is the sum of two continuous functions, Ψ is continuous. We claim that Ψ is constant on every connected component of U . Indeed, consider (a, b) a connected component of U and suppose there exist $x < y$ in (a, b) such that $\Psi(x) \neq \Psi(y)$. Then the set $\{\Psi(z) : z \in [x, y]\}$ must contain the interval $[\Psi(x) \wedge \Psi(y), \Psi(x) \vee \Psi(y)]$, in contradiction with the fact that, when $x \in U$, Ψ takes value in the countable set $\{S(A_j) : j \in J\}$. This concludes the proof of our claim, which is equivalent to (2.14). By definition of U , one trivially gets (2.15).

Let us prove that U is nonempty and that the interior of $\mathbb{R} \setminus U$ is nonempty. We start with the second claim. The function Φ is Lipschitz and consequently it is absolutely continuous. This implies that it is almost everywhere differentiable, its derivative is Lebesgue locally integrable and moreover it holds

$$\int_a^b \nabla \Phi(y) \, dy = \Phi(b) - \Phi(a),$$

for any $a, b \in \mathbb{R}$. From the previous analysis we know that Φ is differentiable on U where it holds $\nabla \Phi = \nabla S$. Recall that U is open and $\mathbb{R} \setminus U$ is closed. If we write A for the interior part of $\mathbb{R} \setminus U$, then $B := (\mathbb{R} \setminus U) \setminus A$ consists of a countable set of points and in particular has zero Lebesgue measure. If A was empty, since B has zero Lebesgue measure, we would conclude that $\mathbb{R} \setminus U = A \cup B$ has zero Lebesgue measure. In particular, we would get

$$0 = \Phi(1) - \Phi(0) = \int_0^1 \nabla \Phi(y) \, dy = \int_{[0,1] \cap U} \nabla S(y) \, dy = \int_{[0,1]} \nabla S(y) \, dy = S(1) \quad (6.4)$$

in contradiction with the fact that $S(1) \neq 0$.

Let us show that U is also nonempty. We discuss the case $S(1) > 0$. The case $S(1) < 0$ can be treated by similar arguments. Since S is not monotone, it must have local minima. We claim that $x \in U$ whenever x is a local minimum point for S . Indeed, since $S(y+1) = S(y) + S(1)$ for all $y \in \mathbb{R}$, also $x+1$ is a local minimum point for S . Since S is not flat (otherwise it would be monotone), there must be a point $z \in (x, x+1)$ such that $S(z) > S(x+1)$. This implies that $\max_{y \in [x, x+1]} S(y) > S(x+1)$, which is equivalent to $x \in U$, since for $S(1) > 0$ the definition of U reads

$$U = \left\{ x \in \mathbb{R} : \Phi(x) = S(x) - \max_{y \in [x, x+1]} S(y) \neq S(x) - S(x+1) \right\}.$$

Let us finally prove (2.16), where only the second identity is nontrivial. Due to the definition (6.3) of U , it must be $\Phi(x) < \min\{0, -S(1)\}$ for all $x \in U$. On the other hand, due to (2.15), $\Phi(x) = \min\{0, -S(1)\}$ for all x in the nonempty set $\mathbb{R} \setminus U$. These considerations trivially imply (2.16).

7. Proof of Theorem 2.4

The proof of Theorem 2.4 is based on the following fact:

Lemma 7.1. *Consider a function $\varphi : \mathbb{T} \rightarrow \mathbb{R}$ that satisfies the following properties:*

- (a) *It is continuous.*
- (b) *There exists an open subset $O \subseteq \mathbb{T}$ where it is differentiable and moreover $\nabla \varphi(x) = F(x)$ for any $x \in O$.*
- (c) *On every connected component of $\mathbb{T} \setminus O$ it is constant.*
- (d) *Calling (o_j^-, o_j^+) , $j \in J$, the countable disjoint maximal connected components of O , it holds $F(o_j^-) \leq 0$ and $F(o_j^+) \geq 0$ for any $j \in J$.*

Then, φ is a viscosity solution of (2.20) for any Hamiltonian H satisfying hypotheses (A) and (B) of Theorem 2.4.

Proof. We first compute the sub- and superdifferential of φ by simply assuming (a), (b), (c) (assuming (d) would not change much, and we prefer to make the computation only under (a), (b), (c) since more instructive).

Trivially, φ is differentiable at x if and only if $D^+ \varphi(x) = D^- \varphi(x) = \{a\}$ for some value $a \in \mathbb{R}$, and in this case it holds $a = \nabla \varphi(x)$. In particular, for any $x \in O$ it must be $D^+ \varphi(x) = D^- \varphi(x) = \{F(x)\}$, and similarly for any x in the interior part of $\mathbb{T} \setminus O$ it must be $D^+ \varphi(x) = D^- \varphi(x) = \{0\}$. The nontrivial cases come from $x = o_j^\pm$.

- If $x = o_j^-$, since $\varphi(y) - \varphi(x) = \int_x^y F(z) dz$ for y in a small right neighborhood of x , given $p \in \mathbb{R}$ it holds

$$\lim_{y \downarrow x} \frac{\varphi(y) - \varphi(x) - p(y-x)}{|y-x|} = F(x) - p. \quad (7.1)$$

- If $x = o_j^+$ then, by the same argument, we get for any $p \in \mathbb{R}$ that

$$\lim_{y \uparrow x} \frac{\varphi(y) - \varphi(x) - p(y-x)}{|y-x|} = -F(x) + p. \quad (7.2)$$

- If $x = o_j^-$ is not an accumulation point of ∂O , then φ is constant on a small left neighborhood of x and therefore

$$\lim_{y \uparrow x} \frac{\varphi(y) - \varphi(x) - p(y-x)}{|y-x|} = p. \quad (7.3)$$

- Similarly if $x = o_j^+$ is not an accumulation point of ∂O , then

$$\lim_{y \downarrow x} \frac{\varphi(y) - \varphi(x) - p(y-x)}{|y-x|} = -p. \quad (7.4)$$

- If $x = o_j^-$ is an accumulation point of ∂O and $F(o_j^-) > 0$, then for any y in a small left neighborhood of x it holds

$$0 \geq \varphi(y) - \varphi(x) \geq \int_x^y F(z) dz. \quad (7.5)$$

Indeed, by assumption $F(o_j^-) > 0$ and therefore F is positive on a small left neighborhood of x . Hence, on a small left neighborhood φ is non decreasing. Moreover, a part a countable set of points, φ is differentiable and $\nabla\varphi \leq \max\{0, F\} = F$. This leads to the second inequality in (7.5). As a consequence, it holds

$$p \geq \limsup_{y \uparrow x} \frac{\varphi(y) - \varphi(x) - p(y-x)}{|y-x|} \geq \liminf_{y \uparrow x} \frac{\varphi(y) - \varphi(x) - p(y-x)}{|y-x|} \geq -F(x) + p. \quad (7.6)$$

- If $x = o_j^-$ is an accumulation point of ∂O and $F(o_j^-) < 0$, by similar arguments we get for any y in a small left neighborhood of x that

$$\int_x^y F(z) dz \geq \varphi(y) - \varphi(x) \geq 0,$$

and therefore

$$-F(x) + p \geq \limsup_{y \uparrow x} \frac{\varphi(y) - \varphi(x) - p(y-x)}{|y-x|} \geq \liminf_{y \uparrow x} \frac{\varphi(y) - \varphi(x) - p(y-x)}{|y-x|} \geq p. \quad (7.7)$$

- If $x = o_j^-$ is an accumulation point of ∂O and $F(o_j^-) = 0$, we can proceed as follows. On a left neighborhood of x minus a countable set of points, φ is differentiable and $\nabla\varphi(z) \in \{0, F(z)\}$. This implies that

$$-\int_x^y |F(z)| dz \geq \varphi(y) - \varphi(x) \geq \int_x^y |F(z)| dz.$$

Since $F(x) = 0$, the above bounds trivially imply that

$$\lim_{y \uparrow x} \frac{\varphi(y) - \varphi(x)}{y-x} = 0. \quad (7.8)$$

• Formulas similar to (7.6), (7.7) and (7.8) are valid if $x = o_j^+$ is an accumulation point of ∂O , $F(o_j^+) > 0$, $F(o_j^+) < 0$ and $F(o_j^+) = 0$ respectively.

The above computations allow us to treat the several possible cases.

Case 1: $x = o_j^- = o_i^+$ for some $i, j \in J$. By (7.1) and (7.2), φ is differentiable at x and moreover $D^+\varphi(x) = D^-\varphi(x) = \{F(x)\}$.

Case 2: $x = o_j^-$ for some $j \in J$ and x is not an accumulation point of ∂O . We claim that

$$\begin{cases} D^-\varphi(x) = [0, F(x)], & D^+\varphi(x) = \emptyset & \text{if } F(x) > 0, \\ D^-\varphi(x) = \emptyset, & D^+\varphi(x) = [F(x), 0] & \text{if } F(x) < 0, \\ \nabla\varphi(x) = D^-\varphi(x) = D^+\varphi(x) = \{0\} & & \text{if } F(x) = 0. \end{cases}$$

Indeed, due to (7.1) and (7.3) we conclude that $p \in D^+\varphi(x)$ if and only if $F(x) - p \leq 0$ and $p \leq 0$, while $p \in D^-\varphi(x)$ if and only if $F(x) - p \geq 0$ and $p \geq 0$. Then the claim follows by distinguishing on the sign of $F(x)$.

Case 3: $x = o_j^+$ for some $j \in J$ and x is not an accumulation point of ∂O . We claim that

$$\begin{cases} D^-\varphi(x) = \emptyset, & D^+\varphi(x) = [0, F(x)] & \text{if } F(x) > 0, \\ D^-\varphi(x) = [F(x), 0], & D^+\varphi(x) = \emptyset & \text{if } F(x) < 0, \\ \nabla\varphi(x) = D^-\varphi(x) = D^+\varphi(x) = \{0\} & & \text{if } F(x) = 0. \end{cases}$$

Indeed, by (7.2) and (7.4) it holds: $p \in D^+\varphi(x)$ if and only if $-F(x) + p \leq 0$ and $-p \leq 0$, while $p \in D^-\varphi(x)$ if and only if $-F(x) + p \geq 0$ and $-p \geq 0$.

Case 4: $x = o_j^-$ and $x \in \partial O$ is an accumulation point of ∂O . We claim that

$$\begin{cases} D^-\varphi(x) \subseteq [0, F(x)], & D^+\varphi(x) \subseteq \{F(x)\} & \text{if } F(x) > 0, \\ D^-\varphi(x) \subseteq \{F(x)\}, & D^+\varphi(x) \subseteq [F(x), 0] & \text{if } F(x) < 0, \\ D^-\varphi(x) = D^+\varphi(x) = \{0\} & & \text{if } F(x) = 0. \end{cases}$$

Indeed, if $F(x) > 0$ then, by (7.1) and (7.6), if $p \in D^+\varphi(x)$ then $F(x) - p \leq 0$ and $-F(x) + p \leq 0$, while if $p \in D^-\varphi(x)$ then $F(x) - p \geq 0$ and $p \geq 0$. If $F(x) < 0$, by (7.1) and (7.7), if $p \in D^+\varphi(x)$ then $F(x) - p \leq 0$ and $p \leq 0$, while if $p \in D^-\varphi(x)$ then $F(x) - p \geq 0$ and $-F(x) + p \geq 0$. If $F(x) = 0$ then the thesis follows from (7.1) and (7.8).

Case 5: $x = o_j^+$ and $x \in \partial O$ is an accumulation point of ∂O . Similarly to Case 4, one obtains

$$\begin{cases} D^-\varphi(x) \subseteq \{F(x)\}, & D^+\varphi(x) \subseteq [0, F(x)] & \text{if } F(x) > 0, \\ D^-\varphi(x) \subseteq [F(x), 0], & D^+\varphi(x) \subseteq \{F(x)\} & \text{if } F(x) < 0, \\ D^-\varphi(x) = D^+\varphi(x) = \{0\} & & \text{if } F(x) = 0. \end{cases}$$

We have now all the tools to show that φ is a viscosity solution of (2.20) at every $x \in \mathbb{T}$, adding assumption (d).

First of all we consider a point $x \in \mathbb{T}$ such that $F(x) = 0$. In this case we proved that φ is differentiable at x and moreover $\nabla\varphi(x) = 0$. As consequence we obtain that the Hamilton–Jacobi equation is satisfied at x due to the fact that $D^+\varphi(x) = D^-\varphi(x) = \{0\}$ and moreover $H(x, 0) = 0$ from the hypothesis (B).

We now consider the case $F(x) \neq 0$. If φ is differentiable at x then either $D^+\varphi(x) = D^-\varphi(x) = \{F(x)\}$ or $D^+\varphi(x) = D^-\varphi(x) = \{0\}$. In both cases by hypothesis (B) we have that φ is a viscosity solution at x . If φ is not differentiable at x as a direct consequence of the previous results, the definitions and all the assumptions (a), (b), (c) and (d), we have that φ is a viscosity solution at x if the following implications holds:

$$p \in [F(x) \wedge 0, F(x) \vee 0] \implies H(x, p) \leq 0 \tag{7.9}$$

and

$$p \in (-\infty, F(x) \wedge 0] \cup [F(x) \vee 0, +\infty) \implies H(x, p) \geq 0. \tag{7.10}$$

We next show that indeed (7.9) and (7.10) are consequences of the hypotheses (A) and (B).

Let us consider first (7.9). Take $p \in [F(x) \wedge 0, F(x) \vee 0]$, then there exists $c \in [0, 1]$ such that $p = c0 + (1 - c)F(x)$. From hypotheses (A) and (B) we deduce immediately

$$H(x, p) \leq cH(x, 0) + (1 - c)H(x, F(x)) = 0.$$

We discuss now (7.10). Take for example the case $F(x) < 0$ and $p \in (-\infty, F(x)]$. Consider an arbitrary $w \in (F(x), 0)$ and the corresponding $c \in (0, 1]$ such that $F(x) = cp + (1 - c)w$. From hypotheses (A) and (B) we have

$$0 = H(x, F(x)) \leq cH(x, p) + (1 - c)H(x, w),$$

and from (7.9) we deduce

$$H(x, p) \geq \frac{c - 1}{c}H(x, w) \geq 0.$$

The remaining cases can be treated similarly. □

We can finally conclude:

Proof of Theorem 2.4. We only need to show that the function Φ satisfies conditions (a), (b), (c) and (d) of Lemma 7.1.

The validity of conditions (a), (b) and (c) follows directly from Theorem 2.3 with the identification $O = U$. Let us show that also condition (d) is satisfied. To this aim, we consider a maximal connected component (u_i^-, u_i^+) of U . Then by definition we have $\Phi(u_i^-) = \Phi(u_i^+) = \min\{0, -S(1)\}$ and $\Phi(u) < \min\{0, -S(1)\}$ for any $u \in (u_i^-, u_i^+)$. In particular, for any $u \in (u_i^-, u_i^+)$ we can write

$$0 > \Phi(u) - \min\{0, -S(1)\} = \Phi(u) - \Phi(u_i^-) = \int_{u_i^-}^u F(z) dz, \quad (7.11)$$

$$0 > \Phi(u) - \min\{0, -S(1)\} = \Phi(u) - \Phi(u_i^+) = \int_{u_i^+}^u F(z) dz. \quad (7.12)$$

Due to the continuity of F , the above expressions imply that $F(u_i^-) \leq 0$ and $F(u_i^+) \geq 0$. □

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