SOLVABLE VARIATIONAL PROBLEMS IN NON EQUILIBRIUM STATISTICAL MECHANICS

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The general Framework

Non equilibrium models of statistical mechanics

Presence of fluxes (matter, energy,...) in the stationary state

Non reversible stochastic models

Large deviations analysis $\Longrightarrow$ Variational problems

Large deviations

X_N sequence of random variables taking values on a metric space M satisfies a Large deviations Principle (LDP) with speed $\alpha(N)$ and rate $I : M \to \mathbb{R}^+ \cup \{+\infty\}$ if

$$\limsup_{N \to +\infty} \frac{1}{\alpha(N)} \log \mathbb{P}\left(X_N \in C\right) \leq - \inf_{x \in C} I(x)\,, \qquad \forall\, C \text{ closed}$$

$$\liminf_{N \to +\infty} \frac{1}{\alpha(N)} \log \mathbb{P}\left(X_N \in O\right) \geq - \inf_{x \in O} I(x)\,, \qquad \forall\, O \text{ open}$$

$$\boxed{\mathbb{P}\Big(X_N \sim x\Big) \simeq e^{-\alpha(N) I(x)}}$$

Examples

- $X_N \in \mathbb{R}^d$ Gaussian

$$\frac{1}{Z_N} e^{-\frac{N}{2}\left(x, C^{-1}x\right)} dx$$

satisfies a LDP with speed N and rate

$$I(x) = \frac{1}{2}\left(x, C^{-1}x\right)$$

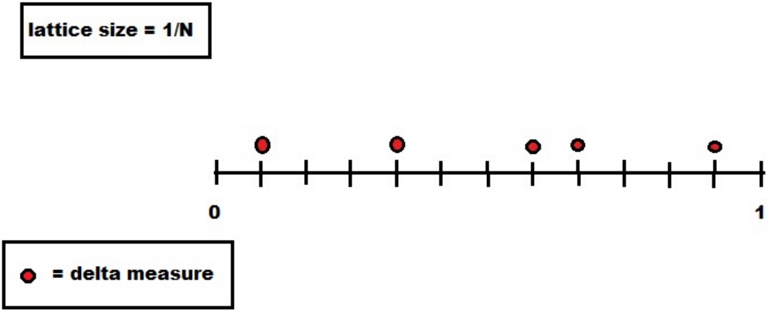
- A gas of independent coins

$$\eta = \Big(\eta(1), \ldots, \eta(N)\Big) \in \{0,1\}^N \, , \qquad \text{i.i.d. Bernoulli}$$

Examples

EMPIRICAL MEASURE

$$\eta \Longrightarrow \pi_N(\eta) := \frac{1}{N}\sum_{i=1}^{N} \eta(i)\delta_{\frac{i}{N}}$$



LD for the empirical measure

$\pi_N(\eta) \in \mathcal{M}^+([0,1])$, random positive measure, satisfies LDP

$$\mathbb{P}\big(\pi_N(\eta) \sim \rho(x)dx\big) \simeq e^{-NI(\rho)}$$

the rate function is

$$\boxed{I(\rho) = \int_0^1 h_m\big(\rho(x)\big)\, d\, x}$$

where

$$h_m(\rho) = \rho \log \frac{\rho}{m} + (1-\rho) \log \frac{1-\rho}{1-m}$$

Contraction

Average number of coins

$$\frac{1}{N}\sum_i \eta(i) = \int_{[0.1]} d\pi_N(\eta)$$

satisfies LDP

$$\mathbb{P}\left(\frac{1}{N}\sum_i \eta(i) \sim y\right) \simeq e^{-NJ(y)}$$

BY CONTRACTION

$$\boxed{J(y) = \inf_{\left\{\rho\,:\,\int_0^1 \rho(x)\,dx = y\right\}} I(\rho)}$$

A simple non equilibrium model

CONTINUOUS TIME MARKOV CHAINS WITH
"EXPONENTIALLY SMALL" RATES OF TRANSITION

V finite set $x, y, z \in V$; $\quad N \sim \frac{1}{T}$, T =temperature

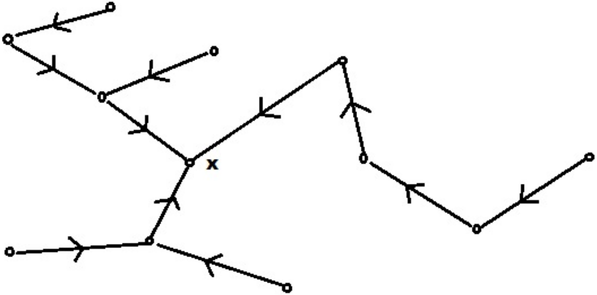
TRANSITION RATES

$$r_N(y,z) = e^{-NR(y,z)}$$

INVARIANT MEASURE

$$\mu_N(x) = \frac{\sum_{\mathcal{G}\in\mathbb{G}_x} \prod_{(y,z)\in\mathcal{G}} r_N(y,z)}{Z_N}$$

The graphs in $\mathbb{G}_x$



A simple non equilibrium model

$$\mu_N(x) \simeq e^{-NW(x)}$$

$$\boxed{W(x) = \min_{\mathcal{G}\in\mathbb{G}_x} \left\{ \sum_{(y,z)\in\mathcal{G}} R(y,z) \right\} + \text{constant}}$$

The reversible case (Example)

$$R(y,z) = \frac{H(z) - H(y)}{2}$$

Combinatorial optimization problem is solved (tricky)

$$W = H + \text{constant}$$

From dynamic to static LD

Stochastic differential equation in $\mathbb{R}^d$

$$dX^\epsilon(t) = b\left(X^\epsilon(t)\right) dt + \sqrt{\epsilon} dW(t)$$

Dynamic sample path LD

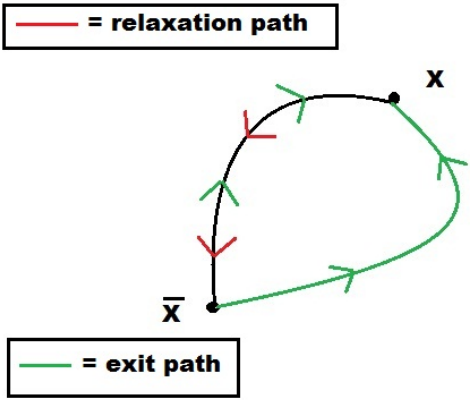
$$\mathbb{P}\left(X^\epsilon(t) \sim x(t)\,, t \in [T_1, T_2]\right) \simeq e^{-\frac{\epsilon^{-1}}{2} \int_{T_1}^{T_2} |\dot{x}-b(x)|^2 dt}$$

$$\begin{cases} b = \text{ globally attractive vector field} \\ \bar{x} \text{ such that } b\left(\bar{x}\right) = 0 \text{ unique equilibrium point} \end{cases}$$

Invariant measure μ^ϵ satisfies LD with rate the QUASIPOTENTIAL

$$V(x) = \inf_{T>0} \inf_{\{x(t)\,:\,x(-T)=\bar{x}\,,x(0)=x\}} I_{[-T,0]}(x(t))$$

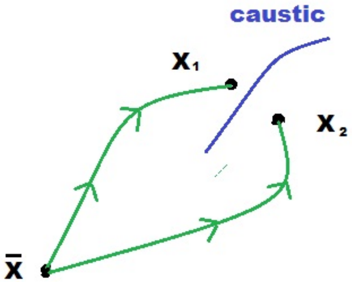
The quasipotential



The quasipotential

$$\boxed{b = -\nabla S \Longleftrightarrow \text{reversible} \quad \Longrightarrow \quad V = 2S}$$

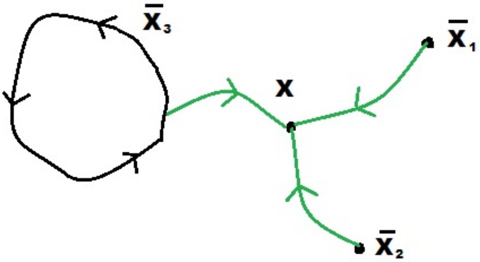
In general V not differentiable (phase transitions, WASEP)



The general case

If b has several stable attractors the quasipotential becomes

$$\boxed{V(x) = \inf_i W_i + V_i(x)}$$



A solvable case

A. Faggionato, D.G. (2012)

Stochastic differential equation on $\mathbb{S}^1$ (unit circle)

$$dX^\epsilon(t) = b\left(X^\epsilon(t)\right) dt + \sqrt{\epsilon} dW(t)$$

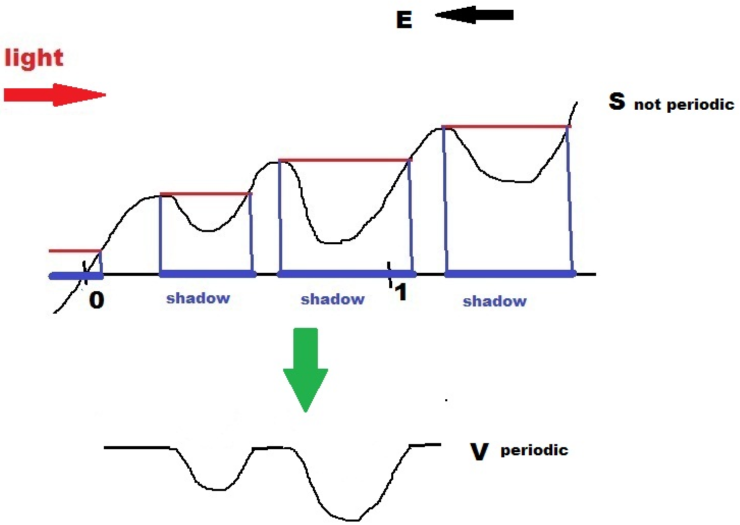
b is periodic of period one

$$S(x) := -2 \int_0^x b(y) dy$$

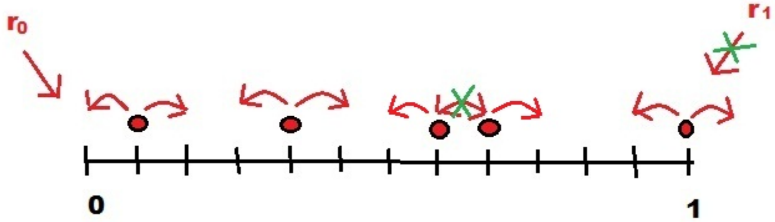
S periodic $\Longleftrightarrow \int_0^1 b(y)dy = 0 \Longleftrightarrow$ reversible $\Longrightarrow V = S$

Add an external field $b \Longrightarrow b + E$

Sunshine transformation



Boundary driven 1-d simple exclusion



Scaling limit

Diffusive rescaling

$$L_N \Longrightarrow N^2 L_N$$

Hydrodynamic scaling limit

$$\pi_N(\eta_t) \overset{N\to+\infty}{\to} \rho(x,t)dx$$

$$\begin{cases} \rho_t = \Delta\rho \\ \rho(0,t) = \rho_{r_0} \\ \rho(1,t) = \rho_{r_1} \end{cases}$$

Large deviations and quasipotential

BDGJL (2002)

DYNAMIC LARGE DEVIATIONS

$$\mathbb{P}\Big(\pi_N(\eta_t) \sim \rho(x,t)dx\ ,\ t \in [T_1, T_2]\Big) \simeq e^{-N I_{[T_1,T_2]}(\rho)}$$

$$\begin{cases} I_{[T_1,T_2]}(\rho) = \frac{1}{4}\int_{T_1}^{T_2} dt \int_0^1 dx\, \rho(1-\rho)\,(\nabla H)^2 \\ \partial_t \rho = \Delta\rho - \nabla\cdot\left(\rho(1-\rho)\nabla H\right)\ , \qquad H(0,t) = H(1,t) = 0 \end{cases}$$

THE QUASIPOTENTIAL

$r_0 = r_1 \implies$ reversible , gas of independent coins

$r_0 \neq r_1 \implies$ not reversible, long range correlations

$$\begin{cases} V(\rho) = \int_0^1 \left[h_f(\rho) + \log\frac{\nabla f}{\nabla \bar\rho}\right] d\,x \\ \frac{\Delta f}{(\nabla f)^2} f(1-f) + f = \rho\,, \qquad f(0) = \rho_{r_0}\ ,\ f(1) = \rho_{r_1} \end{cases}$$

The minimization path

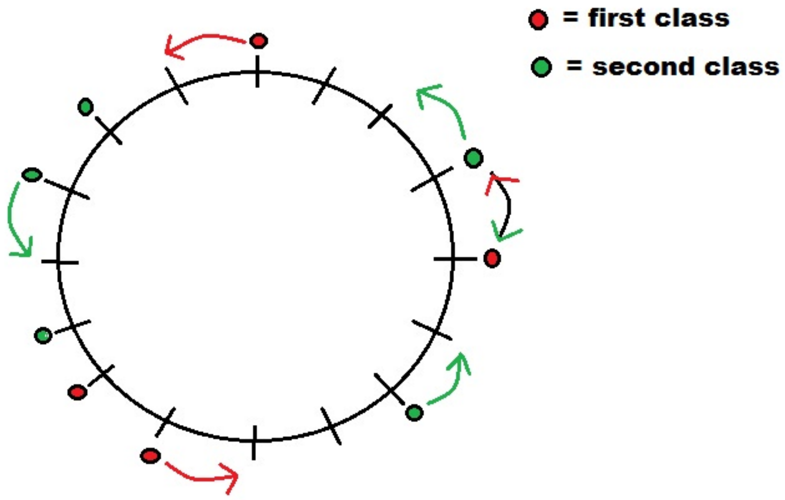
The minimizer for $V(\rho_0)$ is the time reversal of the following coupled differential problem

$$\begin{cases} \partial_t \rho = \Delta\rho - \nabla \cdot \left(\frac{\rho(1-\rho)}{f(1-f)} \nabla f \right) \\ \rho(u,0) = \rho_0(u) \\ f(1-f)\frac{\Delta f}{(\nabla f)^2} + f = \rho \\ \text{and b.c.} \end{cases}$$

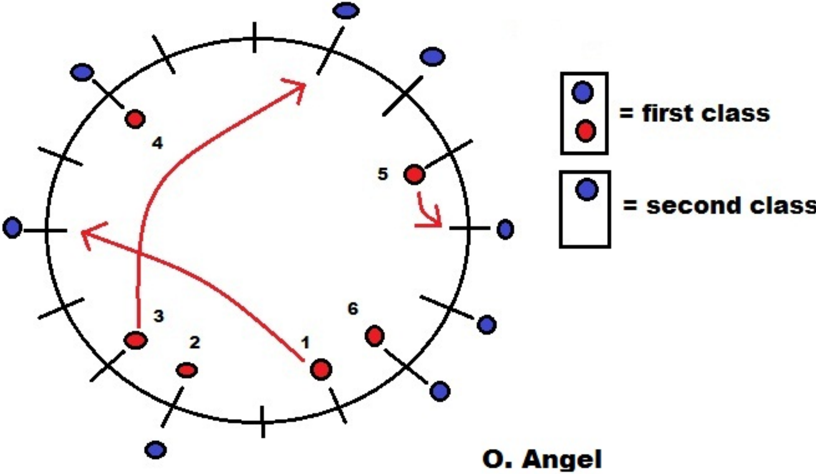
A computation (magic transformation!) shows it is equivalent to

$$\begin{cases} \partial_t f = \Delta f \\ f(1-f)\frac{\Delta f}{(\nabla f)^2} + f = \rho \\ \rho(u,0) = \rho_0(u) \\ \text{and b.c.} \end{cases}$$

2-class TASEP



The invariant measure



Collapsing particles

$$(\tilde{\eta}_1, \tilde{\eta}_T) \,:\, \sum_x \tilde{\eta}_1(x) \leq \sum_x \tilde{\eta}_T(x) \implies (\eta_1, \eta_T) = \mathcal{C}\big[\,(\tilde{\eta}_1, \tilde{\eta}_T)\big)\,\big]$$

Flux across bond $(x, x+1)$

$$J(x) = \sup_y \Big[\sum_{z\in[y,x]} \tilde{\eta}_1(z) - \tilde{\eta}_T(z)\Big]_+$$

Collapsing measures

D.G. (08)

$$(\tilde\rho_1, \tilde\rho_T)) \ : \ \int_{\mathbb{S}^1} d\tilde\rho_1 \leq \int_{\mathbb{S}^1} d\tilde\rho_T \implies (\rho_1, \rho_T) = \mathcal{C}\left[\,(\tilde\rho_1, \tilde\rho_T))\,\right]$$

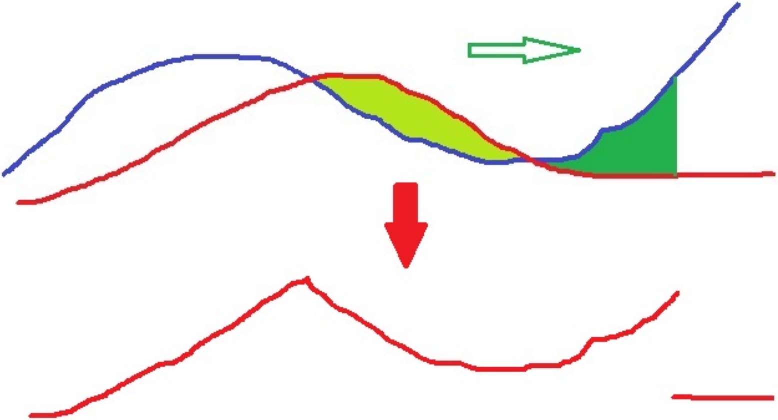
Definition

$$\boxed{\int_{(a,b]} d\rho_1 = \int_{(a,b]} d\tilde\rho_1 + J(a) - J(b)}$$

where

$$J(x) := \sup_y \left[\int_{(y,x]} d\tilde\rho_1 - \int_{(y,x]} d\tilde\rho_2\right]_+$$

Collapsing measures



Large deviations

LD for the $(\tilde{\eta}_1, \tilde{\eta}_T)$ variables

$$\tilde{V}(\tilde{\rho}_1, \tilde{\rho}_T) = \int_{\mathbb{S}^1} \left[h_{m_1}(\tilde{\rho}_1) + h_{m_2}(\tilde{\rho}_T)\right] dx$$

LD for the SNS (not convex!)

$$\begin{aligned} V(\rho_1, \rho_T) &= \inf_{\{(\tilde{\rho}_1, \tilde{\rho}_T) : \mathcal{C}[(\tilde{\rho}_1, \tilde{\rho}_T)] = (\rho_1, \rho_T)\}} \tilde{V}(\tilde{\rho}_1, \tilde{\rho}_T) \\ &= \int_{\mathbb{S}^1} \left[h_{m_1}(\hat{\rho}_1) + h_{m_2}(\rho_T)\right] dx \end{aligned}$$

On any (a, b) where $\rho_1 = \rho_T$

$$\int_a^x \hat{\rho}_1(y) dy = CE\left[\int_a^x \rho_1(y) dy\right]$$