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LETTER

Lagrangian phase transitions in nonequilibrium thermodynamic systems

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Abstract. In previous papers we have introduced a natural nonequilibrium free energy by considering the functional describing the large fluctuations of stationary nonequilibrium states. While in equilibrium this functional is always convex, in nonequilibrium this is not necessarily the case. We show that in nonequilibrium a new type of singularity can appear that is interpreted as a phase transition. In particular, this phenomenon occurs for the one-dimensional boundary driven weakly asymmetric exclusion process when the drift due to the external field is opposite to that due to the external reservoirs and is strong enough.

Keywords: stochastic particle dynamics (theory), stationary states, large deviations in non-equilibrium systems

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1. Introduction

Irreversible nonequilibrium phenomena have been central in statistical mechanics research in the last few decades. In the last ten years the authors have developed a new approach to nonequilibrium statistical mechanics inspired and supported by the analysis of stochastic lattice gases [1]–[4]. This theory is applicable to a wide class of thermodynamic systems where diffusion is the dominant mechanism. For example, as shown in [5, 6], this theory leads to the prediction of universality properties for current fluctuations. A basic ingredient of the theory is the so-called quasi-potential, a concept introduced in the analysis of stochastically perturbed dynamical systems [7], which provides a natural definition of a nonequilibrium thermodynamic potential.

In this letter we discuss the occurrence of singularities of the quasi-potential for nonequilibrium systems with infinitely many degrees of freedom. We analyze in detail the weakly asymmetric exclusion process and show analytically that, when the external field is strong, these singularities do appear. The singularities of the quasi-potential are interpreted as nonequilibrium phase transitions. Examples of phenomena of this kind in a finite-dimensional setting have been discussed in the literature and have also been observed in simulations [8]–[10]. The present work is the first example in which a thermodynamic model, that is a system with infinitely many degrees of freedom, is shown to exhibit such a singular behavior.

2. Macroscopic fluctuation theory

The dynamical macroscopic behavior of the system in a d-dimensional volume Λ is described by a nonlinear driven diffusion type equation of the form

$$\rho_t + \nabla \cdot \sigma(\rho) E = \nabla \cdot D(\rho) \nabla \rho, \tag{1}$$

where $\rho = \rho(t, x)$ represents the thermodynamic variable, e.g. the density, and ρ_t is its time derivative. The diffusion coefficient D and the mobility σ are $d \times d$ matrices and E denotes the external field. The transport coefficients D and σ satisfy the local Einstein

relation $D(\rho) = \sigma(\rho)s''(\rho)$ where s is the equilibrium free energy of the homogeneous system. Equation (1) has to be supplemented by the appropriate boundary conditions due to the interaction with the external reservoirs. We denote by $\bar\rho = \bar\rho(x)$ the stationary solution of (1).

The hydrodynamic equation (1) can be derived from an underlying microscopic dynamics through a suitable scaling limit. It represents the typical behavior, as the number N of degrees of freedom diverges, of the empirical density profile $\rho_N(t,x)$ defined as the average number of particles at time t in a macroscopic infinitesimal volume around x. The validity of the local Einstein relationship can then be deduced from the local microscopic detailed balance [11].

The probability that in the time interval $[T_1, T_2]$ the evolution of the variable ρ_N deviates from the solution of the hydrodynamic equation and is close to some trajectory ρ is exponentially small and of the form

$$P\left(\rho_N(t,x) \approx \rho(t,x)\right) \;\approx\; \mathrm{e}^{-NI_{[T_1,T_2]}(\rho)}, \tag{2}$$

where $I_{[T_1,T_2]}(\rho)$ is a functional which vanishes if ρ is a solution of (1). The functional $I_{[T_1,T_2]}(\rho)$ represents the energetic cost necessary for the system to follow the trajectory ρ. In the case of stochastic lattice gases, the expression of $I_{[T_1,T_2]}$ can be obtained from the microscopic dynamics as the large deviation rate functional [1]–[4], [12]. Following [13], we next sketch a purely macroscopic argument which yields the same conclusion. Consider a time dependent variation $F = F(t,x)$ of the external field so that the total applied field is $E + F$. Denote by ρ^F the corresponding solution of (1). By minimizing the energy dissipated by the field F with the constraint that ρ^F equals the prescribed path ρ we obtain that

$$I_{[T_1,T_2]}(\rho) = \tfrac{1}{4}\int_{T_1}^{T_2} \langle F\cdot\sigma(\rho)F\rangle\,\mathrm{d}t, \tag{3}$$

where $\langle\cdot\rangle$ is the integration over space and the optimal field is given by $F = 2\nabla H$, where H is the unique solution to the Poisson equation

$$-2\nabla\cdot\sigma(\rho)\nabla H = \rho_t + \nabla\cdot\sigma(\rho)E - \nabla\cdot D(\rho)\nabla\rho \tag{4}$$

which vanishes at the boundary of Λ for any $t\in[T_1,T_2]$.

The quasi-potential $V(\rho)$ is defined as the minimal cost to reach the density profile ρ starting from the stationary profile $\bar\rho$:

$$V(\rho) = \inf\{I_{(-\infty,0]}(\hat\rho),\, \hat\rho\colon\, \hat\rho(-\infty) = \bar\rho, \hat\rho(0) = \rho\}. \tag{5}$$

Therefore, while $I_{[T_1,T_2]}(\hat\rho)$ measures how much a path $\hat\rho$ is close to the solution of (1), the quasi-potential $V(\rho)$ measures how much a profile ρ is close to the stationary solution $\bar\rho$. Moreover, V is proportional to the total work done by the external field along the optimal time evolution to reach the density profile ρ [13]. In the context of nonequilibrium stationary states of stochastic lattice gases, the quasi-potential gives the asymptotics, as the number of degrees of freedom diverges, of the probability of observing a static fluctuation of the density: $\mu(\rho_N(x)\approx\rho(x)) \approx \mathrm{e}^{-NV(\rho)}$, where μ is the stationary state of the microscopic dynamics. This makes it natural to interpret V as a nonequilibrium free energy. In particular, for equilibrium systems μ has the standard Gibbs form and the quasi-potential coincides with the variation of the free energy.

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3. Hamiltonian picture

By considering the functional $I_{[T_1,T_2]}$ defined in (3) as an action functional, i.e. $I_{[T_1,T_2]}(\rho) = \int_{T_1}^{T_2} \mathbb{L}(\rho,\rho_t)\,\mathrm{d}t$ for a Lagrangian $\mathbb{L}(\rho,\rho_t)$ obtained by solving (4) and expressing the external field F in terms of ρ and ρ_t, the variational problem (5) can be viewed as the minimal action principle of classical mechanics. The corresponding Hamiltonian $\mathbb{H}$ is given by

$$\mathbb{H}(\rho,\pi) = \langle \nabla\pi \cdot \sigma(\rho)\nabla\pi\rangle + \langle \nabla\pi \cdot [\sigma(\rho)E - D(\rho)\nabla\rho]\rangle$$

where at the boundary of Λ the value of ρ is prescribed by the external reservoirs and the momentum π vanishes [2]. The canonical equations associated with the Hamiltonian $\mathbb{H}$ are

$$\begin{aligned} \rho_t + \nabla\cdot\sigma(\rho)E &= \nabla\cdot D(\rho)\nabla\rho - 2\nabla\cdot\sigma(\rho)\nabla\pi \\ \pi_t + E\cdot\sigma'(\rho)\nabla\pi &= -\nabla\pi\cdot\sigma'(\rho)\nabla\pi - D(\rho)\nabla\nabla\pi \end{aligned} \tag{6}$$

in this formula $D(\rho)\nabla\nabla\pi = \sum_{i,j} D_{i,j}(\rho)\partial^2_{x_i,x_j}\pi$.

Recalling that $\bar\rho$ is the stationary solution to (1), $(\bar\rho,0)$ is an equilibrium solution of (6) belonging to the zero energy manifold $\mathbb{H}(\rho,\pi) = 0$. Any solution $\rho(t)$ of the hydrodynamical equation (1) corresponds to a solution $(\rho(t),0)$ of the Hamilton equation (6) which converges, as $t \to +\infty$, to the equilibrium point $(\bar\rho,0)$ and the corresponding action vanishes. The set $\{(\rho,\pi)\colon \pi = 0\}$ is therefore the stable manifold $\mathcal{M}_\mathrm{s}$ associated to the equilibrium position $(\bar\rho,0)$. The unstable manifold $\mathcal{M}_\mathrm{u}$ is defined as the set of points (ρ,π) such that the solution of the canonical equations (6) starting from (ρ,π) converges to $(\bar\rho,0)$ as $t \to -\infty$. By the conservation of the energy, $\mathcal{M}_\mathrm{u}$ is also a subset of the zero energy manifold.

A basic result in Hamiltonian dynamics is the following [14]. Given a closed curve $\gamma = \{(\rho(\alpha),\pi(\alpha)), \alpha\in[0,1]\}$, the integral $\oint_\gamma \langle \pi\,\mathrm{d}\rho\rangle = \int_0^1 \langle \pi(\alpha)\rho_\alpha(\alpha)\rangle\,\mathrm{d}\alpha$ is invariant under the Hamiltonian evolution. This means that, by denoting with $\gamma(t)$ the evolution of γ under the Hamiltonian flow, $\oint_{\gamma(t)}\langle\pi\,\mathrm{d}\rho\rangle = \oint_\gamma\langle\pi\,\mathrm{d}\rho\rangle$. In view of this result, if γ is a closed curve contained in the unstable manifold $\mathcal{M}_\mathrm{u}$ then $\oint_\gamma\langle\pi\,\mathrm{d}\rho\rangle = \lim_{t\to-\infty}\oint_{\gamma(t)}\langle\pi\,\mathrm{d}\rho\rangle = 0$. We can therefore define the pre-potential $W\colon \mathcal{M}_\mathrm{u}\to\mathbb{R}$ by

$$W(\rho,\pi) = \int_\gamma \langle\hat\pi\,\mathrm{d}\hat\rho\rangle, \tag{7}$$

where the integral is carried over a path $\gamma = (\hat\rho,\hat\pi)$ in $\mathcal{M}_\mathrm{u}$ which connects $(\bar\rho,0)$ to (ρ,π). The possibility of defining such a potential is usually referred to by saying that $\mathcal{M}_\mathrm{u}$ is a Lagrangian manifold.

The relationship between the quasi-potential and the pre-potential is given by

$$V(\rho) = \inf\{W(\rho,\pi),\pi\colon (\rho,\pi)\in\mathcal{M}_\mathrm{u}\}. \tag{8}$$

Indeed, fix ρ and consider π such that (ρ,π) belongs to $\mathcal{M}_\mathrm{u}$. Let $(\hat\rho(t),\hat\pi(t))$ be the solution of the Hamilton equation (6) starting from (ρ,π) at $t=0$. Since $(\rho,\pi)\in\mathcal{M}_\mathrm{u}$, $(\hat\rho(t),\hat\pi(t))$ converges to $(\bar\rho,0)$ as $t\to-\infty$. Therefore, the path $\hat\rho(t)$ is a solution of the Euler–Lagrange equations for the action $I_{(-\infty,0]}$, which means that it is a critical path for (5). Since $\mathbb{L}(\hat\rho,\hat\rho_t) = \langle\hat\pi\hat\rho_t\rangle - \mathbb{H}(\hat\rho,\hat\pi)$ and $\mathbb{H}(\hat\rho(t),\hat\pi(t)) = 0$, the action of such a path $\hat\rho(t)$ is given by $I_{(-\infty,0]}(\hat\rho) = W(\rho,\pi)$. The right-hand side of (8) selects among all such paths the one with minimal action.

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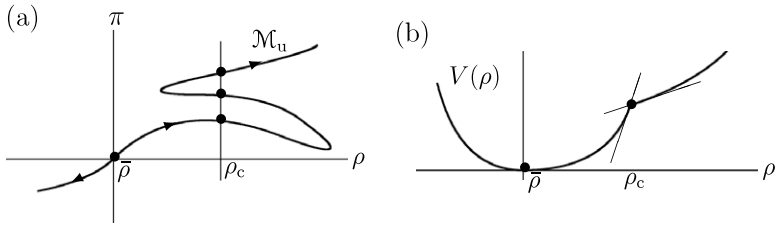

Figure 1. (a) Picture of the unstable manifold. (b) Graph of the quasi-potential. ρ_c is a caustic point.

In the neighborhood of the fixed point $(\bar\rho, 0)$, the unstable manifold $\mathcal{M}_u$ can be written as a graph, namely it has the form $\mathcal{M}_u = \{(\rho, \pi)\colon \pi = m_u(\rho)\}$ for some map m_u. In this case, the infimum on the right-hand side of (8) is trivial and $V(\rho) = W(\rho, m_u(\rho))$. In general, though, this is not true globally and it may happen, for special ρ, that the variational problem on the right-hand side of (8) admits more than a single minimizer (figure 1(a)). In this case there is also more than one minimizer for the variational problem (5). The set of profiles ρ for which the minimizer is not unique is called the caustic. In general, it is a codimension one submanifold of the configuration space. We call the occurrence of this situation a Lagrangian phase transition. In this case, profiles arbitrarily close to each other but lying on opposite sides of the caustic are reached by optimal paths which are not close to each other. This implies that on the caustics the first derivative of the quasi-potential is discontinuous (figure 1(b)). In particular, the occurrence of this phenomenon can be described as a first order phase transition. Of course, there exist also profiles for which the transition becomes of higher order.

Lagrangian phase transitions cannot occur in equilibrium. In this case the quasi-potential is in fact always convex, the unstable manifold is globally a graph, and the occurrence of a first order phase transition is due to a flat part in the quasi-potential. In contrast, in nonequilibrium systems the quasi-potential can be non-convex [15, 16] and Lagrangian phase transitions can arise when projecting the pre-potential W, which is a smooth function on the unstable manifold $\mathcal{M}_u$, onto the configuration space.

4. Microscopic model

We next show that a Lagrangian phase transition occurs in a simple nonequilibrium model, the one-dimensional asymmetric simple exclusion process on a lattice of N sites with open boundaries. Each site i/N, $1 \le i \le N$, is either empty or occupied by a single particle. Each particle independently attempts to jump to its right neighboring site with rate p and to its left neighboring site at rate q; we assume $p > q$. At the boundary sites particles are added and removed: a particle is added at site 1, when the site is empty, at rate ρ_0 and removed, when the site is occupied, at rate $1 - \rho_0$; similarly particles are added to site N at rate ρ_1 and removed at rate $1 - \rho_1$. The phase diagram of the model, corresponding to the typical behavior of the empirical density as $N \to \infty$, can be derived from an algebraic representation of the invariant measure [16, 17]. Such a phase diagram exhibits a phase coexistence when $0 < \rho_0 < \rho_1 < 1$ and $\rho_0 + \rho_1 = 1$. Note that the phase diagram can also be constructed just by looking at the entropic stationary solutions to the inviscid

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Burger's equation $\rho_t + (p-q)[\rho(1-\rho)]_x = 0$ with the boundary conditions $\rho(0) = \rho_0$, $\rho(1) = \rho_1$ [18].

We consider the weakly asymmetric exclusion process which is obtained by choosing $p-q = E/N$ with $E > 0$ and $q = 1$ [19]. We also assume $0 < \rho_0 < \rho_1 < 1$ so that there is a competition between the external field and the boundary conditions. With these choices, the hydrodynamic equation, obtained in the diffusive scaling limit, is (1) with Λ given by the interval $[0,1]$, $D = 1$, $\sigma = \rho(1-\rho)$, and boundary conditions $\rho(t,0) = \rho_0$, $\rho(t,1) = \rho_1$. The unique stationary solution of (1), denoted by $\bar\rho_E$, can be computed explicitly. In particular, in the weakly asymmetric regime the phase diagram does not exhibit any phase coexistence. The Einstein relation holds with $s(\rho) = \rho\log\rho + (1-\rho)\log(1-\rho)$.

To compute the quasi-potential we consider the Hamiltonian flow (6). It is convenient to perform the symplectic change of variables $\varphi = s'(\rho) - \pi$, $\psi = \rho$. In the new variables (φ,ψ) the Hamiltonian $\tilde{\mathbb{H}}(\varphi,\psi) = \mathbb{H}(\psi, s'(\psi) - \varphi)$ reads

$$\tilde{\mathbb{H}}(\varphi,\psi) = \langle \varphi_x \psi(1-\psi)\varphi_x\rangle - \langle [\psi_x + E\psi(1-\psi)]\varphi_x\rangle + E(\rho_1-\rho_0)$$

where we used $\rho(0) = \rho_0$, $\rho(1) = \rho_1$. The corresponding canonical equations are

$$\varphi_t = \varphi_{xx} - (1-2\psi)\varphi_x(E-\varphi_x) \qquad \psi_t = -\psi_{xx} - E[\psi(1-\psi)]_x + 2[\psi(1-\psi)\varphi_x]_x. \tag{9}$$

In the new variables the fixed point $(\bar\rho_E, 0)$ becomes $(s'(\bar\rho_E), \bar\rho_E)$. The associated stable manifold is $\{(\varphi,\psi)\colon \varphi = s'(\psi)\}$. We claim that the unstable manifold is given by

$$\mathcal{M}_{\mathrm{u}} = \left\{ (\varphi,\psi)\colon \psi = \frac{1}{1+\mathrm{e}^{\varphi}} - \frac{\varphi_{xx}}{\varphi_x(E-\varphi_x)}, \quad 0 < \varphi_x < E \right\}. \tag{10}$$

Indeed, pick a point $(\varphi,\psi) \in \mathcal{M}_{\mathrm{u}}$ and let $\hat\varphi$ be the solution to

$$\hat\varphi_t = -\hat\varphi_{xx} + \frac{1-\mathrm{e}^{\hat\varphi}}{1+\mathrm{e}^{\hat\varphi}}\hat\varphi_x(E-\hat\varphi_x)$$

with initial condition $\hat\varphi(0) = \varphi$. Set now

$$\hat\psi = \frac{1}{1+\mathrm{e}^{\hat\varphi}} - \frac{\hat\varphi_{xx}}{\hat\varphi_x(E-\hat\varphi_x)}$$

and observe that $\hat\psi(0) = \psi$ since $(\varphi,\psi) \in \mathcal{M}_{\mathrm{u}}$. Then, a tedious computation that we omit shows that $(\hat\varphi,\hat\psi)$ is a solution to the canonical equations (9) which converge to the fixed point $(s'(\bar\rho_E),\bar\rho_E)$ as $t \to -\infty$. Note that in the variables (φ,ψ) the unstable manifold $\mathcal{M}_{\mathrm{u}}$ is a graph.

The computation of the pre-potential is easily achieved in the new variables (φ,ψ). We start with the generating function of the symplectic transformation. Let $F(\rho,\varphi) = \int_0^1 [s(\rho) - \rho\varphi]\,\mathrm{d}x$ be the so-called free generating function [14, section 48], so that $\pi = \delta F/\delta\rho$, $\psi = -\delta F/\delta\varphi$. Equivalently, $\mathrm{d}F = \langle\pi,\mathrm{d}\rho\rangle - \langle\psi,\mathrm{d}\varphi\rangle$. Hence, for any path $\Gamma = \{\gamma(t), t\in[0,1]\}$ in the phase space

$$\int_\Gamma \langle\pi,\mathrm{d}\rho\rangle = \int_\Gamma \langle\psi,\mathrm{d}\varphi\rangle + F(\gamma(1)) - F(\gamma(0)).$$

Assume now that $\Gamma \subset \mathcal{M}_{\mathrm{u}}$. By (10), we have that

$$\int_\Gamma \langle\psi,\mathrm{d}\varphi\rangle = \int_0^1 [\varphi(t) - \log(1+\mathrm{e}^{\varphi(t)}) + s(\varphi_x(t)/E)]\,\mathrm{d}x\big|_{t=0}^{t=1}.$$

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Therefore, if we define $\mathcal{G}_E$ by

$$\mathcal{G}_E(\rho,\varphi)=\int_0^1 [s(\rho)+s(\varphi_x/E)+(1-\rho)\varphi-\log(1+\mathrm{e}^{\varphi})]\,\mathrm{d}x,$$

the previous identities imply that

$$\int_\Gamma \langle \pi,\mathrm{d}\rho\rangle = \mathcal{G}_E(\rho(1),\varphi(1))-\mathcal{G}_E(\rho(0),\varphi(0)).$$

Hence, by (7), $W_E(\rho,\pi)=\mathcal{G}_E(\rho,\varphi)-\mathcal{G}_E(\bar\rho_E,s'(\bar\rho_E))$, where $(\varphi,\rho)\in\mathcal{M}_{\mathrm{u}}$. Therefore,

$$V_E(\rho)=\inf\{\mathcal{G}_E(\rho,\varphi),\varphi\colon(\varphi,\rho)\in\mathcal{M}_{\mathrm{u}}\}-\mathcal{G}_E(\bar\rho_E,s'(\bar\rho_E)). \tag{11}$$

In the previous formula the condition that $(\varphi,\rho)\in\mathcal{M}_{\mathrm{u}}$ can be dropped since it is equivalent to the condition that φ is a critical point of $\mathcal{G}_E(\rho,\varphi)$.

A similar formula for the quasi-potential V_E in the case $\rho_0>\rho_1$ has been obtained in [19] by combinatorial techniques. An analogous expression for the quasi-potential in terms of a trial functional like $\mathcal{G}_E$ appeared in [12, 15, 16, 20]. However, its intrinsic significance in terms of the Hamilton structure behind the variational problem (5) is new and answers a question raised in [20]. In particular, equation (2) in [20] characterizes the unstable manifold.

5. Lagrangian phase transitions

We next show that, when $\rho_0<\rho_1$ and the external field E is large enough, the weakly asymmetric exclusion process exhibits Lagrangian phase transitions. This is not the case when the external field and the reservoirs push in the same direction. We refer to [21] for the mathematical details.

We start by arguing that, when E is not large, Lagrangian phase transitions do not occur. Let $\varphi_i=s'(\rho_i)=\log[\rho_i/(1-\rho_i)]$, $i=0,1$, be the chemical potentials associated to the boundary reservoirs. When $E=E_0=\varphi_1-\varphi_0$ there is no current and the microscopic dynamics satisfies the detailed balance. Therefore, in this case, the unstable manifold is globally a graph and there exists a unique minimizing path for (5). By perturbing around equilibrium, this is still the case when E is close to E_0.

Consider now the limiting case $E=\infty$ which corresponds to the asymmetric simple exclusion process examined in [16]. In this singular limit the hydrodynamic equation (1) becomes the inviscid Burger's equation and shocks are possible. In this limit the functional $\mathcal{G}_E$ becomes

$$\mathcal{G}(\rho,\varphi)=\int_0^1 [s(\rho)+(1-\rho)\varphi-\log(1+\mathrm{e}^{\varphi})]\,\mathrm{d}x.$$

Since $\mathcal{G}$ is a concave functional of φ, the minimum of $\mathcal{G}(\rho,\varphi)$ is attained when φ is at the boundary of the function space. Since $\varphi(0)=\varphi_0$, $\varphi(1)=\varphi_1$ and φ is increasing, the boundary of the function space is given by the step functions $\varphi^{(y)}(x)=\varphi_0+(\varphi_1-\varphi_0)\vartheta(x-y)$, $y\in[0,1]$, where ϑ is the Heaviside function. The profile $\varphi^{(y)}$ jumps from φ_0 to φ_1 at y. The variational problem for V is therefore reduced to the one-dimensional problem

$$\min_{y\in[0,1]}\int_0^1 [s(\rho)+(1-\rho)\varphi^{(y)}-\log(1+\mathrm{e}^{\varphi^{(y)}})]\,\mathrm{d}x \tag{12}$$

which is equivalent to the expression derived in [16].

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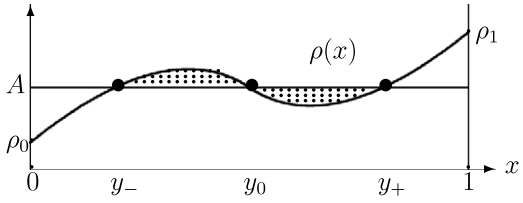

Figure 2. Graph of a caustic density profile for $E = \infty$. The shaded regions have equal area.

It is not difficult to show that, if the density profile ρ is suitably chosen, (12) admits two minimizers. Let

$$A = 1 - \frac{\log(1 + \mathrm{e}^{\varphi_1}) - \log(1 + \mathrm{e}^{\varphi_0})}{\varphi_1 - \varphi_0}$$

and fix a density profile ρ: $[0, 1] \to [0, 1]$ satisfying the following conditions, see figure 2. There exist $0 < y_- < y_0 < y_+ < 1$ such that: $\rho(y_0) = \rho(y_\pm) = A$, $\rho(x) < A$ for $x \in [0, y_-) \cup (y_0, y_+)$, $\rho(x) > A$ for $x \in (y_-, y_0) \cup (y_+, 1)$, and ρ satisfies $\int_{y_-}^{y_+} \rho \, \mathrm{d}x = A(y_+ - y_-)$. It is simple to check that there are two global minimizers for the variational problem (12), which are given by $y_\pm$.

We finally argue that the occurrence of Lagrangian phase transitions persists when the external field E is large. If we consider the density profile $\rho(x)$ in figure 2, $\mathcal{G}_E(\rho, \varphi)$, as a functional of φ, will have two local minima close to $\varphi^{(y_\pm)}$ and only one of them is the global minimizer. However we can modify, depending on E, the density profile ρ in such a way that the two local minima are brought back at the same level. In view of (10), two optimal paths for the variational problem (5) can be constructed by the following algorithm. Given the density profile $\rho(x)$, let $\varphi^\pm(x)$ be two minimizers for the variational problem (11) and set $F_0^\pm = \mathrm{e}^{\varphi^\pm}/(1 + \mathrm{e}^{\varphi^\pm})$. Denote by $F^\pm = F^\pm(t, x)$ the solution of the viscous Burger's equation $F_t + E(F(1-F))_x = F_{xx}$ with boundary conditions $F(t, 0) = \rho_0$, $F(t, 1) = \rho_1$ and initial condition $F(0, x) = F_0^\pm(x)$. Set $u^\pm = s'(F^\pm)$ and define $v^\pm$ by

$$v^\pm = \frac{1}{1 + \mathrm{e}^{u^\pm}} - \frac{u_{xx}^\pm}{u_x^\pm (E - u_x^\pm)}.$$

Then $v^\pm(0) = \rho$ and $v^\pm(t)$ converges to $\bar{\rho}_E$ as $t \to +\infty$. The paths $v^\pm$ reversed in time are two optimal paths for the variational problem (5).

6. Discussion

We conclude with some remarks on the possibility of observing Lagrangian phase transitions. In noisy electronic devices with a finite number of degrees of freedom, optimal paths have been experimentally observed [22, 23]. In Langevin equations with noise, Lagrangian singularities have been observed in simulations [9, 10]. In this letter we have shown analytically that they occur in a simple model with infinitely many degrees of freedom. In thermodynamic systems the thermal fluctuations are very small and the direct observation of Lagrangian phase transitions does not appear feasible, as it would require an extremely long time. On the other hand, the problem of large fluctuations admits an interpretation as a control problem [24]. This means that rather than considering

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the optimal path, we look for the field driving the system from the stationary state to a chosen profile with the minimal energetic cost. The Lagrangian phase transition then corresponds to the existence of two different optimal fields dissipating the same energy. In principle, these two fields can be theoretically calculated and an experiment can be designed to check the predictions.

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