INTERACTING PARTICLE SYSTEMS OUT OF EQUILIBRIUM

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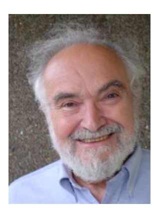
(EURANDOM 2011)
joint with







Continuous time Markov chains

State space: configurations of particles

$\eta =$ configuration, $\eta \in M^{\Lambda_N} = X_N$, $N \in \mathbb{N}$ is a parameter

$\Lambda_N \subseteq \mathbb{Z}^d$ or $\Lambda_N = (\mathbb{Z}/N\mathbb{Z})^d = \mathbb{T}_N^d =$ d-dimensional discrete torus

$M \subseteq \mathbb{N} \bigcup \{0\}$, ex. $M = \{0, 1\}$ (exclusion rule)

$\eta_x = k \Leftrightarrow$ there are k particles at site $x \in \Lambda_N$

Stochastic evolution: $\eta(t) =$ configuration of particles at time t.

Generator:

$$L_N f(\eta) = \sum_{\eta'} c(\eta, \eta') \Big(f(\eta') - f(\eta) \Big)$$

Particles are indistinguishable

Out of equilibrium $\Leftrightarrow$ not reversible

$\mathbb{P}_\eta(\cdot) =$ probability measures induced on $D([0, T], X_N)$ with initial condition η at time 0. When the initial condition is distributed according to ν we call it $\mathbb{P}_\nu(\cdot)$

$\mu_N =$ invariant measure

$$\mathbb{P}_{\mu_N}\Big(\eta(t) = \eta\Big) = \mu_N(\eta) \qquad \forall t$$

Main problems:

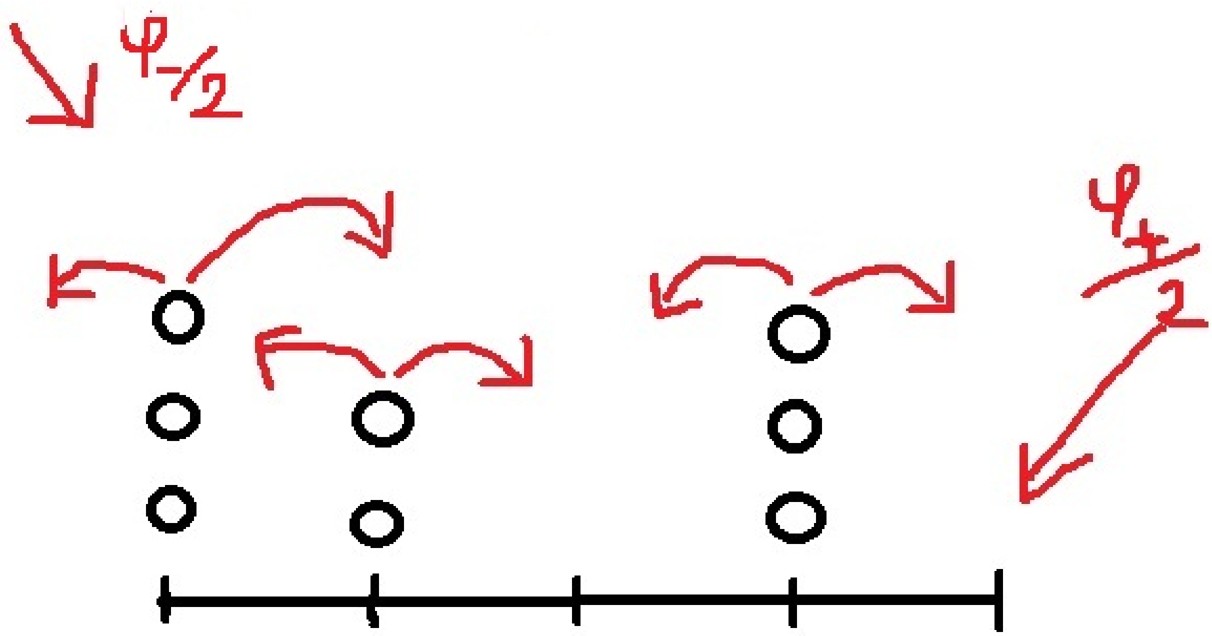
determine μ_N

determine its asymptotic behavior when $N \to +\infty$ in a large deviations regime.

FIRST EXAMPLE

1-d boundary driven ZERO RANGE

$$\Lambda_N = \{1, 2, \ldots N\}\ , \qquad \eta_x \in \mathbb{N}$$



dynamics on the bulk

$g : \mathbb{N} \bigcup \{0\} \to \mathbb{R}^+$ such that $g(0) = 0$ + suitable growth conditions

$$\eta \to \eta - \delta_x + \delta_{x\pm 1} \qquad \text{with rate} \qquad \frac{g(\eta_x)}{2}$$

$$\text{Reversibility} \Leftrightarrow \qquad \varphi_- = \varphi_+$$

Always product invariant measures

$$\mu_N(\eta) = \prod_{x=1}^{N} \frac{\varphi_x^{\eta_x}}{Z(\varphi_x) g(\eta_x)!}$$

where $g(k)! := g(k)g(k-1)\ldots g(2)g(1)$ and φ solves

$$\begin{cases} \varphi_x = \frac{1}{2}\Big(\varphi_{x-1} + \varphi_{x+1}\Big) \\ \varphi_0 = \varphi_-\,,\ \varphi_{N+1} = \varphi_+ \end{cases}$$

The equation is equivalent to

$$\frac{1}{2}\Big(\varphi_x - \varphi_{x+1}\Big) = \text{constant} = j$$

When N is large $\varphi_x \sim \Phi\left(\frac{x}{N}\right)$, where

$$\begin{cases} \Delta\Phi = 0 \\ \Phi(0) = \varphi_- \,, \; \Phi(1) = \varphi_+ \end{cases}$$

Product invariant measure is very general for Z.R.: for example it holds in any dimension and/or when

$$\eta \to \eta - \delta_x + \delta_{x+1} \qquad \text{with rate } g(\eta_x) r_x$$

$$\eta \to \eta - \delta_x + \delta_{x-1} \qquad \text{with rate } g(\eta_x) l_x$$

LARGE DEVIATIONS

$\{\nu_n\}_{n\in\mathbb{N}}$ sequence of probability measures on X a Polish metric space satisfies a LDP with rate I if

$$\limsup_{n\to+\infty}\frac{1}{n}\log\nu_n(C)\leq-\inf_{y\in C}I(y)\,,\qquad C \text{ closed}$$

$$\liminf_{n\to+\infty}\frac{1}{n}\log\nu_n(O)\geq-\inf_{y\in O}I(y)\,,\qquad O \text{ open}$$

A random element X_n satisfies a LDP if the corresponding law satisfies a LDP

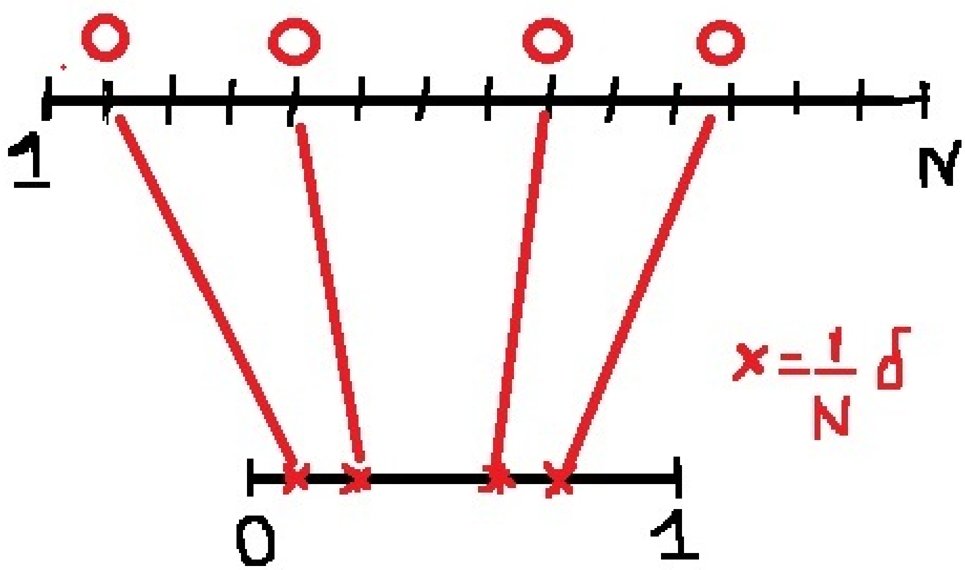
$$\mathbb{P}(X_n\sim a)\simeq e^{-nI(a)}$$

A typical situation is $\nu_n\stackrel{w}{\to}\delta_{\overline{x}}$, and $I(x)\geq 0$ with $I(x)=0\Leftrightarrow x=\overline{x}$

LD asymptotic for Zero Range

$\eta \to \pi_N(\eta) =$ Empirical measure

$$\pi_N(\eta) := \frac{1}{N}\sum_{x=1}^{N} \eta_x \delta_{\frac{x}{N}} \ \in \mathcal{M}^+([0,1])$$



(Coarse graining)

When η is distributed according to μ_N then $\pi_N(\eta)$ satisfies a LDP on $\mathcal{M}^+([0,1])$ endowed with the weak topology

Gärtner-Ellis $\Rightarrow$ rate function S

$$P(f) := \lim_{N \to +\infty} \frac{1}{N} \log \mathbb{E}_{\mu_N} \left(e^{N \int_0^1 f d\pi_N(\eta)} \right)$$

$$S(\rho) = \sup_f \left\{ \int_{[0,1]} f d\rho - P(f) \right\}$$

in particular is convex

$$
\begin{aligned}
P(f) &= \lim_{N\to+\infty} \frac{1}{N} \log \mathbb{E}_{\mu_N}\left(e^{N\int_0^1 f d\pi_N(\eta)}\right) \\
&= \lim_{N\to+\infty} \frac{1}{N} \log \prod_{x=1}^{N} \mathbb{E}_{\mu_N}\left(e^{f\left(\frac{x}{N}\right)\eta_x}\right) \\
&= \lim_{N\to+\infty} \frac{1}{N} \sum_{x=1}^{N} \log \frac{Z\left(\varphi_x e^{f\left(\frac{x}{N}\right)}\right)}{Z(\varphi_x)} \\
&= \int_0^1 \log \frac{Z\left(\Phi(u) e^{f(u)}\right)}{Z(\Phi(u))} du
\end{aligned}
$$

and then

$$
S(\rho) = \int_0^1 \rho(u) \log \frac{\psi(\rho(u))}{\Phi(u)} - \log \frac{Z(\psi(\rho(u)))}{Z(\Phi(u))} du
$$

if $\rho = \rho(u)du$ and $S(\rho) = +\infty$ otherwise. This means

$$
S(\rho) = \int_0^1 s(\rho(u), u) du
$$

LDP for Gibbs measures

(Comets, Föllmer, Lanford, Olla,...) $\Lambda_N = \mathbb{T}_N$, finite range interactions

$$\mu_N^\lambda(\eta) = \frac{e^{-H_\lambda(\eta)}}{Z_N(\lambda)}$$

$$H_\lambda(\eta) = \lambda \sum_{x \in \mathbb{T}_N} \eta_x + J \sum_{x \in \mathbb{T}_N} \eta_x \eta_{x+1}$$

Pressure

$$P(\lambda) := \lim_{N \to +\infty} \frac{1}{N} \log Z_N(\lambda)$$

Gärtner-Ellis

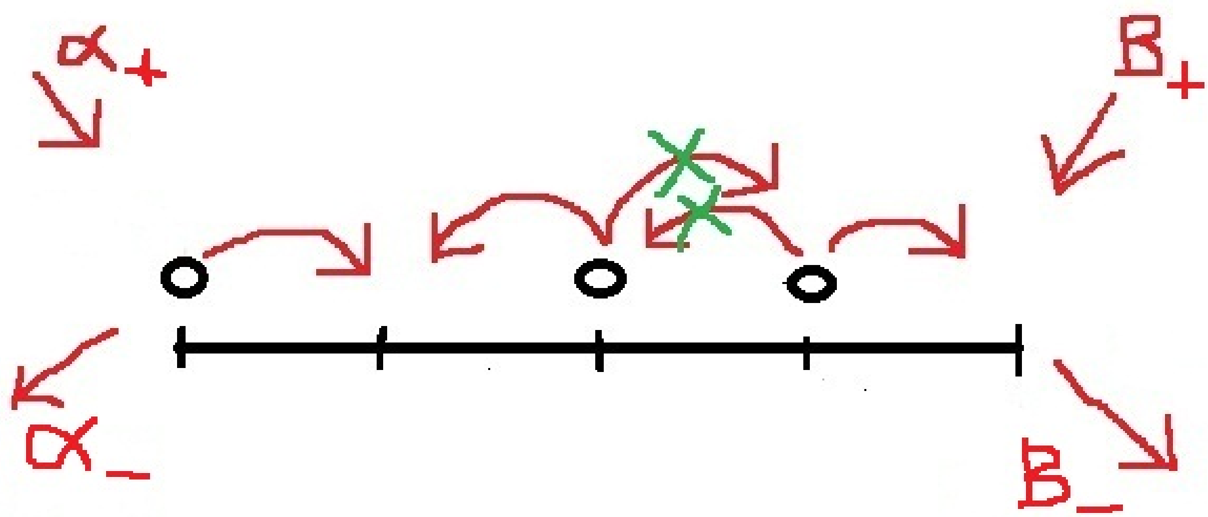
$$s_{\lambda_0}(\alpha) = \sup_\lambda \left\{ \alpha\lambda - \Big(P(\lambda + \lambda_0) - P(\lambda_0) \Big) \right\}$$

convex. Then $\pi_N(\eta)$, when η is distributed according to $\mu_N^{\lambda_0}$, satisfies LDP with rate

$$S_{\lambda_0}(\rho) = \begin{cases} \int_0^1 s_{\lambda_0}(\rho(u))du & \rho = \rho(u)du \\ +\infty & \text{otherwise} \end{cases}$$

Second model

1-d boundary driven SEP



In the bulk exchange dynamics

$\eta \to \eta^{x,x+1}$ with rate 1, $\qquad x = 1, \dots, N-1$

where

$$\eta_z^{x,x+1} = \begin{cases} \eta_z & \text{if } z \neq x, x+1 \\ \eta_x & \text{if } z = x+1 \\ \eta_{x+1} & \text{if} z = x \end{cases}$$

Reversibility : $\alpha_+ = \beta_+$ and $\alpha_- = \beta_-$. In this case product invariant measure

$$\mu_N(\eta) = \prod_{x=1}^{N} p^{\eta_x}(1-p)^{1-\eta_x}$$

where $p = \frac{\alpha_+}{\alpha_+ + \alpha_-} = \frac{\beta_+}{\beta_+ + \beta_-}$. A special case: $\alpha_+ = \beta_- = 0$, (Kingmann 1969!!)

$$\mu_N(\eta_{x_1} = 1, \dots \eta_{x_k} = 1) =$$

$$\frac{(A-k-x_1)(A-k+1-x_2)\dots(A-1-x_k)}{(B-k)(B-k+1)\dots(B-1)}$$

where $1 \leq x_1 < x_2 < \cdots < x_k \leq N$ and $A = N + 1 + \frac{1}{\beta_+}$, $B = 1 + \frac{1}{\alpha_-} + \frac{1}{\beta_+}$

from now on $\alpha_+ = (1 - \alpha_-) = \alpha$ and $\beta_+ = (1 - \beta_-) = \beta$, $\alpha, \beta \in [0, 1]$

Duality

$$\mu_N(\eta_{x_1} = 1, \ldots \eta_{x_k} = 1) =$$

$$\sum_{j=0}^{k} \mathbb{P}\left(x_1 \ldots x_k \stackrel{\text{left}}{\to} j\right) (\alpha)^j (\beta)^{k-j}$$

where $P\left(x_1 \ldots x_k \stackrel{\text{left}}{\to} j\right)$ denotes the probability that of k particles starting at $1 \leq x_1 < x_2 < \cdots < x_k \leq N$ and evolving in exclusion, j will be absorbed at 0 and $k - j$ at $N + 1$.

Long range correlations

Closed expression for 1-marginals solving

$$\mu_N(L\eta_x) = 0\,, x = 1, \ldots, N$$

$$\mu_N(\eta_x) = \overline{\rho}\left(\frac{x}{N}\right)$$

where $\overline{\rho}(u) = \alpha + u(\beta - \alpha)$

Closed expression for correlation functions (Spohn 1983) solving $\mu_N\Big(L(\eta_x\eta_y)\Big) = 0$.

For $1 \le x < y \le N$

$$\begin{aligned} C_N(x,y) :&= \mu_N(\eta_x\eta_y) - \mu_N(\eta_x)\mu_N(\eta_y) = \\ &= \frac{(\alpha-\beta)^2}{N-1}\Delta^{-1}\left(\frac{x}{N},\frac{y}{N}\right) \end{aligned}$$

where $\Delta^{-1}(u,v)$ is the Green function of the Laplacian on $[0,1]$ with Dirichelet boundary conditions

$$\Delta^{-1}(u,v) = -u(1-v) \qquad 0 \le u \le v \le 1$$

In particular negatively correlated.

Matrix representation

(Derrida, Scütz,...)

$$\mu_N(\eta) = \frac{w_N(\eta)}{\sum_{\eta'} w_N(\eta')}$$

where

$$w_N(\eta) = \langle l| \prod_{x=1}^{N} (D\eta_x + E(1-\eta_x))|r\rangle$$

where the matrices E, D and the vectors $\langle l|, |r\rangle$ satisfy

$$\frac{1}{2}(DE - ED) = D + E$$

$$\frac{1}{2}\langle l| \left(\alpha E + (1-\alpha)D\right) = \langle l|$$

$$\frac{1}{2}\left((1-\beta)E + \beta D\right)|r\rangle = |r\rangle$$

LDP for the empirical measure

when η is distributed according to μ_N then $\pi_N(\eta)$ satisfy a LDP in $\mathcal{M}^+([0,1])$ with rate

$$S(\rho) = \int_0^1 \left[h_{F(u)}(\rho(u)) + \log \frac{F'(u)}{\bar{\rho}'(u)} \right] du$$

where

$$h_F(\rho) := \rho \log \frac{\rho}{F} + (1-\rho) \log \frac{(1-\rho)}{(1-F)}$$

and F is monotone and solves

$$\begin{cases} F(1-F)\frac{F''}{(F')^2} + F = \rho \\ F(0) = \alpha, \;\; F(1) = \beta \end{cases}$$

In particular S is not additive

$$S_{[0,c]}(\rho) + S_{[c,1]}(\rho) \neq S_{[0,1]}(\rho)$$

$$S(\rho) = \sup_F \mathcal{G}(\rho, F)$$

(Shannon) Entropy

(Bahadoran, Derrida-Lebowitz-Speer) Let

$$\overline{\mu}_N(\eta) := \prod_{x=1}^{N} \overline{\rho}\left(\frac{x}{N}\right)^{\eta_x} \left(1 - \overline{\rho}\left(\frac{x}{N}\right)\right)^{1-\eta_x}$$

Then

$$\begin{aligned} & \lim_{N\to+\infty} \frac{-1}{N} \sum_{\eta} \mu_N(\eta) \log \mu_N(\eta) \\ = & \lim_{N\to+\infty} \frac{-1}{N} \sum_{\eta} \overline{\mu}_N(\eta) \log \overline{\mu}_N(\eta) \\ = & -\int_0^1 du\, h(\overline{\rho}(u)) \end{aligned}$$

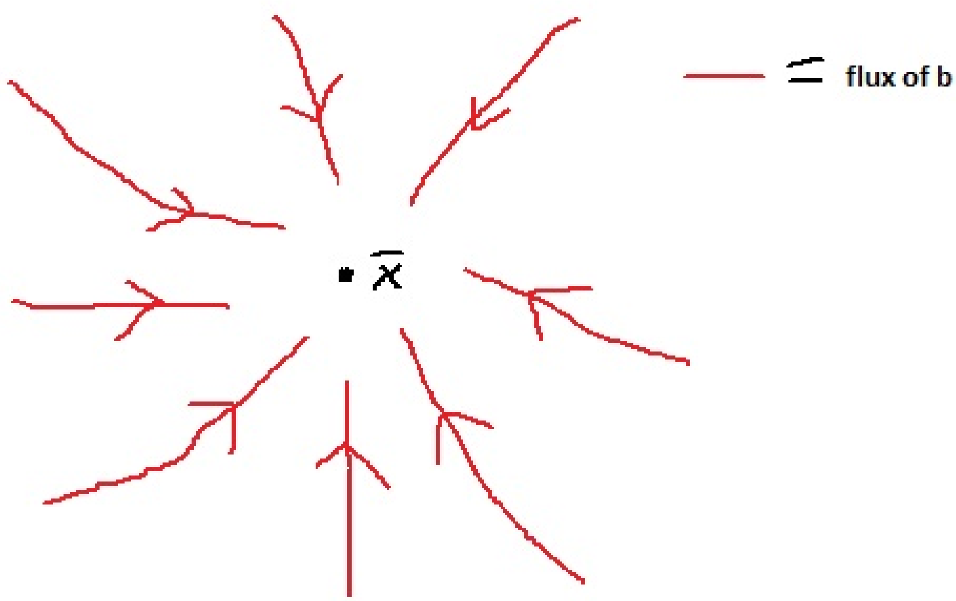
where

$$h(\rho) := \rho \log \rho + (1-\rho) \log(1-\rho)$$

Freidlin-Wentzell theory

$$dX_t^\epsilon = b(X_t^\epsilon)dt + \sqrt{\epsilon}dB_t$$

$X_t^\epsilon \in \mathbb{R}^n$, $B_t =$ n-dimensional Brownian motion, $b =$ globally attractive vector field ($\epsilon \Leftrightarrow N^{-1}$)



Sample path LDP
(Dynamic LDP)

$$\mathbb{P}\left(X_t^\epsilon \sim x_t\,;\, t \in [0,T]\right) \simeq e^{-\epsilon^{-1} I_{[0,T]}(x)}$$

where

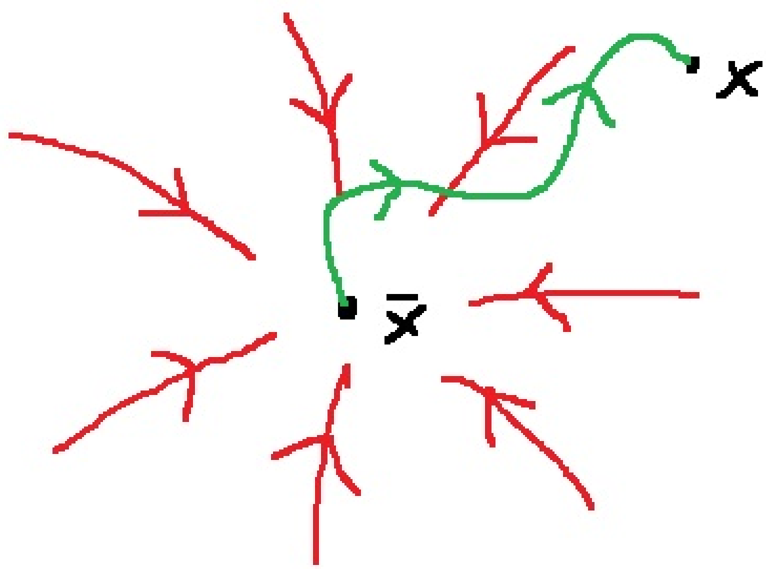
$$I_{[T_1,T_2]}(x) = \frac{1}{2}\int_{T_1}^{T_2} |\dot{x}(s) - b(x(s))|^2\, ds$$

$$\mu_\epsilon = \text{ invariant measure}$$

Main Issue

μ_ϵ satisfies a LDP (static LDP) on $\mathbb{R}^n$ with rate functional $W =$ QUASIPOTENTIAL

$$W(x) = \inf_{T} \inf_{\{y(s) : y(-T) = \bar{x},\ y(0) = x\}} I_{[-T,0]}(y)$$



Hamilton-Jacobi equation

$$I_{[T_1,T_2]}(x) = \int_{T_1}^{T_2} \mathcal{L}(\dot{x}(s), x(s))ds$$

$$\mathcal{L} = \text{ Lagrangian}$$

$$\mathcal{H}(p,x) = \sup_y \{p \cdot y - \mathcal{L}(y,x)\}$$

$$\mathcal{H} = \text{ Hamiltonian}$$

Then W solves (weakly) the stationary Hamilton-Jacobi equation

$$\mathcal{H}(\nabla W(x), x) = 0$$

If $b = -\nabla U$ (reversible) then $W = 2U$. In general W is not differentiable (Lagrangian phase transitions)

Example: 1-d torus

(Faggionato-G.)

$$dX_t^\epsilon = b(X_t^\epsilon)dt + \sqrt{\epsilon}dB_t$$

W = quasipotential + combinatorial optimization

$$\int_0^1 b(u)du = 0 \ \Rightarrow \ \text{reversible}$$

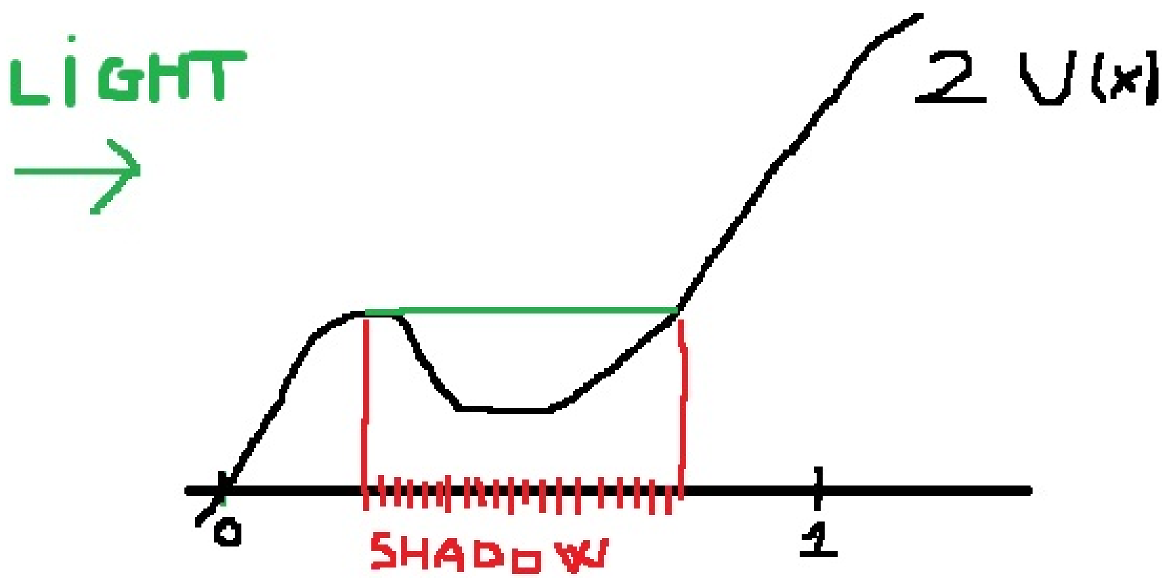
$$U(u) = -\int_0^u b(z)dz$$

$$W(u) = 2U(u) + c$$

If $\int_0^1 b(u)\, du \neq 0$

$$W(u) = \mathcal{S}[2U](u) + c$$

$$\mathcal{S} \ = \ \text{sunshine transformation}$$


LIGHT
2 V(x)
0
SHADOW
1


SHADOW
0
1
W(x)

Hydrodynamic scaling limit
(Exclusion)

Initial condition η such that $\pi_N(\eta) \to \rho_0(u)du$

$$\int_{[0,1]} f d\pi_N(\eta) \stackrel{N\to+\infty}{\to} \int_{[0,1]} f(u)\rho_0(u)du$$

$$L_N \to N^2 L_N \qquad \text{Diffusive rescaling}$$

$$\pi_N(\eta(t)) \to \rho(u,t)du$$

where

$$\begin{cases} \rho_t = \frac{1}{2}\rho_{uu} \\ \rho(u,0) = \rho_0(u) \\ \rho(0,t) = \alpha\,, \rho(1,t) = \beta \end{cases}$$

Law of $\pi_N(\eta)$ converges weakly to δ_ρ on $D([0,T],\mathcal{M}^+)$

Dynamic LDP
(Kipnis-Olla-Varadhan)

$L_N \to L_N^H$ weakly asymmetric perturbation

exchange $\eta \to \eta^{x,y}$ with rate

$$c^H_{\{x,y\}}(\eta) := c_{\{x,y\}}(\eta) e^{(\eta_x - \eta_y)\left(H(\frac{y}{N},t) - H(\frac{x}{N},t)\right)}$$

$H : [0,1] \times [0,T] \to \mathbb{R},\ H(0,t) = H(1,t) = 0$

$$c^H_{\{x,y\}}(\eta) := c_{\{x,y\}}(\eta) + O\left(\frac{1}{N}\right)$$

$$\frac{d\mathbb{P}^H_{\eta,N}}{d\mathbb{P}_{\eta,N}} \leftarrow \text{Graphical construction}$$

Upper bound: Chebyshev inequality arguments

Lower bound: a relative entropy computation

$$\lim_{N \to +\infty} \frac{1}{N} \text{Ent}\left(\mathbb{P}^H_{\eta,N} | \mathbb{P}_{\eta,N}\right)$$

Key step: superexponential replacement Lemma

The rate functional

Variational representation

If $\rho(u,t)$ is smooth and far from 0 and 1

$$I_{[T_1,T_2]}(\rho) = \frac{1}{2}\int_{T_1}^{T_2} dt \int_0^1 du\, \rho(u,t)(1-\rho(u,t))H_u^2(u,t)$$

where

$$\begin{cases} \rho_t = \frac{1}{2}\rho_{uu} - (\rho(1-\rho)H_u)_u \\ H(0,t) = H(1,t) = 0 \end{cases}$$

if $\rho(0,t) = \alpha$, $\rho(1,t) = \beta$ and $+\infty$ otherwise

The quasipotential

(Bodineau-Giacomin, Farfan): Exclusion processes: the quasipotential associated to the dynamic rate function from hydrodynamic rescaling coincides with LD rate function for the invariant measure

Generalized Onsager-Machlup symmetry

Particle systems satisfying a dynamic and static LDP

$$\mathbb{P}_{\mu_N}\Big(\pi_N(\eta(t)) \sim \rho(u,t)\,, t \in [-T,0]\Big)$$
$$= \mathbb{P}^*_{\mu_N}\Big(\pi_N(\eta(t)) \sim \rho(u,-t)\,, t \in [0,T]\Big)$$

where $\mathbb{P}^*_{\mu_N}(\cdot)$= law of the stationary time reversed (adjoint) process. At LD level using Markov property we get

$$e^{-NS(\rho(-T))}e^{-NI_{[-T,0]}(\rho)} = e^{-NS(\rho(0))}e^{-NI^*_{[0,T]}(\theta\rho)}$$

where I^* is the dynamic rate functional of the adjoint process and θ is the time-reversal

$$\theta\rho(u,t) := \rho(u,-t)$$

$$S(\rho(-T)) + I_{[-T,0]}(\rho) = S(\rho(0)) + I^*_{[0,T]}(\theta\rho)$$

if $\rho(-T) = \overline{\rho}$, since $S(\overline{\rho}) = 0$ it becomes

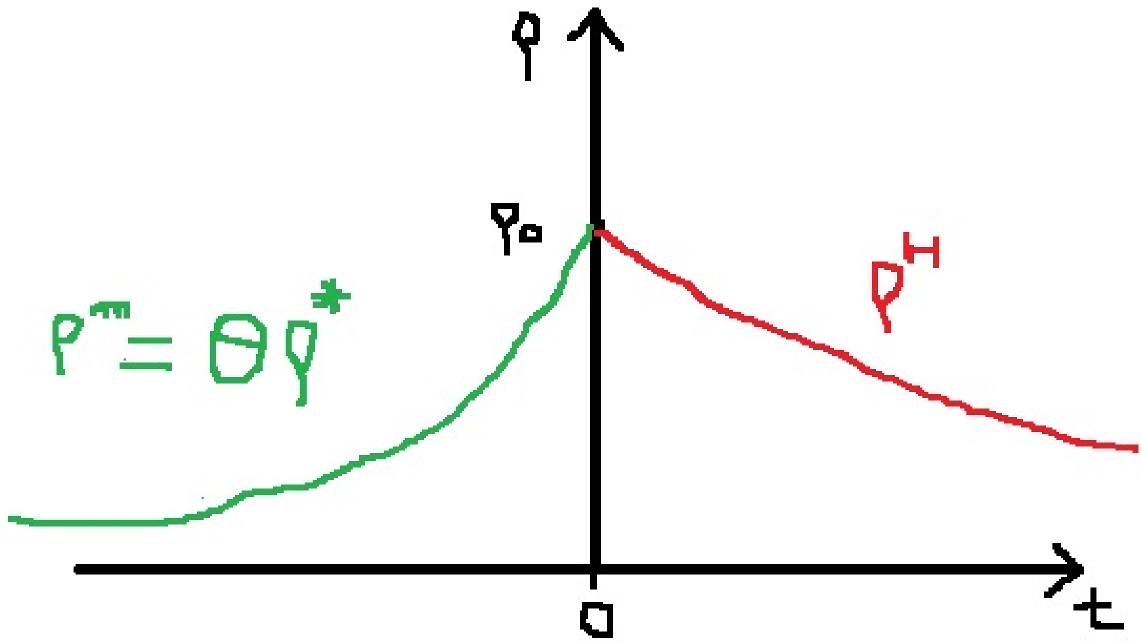
$$I_{[-T,0]}(\rho) = S(\rho(0)) + I^*_{[0,T]}(\theta\rho)$$

consequently since $I^* \geq 0$

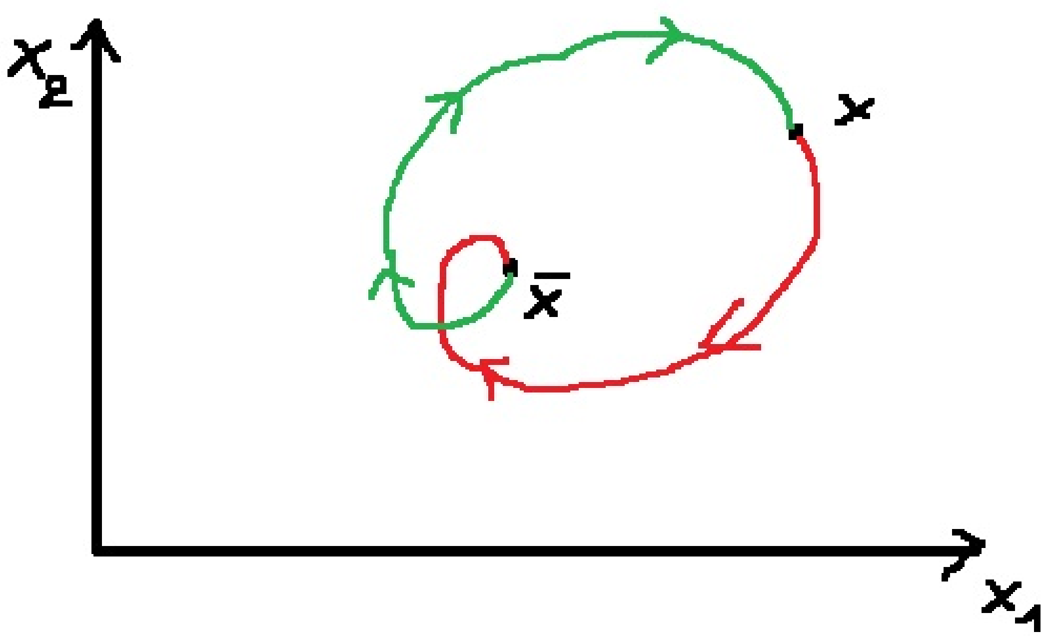
$$\inf_T \inf_{\{\rho:\rho(-T)=\overline{\rho}\ \rho(0)=\rho_0\}} I_{[-T,0]}(\rho) \geq S(\rho_0)$$

Let ρ^* such that $I^*(\rho^*) = 0$. The minimizer for the quasipotential ρ^m is such that

$$\rho^m = \theta\rho^*$$



ρ
ρ₀
ρ^m = θρ*
ρ^H
0
t



x₂
x
x̄
x₁

How to compute the quasipotential (Exclusion)

Reversible case $\alpha{=}\beta{=}c$. Rate function for the invariant measure

$$S(\rho) = \int_0^1 h_c(\rho(u))\, d\,u$$

Hydrodynamic limit ρ^* associated to $\mathbb{P}^*$ solves the heat equation. We expect

$$\rho^m = \theta\rho^* \Rightarrow \;\; \rho^m_t = -\frac{1}{2}\rho^m_{uu}$$

Let

$$\Gamma[\rho] := \log\frac{\rho}{1-\rho} - \log\frac{c}{1-c}$$

it holds

$$S(\rho(T_2)) - S(\rho(T_1)) = \int_{T_1}^{T_2} d\,t \int_0^1 d\,u \Gamma[\rho(t)](u)\rho_t(u,t)$$

A computation gives

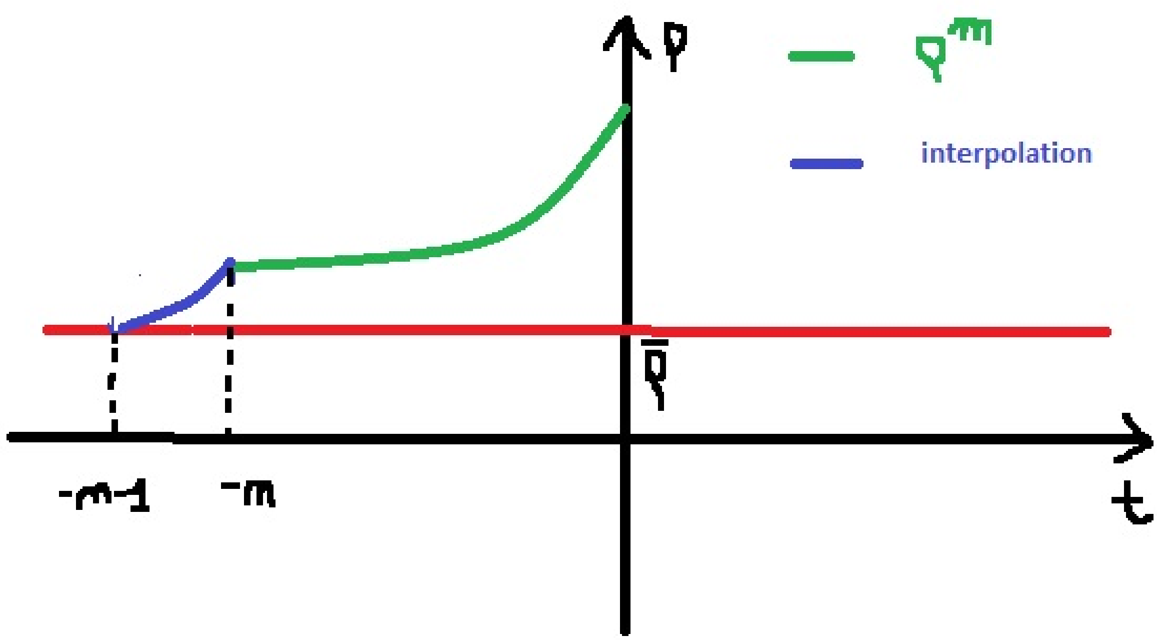
$$I_{[-T,0]}(\rho) = S(\rho(0)) - S(\rho(-T))$$
$$+\frac{1}{2}\int_{-T}^{0} d\,t \int_0^1 d\,u (H-\Gamma)^2_u \rho(1-\rho)$$

If $\rho(-T) = \bar{\rho} \;\Rightarrow\; S(\rho(-T)) = 0 \;\Rightarrow I_{[-T,0]}(\rho) \geq S(\rho(0))$. Minimizing sequence

$$\rho^m \Leftrightarrow H^m = \Gamma[\rho^m]$$

for any $n \in \mathbb{N}$

$$\rho_n(t) = \begin{cases} \rho^m(t) & t \in [-n, 0] \\ \text{interpolation} & t \in [-n-1, -n] \end{cases}$$



$$I_{[-n-1,0]}(\rho_n) = I_{[-n,0]}(\rho^m) + I_{[-n-1,-n]}(\rho_n)$$
$$= S(\rho(0)) - S(\rho^m(-n)) + \epsilon(n) \stackrel{n\to+\infty}{\to} S(\rho(0))$$

$\Rightarrow$ S is the quasipotential

not reversible $\alpha \neq \beta$

$$\Gamma[\rho] := \log \frac{\rho}{1-\rho} - \log \frac{F[\rho]}{1-F[\rho]}$$

where $F[\rho]$ is the unique monotone solution of

$$\begin{cases} F(1-F)\frac{F''}{(F')^2} + F = \rho \\ F(0) = \alpha, \;\; F(1) = \beta \end{cases}$$

The rate function of μ_N is $S(\rho) = \mathcal{G}(\rho, F[\rho])$ A computation

$$S(\rho(T_2)) - S(\rho(T_1)) = \int_{T_1}^{T_2} dt \int_0^1 du \Gamma[\rho(t)](u)\rho_t(u,t)$$

$$\frac{dS(\rho(t))}{dt} = \int_0^1 d\,u\,\left[\frac{\delta\mathcal{G}}{\delta\rho}\rho_t + \frac{\delta\mathcal{G}}{\delta F}F_t\right]$$

and

$$\begin{cases} \frac{\delta\mathcal{G}}{\delta\rho}(\rho, F[\rho]) = \Gamma[\rho] \\ \frac{\delta\mathcal{G}}{\delta F}(\rho, F[\rho]) = 0 \end{cases}$$

A computation

$$I_{[-T,0]}(\rho) = S(\rho(0)) - S(\rho(-T))$$
$$+\frac{1}{2}\int_{-T}^{0} d\,t \int_0^1 d\,u(H-\Gamma)_u^2\rho(1-\rho)$$

This suggests

$$\rho^m \quad \Leftrightarrow \quad H^m = \Gamma[\rho^m]$$

The minimizer ρ^m solves

$$\begin{cases} \rho^m_t = \frac{1}{2}\rho^m_{uu} - \Big(\rho^m(1-\rho^m)(\Gamma[\rho^m])_u\Big)_u \\ \rho^m(u,0) = \rho_0(u) \text{ and b.c.} \end{cases}$$

It is the following coupled differential problem

$$\begin{cases} \rho^m_t = -\frac{1}{2}\rho^m_{uu} - \left(\frac{\rho^m(1-\rho^m)}{F(1-F)}F_u\right)_u \\ \rho^m(u,0) = \rho_0(u) \\ F(1-F)\frac{F_{uu}}{(F_u)^2} + F = \rho^m \\ \text{and b.c.} \end{cases}$$

A computation shows it is equivalent to

$$\begin{cases} F_t = -\frac{1}{2}F_{uu} \\ F(1-F)\frac{F_{uu}}{(F_u)^2} + F = \rho^m \\ \rho^m(u,0) = \rho_0(u) \\ \text{and b.c.} \end{cases}$$

KMP model

(Kipnis-Marchioro-Presutti)

$\Lambda_N = \{1, 2, \ldots N\}$ and $\eta_x \in \mathbb{R}^+$

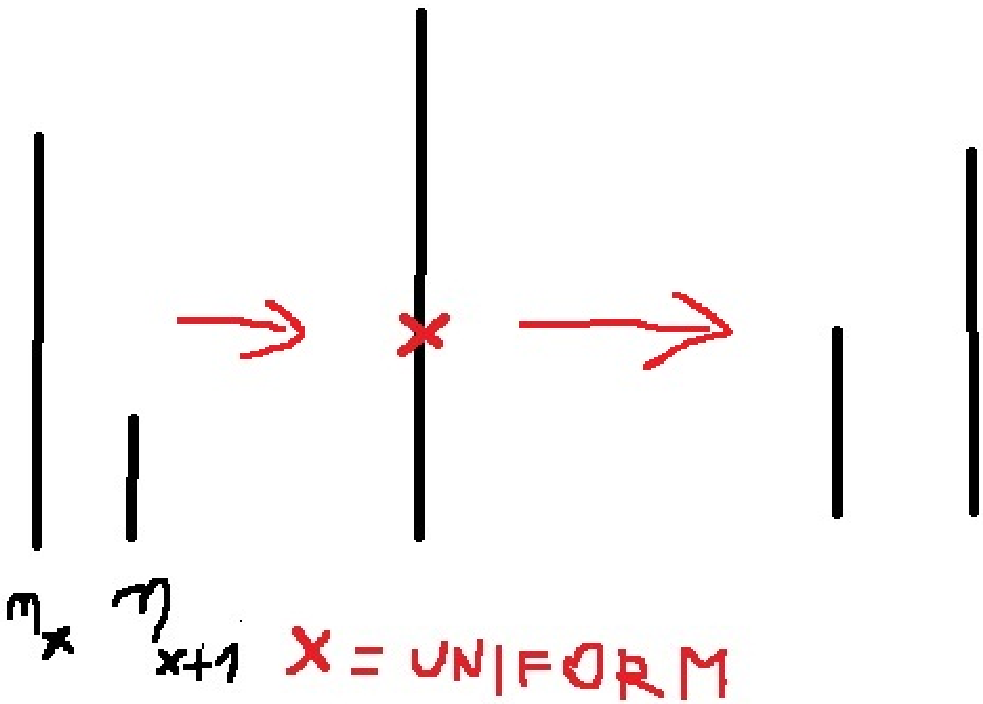
$\eta_x =$ Energy of the harmonic oscillator located at x

Stochastic dynamics

in the bulk at rate 1

$$(\eta_x, \eta_{x+1}) \to (U(\eta_x + \eta_{x+1}), (1 - U)(\eta_x + \eta_{x+1}))$$

$U =$ uniform on $[0, 1]$


η_x η_{x+1}
X = UNIFORM

at the boundary

$$\eta_1 \to U(\eta_1 + X_{\tau_L}) \qquad \mathcal{L}(X_{\tau_L}) = \frac{e^{-\frac{u}{\tau_L}}}{\tau_L} du$$

$$\eta_N \to U(\eta_N + X_{\tau_R}) \qquad \mathcal{L}(X_{\tau_R}) = \frac{e^{-\frac{u}{\tau_R}}}{\tau_R} du$$

Reversibility $\Leftrightarrow \tau_L = \tau_R = \tau$, product invariant measure

$$\mu_N(\eta) d\eta = \prod_{x=1}^{N} \frac{e^{-\frac{\eta_x}{\tau}}}{\tau} d\eta_x$$

If $\tau_L \neq \tau_R$ not reversible, no matrix representation

Hydrodynamic diffusive rescaling

when $L_N \to N^2 L_N$ then $\pi_N(\eta)$ converges to

$$\begin{cases} \rho_t = \frac{1}{2}\rho_{uu} \\ \rho(u,0) = \rho_0(u) \\ \rho(0,t) = \tau_L \quad \rho(1,t) = \tau_R \end{cases}$$

Dynamic LDP

$$I_{[T_1,T_2]}(\rho) = \frac{1}{2}\int_{T_1}^{T_2}\int_0^1 \rho^2(u,t)H_u^2(u,t)\,du$$

where

$$\begin{cases} \rho_t = \frac{1}{2}\rho_u u - (\rho^2 H_u)_u \\ H(0,t) = H(1,t) = 0 \end{cases}$$

if $\rho(0,t) = \tau_L$, $\rho(1,t) = \tau_R$ and $I = +\infty$ otherwise

Static LDP

When η is distributed according to $\mu_N(\eta)d\eta$ then $\pi_N(\eta)$ satisfies a LDP with rate

$$S(\rho) = \mathcal{G}(\rho, F[\rho])$$

where

$$\mathcal{G}(\rho, F) := \int_0^1 K_{F(u)}(\rho(u)) - \log \frac{F'(u)}{\overline{\rho}'(u)} du$$

$$K_F(\rho) := \frac{\rho}{F} - 1 - \log \frac{\rho}{F}$$

and $F[\rho]$ is the unique monotone solution to

$$\begin{cases} F^2 \frac{F''}{(F')^2} - F = -\rho \\ F(0) = \tau_L \quad F(1) = \tau_R \end{cases}$$

It holds

$$S(\rho) = \inf_F \mathcal{G}(\rho, F)$$

not convex!

A physicist proof

Infinite dimensional Hamilton-Jacobi equation for S

$$\frac{1}{2}\langle\nabla\frac{\delta S}{\delta\rho},\rho^2\nabla\frac{\delta S}{\delta\rho}\rangle+\langle\frac{\delta S}{\delta\rho},\Delta\rho\rangle=0$$

Search for a solution of the form

$$\frac{\delta S}{\delta\rho}=\frac{1}{F}-\frac{1}{\rho}$$

A few smart integrations by parts give

$$\left\langle\frac{(\rho-F)}{F^4},\left(F^2\Delta F+(\rho-F)(\nabla F)^2\right)\right\rangle=0$$

if $F=F[\rho]$ H-J is satisfied. If $S(\rho)=\mathcal{G}(\rho,F[\rho])$ then

$$\frac{\delta S}{\delta\rho}=\frac{\delta\mathcal{G}(\rho,F[\rho])}{\delta\rho}+\frac{\delta\mathcal{G}(\rho,F[\rho])}{\delta F}\frac{\delta F[\rho]}{\delta\rho}$$
$$=\frac{1}{F[\rho]}-\frac{1}{\rho}$$

a solution

contraction principle? (yes)

F= hidden temperature profile

A toy model for the SNS

$(U_1, \dots, U_N)$ i.i.d. uniform random variables on $[\tau_L, \tau_R]$

$(U_{[1]}, \dots, U_{[N]})$ the order statistics and $\gamma_N(u)du$ the corresponding law. Define

$$\nu_N(\eta, u)d\eta du := \gamma_N(u)du \prod_{x=1}^{N} \frac{e^{-\frac{\eta_x}{u_x}}}{u_x} d\eta_x$$

probability measure on $[\tau_L, \tau_R]^N \times (\mathbb{R}^+)^N$. The toy measure is

$$\mu_N^T(\eta)d\eta := \left(\int_{[\tau_L, \tau_R]^N} \nu_N(\eta, u)du \right) d\eta$$

a mixture of product of exponentials

$$\pi_N(U) := \frac{1}{N}\sum_{x=1}^{N} U_{[x]}\delta_{\frac{x}{N}}$$

When (U,η) is distributed according to $\nu_N(u,\eta)du d\eta$ then $\Big(\pi_N(U),\pi_N(\eta)\Big)$ satisfies a LDP with rate

$$\mathbb{P}\Big((\pi_N(U),\pi_N(\eta)) \sim (F,\rho)\Big) \simeq e^{-N\mathcal{G}(\rho,F)}$$

It follows by

$$\begin{aligned} & \mathbb{P}\Big((\pi_N(U),\pi_N(\eta)) \sim (F,\rho)\Big) \\ = \; & \mathbb{P}\Big(\pi_N(U) \sim F\Big)\mathbb{P}\Big(\pi_N(\eta) \sim \rho \Big| \pi_N(U) \sim F\Big) \end{aligned}$$

$$\mathbb{P}\Big(\pi_N(U) \sim F\Big) \simeq e^{-N\left(-\int_0^1 \log \frac{F'(u)}{\overline{\rho}'(u)}\right)} \qquad \text{(A)}$$

$$\mathbb{P}\Big(\pi_N(\eta) \sim \rho \Big| \pi_N(U) \sim F\Big) \simeq e^{-N\int_0^1 K_{F(u)}(\rho(u))du} \quad \text{(B)}$$

(A): i.i.d exponentials conditioned to have fixed sum $\rightarrow$ order statistics of uniforms

(B): by LDP for product measures

An exact result

1 Oscillator $N = 1$

$$\mu_1(\eta_1)d\eta_1 = \frac{1}{\pi}\left(\int_{\tau_L}^{\tau^R} \frac{e^{-\frac{\eta_1}{u}}\,du}{u\sqrt{(\tau_R - u)(u - \tau_L)}}\right)d\eta_1$$

It means $\gamma_1(u)du =$ arcsine distribution and not uniform.

$N > 1$?

External fields

Stochastic lattice gas $\eta_x \in \{0,1\}$ on $\mathbb{T}_N^d$

$c_{\{x,y\}}(\eta) =$ rate of exchange (only n.n.)

Detailed balance

$$e^{-H(\eta)} c_{\{x,y\}}(\eta) = e^{-H(\eta^{x,y})} c_{\{x,y\}}(\eta^{x,y})$$

H = Hamiltonian: translation invariant finite range. High temperature regime.

$$\mu_N(\eta) = \frac{e^{-H(\eta)}}{Z_N}$$

reversible, also the canonical.

Switch on a vector field E, function on ordered edges

$$E_{xy} = -E_{yx}$$

discrete vector field

$$c \to c^E$$

$$c^E_{\{x,y\}}(\eta) := c_{\{x,y\}}(\eta) e^{E_{xy}(\eta_x - \eta_y)}$$

$E_{xy}(\eta_x - \eta_y) =$ work done by the field

Microscopic analysis

1) E is a gradient vector field

$$E_{xy} = \frac{1}{2}(V_y - V_x)$$

$$\mu^E_N(\eta) = \frac{e^{-H^E(\eta)}}{Z^E_N}$$

$$H^E(\eta) = H(\eta) - \sum_{x \in \mathbb{T}^d_N} V_x \eta_x$$

reversible!

2) E constant vector field $E_{x,x+e_i} = E_i$, $i = 1, \dots d$

2A) c determine a gradient model: exists a cylindric function $h(\eta)$ such that average current through the edge (x, y)

$$c_{\{x,y\}}(\eta)(\eta_x - \eta_y) = \tau_y h(\eta) - \tau_x h(\eta)$$

Exclusion, zero range, KMP, all gradient

$$\mu_N^E(\eta) = \mu_N(\eta)$$

not reversible (ex: asymmetric exclusion has Bernoulli product as invariant measure)

$2B)$ c is not gradient $\to \mu_N^E = ?$

Conjectures (Garrido-Lebowitz-Maes-Spohn)

$i)$ For any density $\bar{\rho}$ there exists a unique translation invariant thermodynamic limit $\mu_{\bar{\rho}}^E$

$ii)$ $d = 1$ then $\mu_{\bar{\rho}}^E$ has pair correlations decaying exponentially

$iii)$ $d > 1$ pair correlations decay as a power law

3) Divergence free fields

$$G_{xy} := e^{E_{xy}} - e^{E_{yx}}$$

divergence free discrete vector field + gradient model

$$\mu_N^E(\eta) = \mu_N(\eta)$$

not reversible

Macroscopic analysis

(Varhadan-Yau) hydrodynamic limit non gradient models, diffusive scaling; (Quastel) dynamic LDP

(Bertini-Faggionato-G) hydrodynamic limit and dynamic LDP for weakly asymmetric models

$$E_{xy} \sim \frac{E(u)}{N}$$

Hydrodynamic equation

$$\rho_t = \nabla \cdot (D(\rho)\nabla\rho) - \nabla \cdot (\sigma(\rho)E)$$

$D(\rho)$ = diffusion matrix, variational representation

$\sigma(\rho)$ = mobility

$$D(\rho) = \sigma(\rho)s''(\rho)$$

Einstein relation

1) If $E(u) = -\frac{1}{2}\nabla U(u)$ is gradient, when η is distributed according to μ_N^E $\pi_N(\eta)$ satisfies a LDP with rate

$$S^E(\rho) = \int_{\mathbb{T}^d} s_{V(u)}(\rho(u))\, du$$

minimizer for the quasipotential

$$\rho^m = \theta\rho^E$$

reversible, Onsager-Machlup symmetry

2) and 3) When η is distributed according to μ_N^E then $\pi_N(\eta)$ satisfy a LDP with rate

$$S^E(\rho) = S(\rho)$$

minimizer for the quasipotential

$$\rho^m = \theta\rho^{-E}$$

not reversible, generalized Onsager-Machlup symmetry

For the case 3) D and σ must be multiple of the identity

Lagrangian phase transitions
(diffusions)

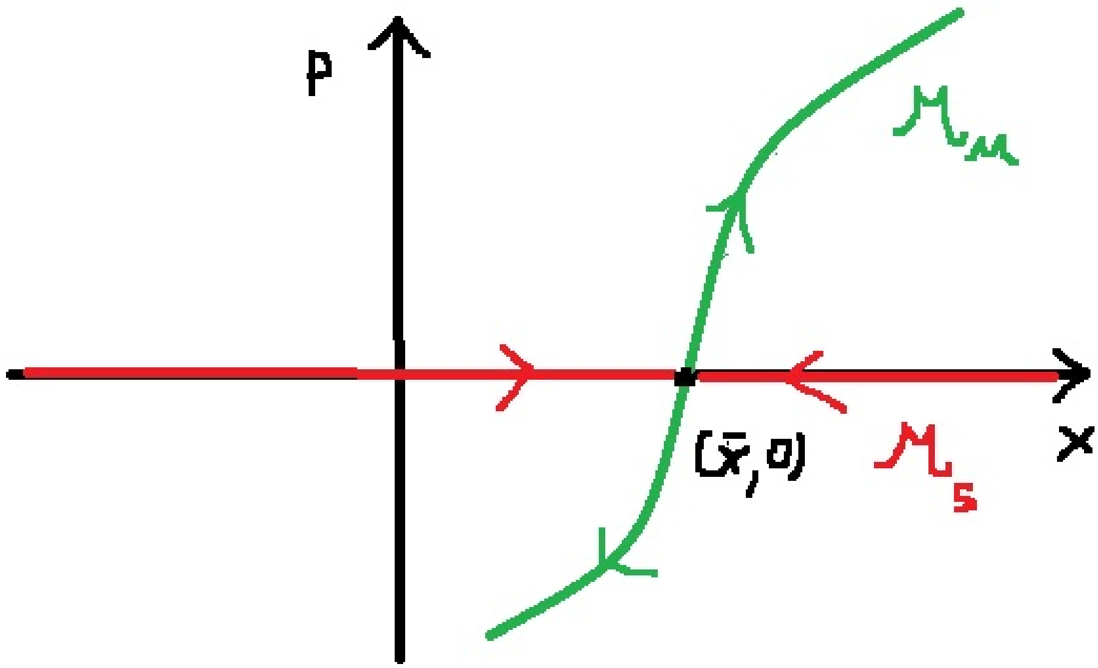
Critical trajectories for computing W the quasipotential

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = \frac{\partial \mathcal{L}}{\partial x}$$

Euler-Lagrange equations

$$\left(x(t), p(t) := \frac{\partial \mathcal{L}}{\partial \dot{x}}(x(t), \dot{x}(t))\right) \Rightarrow \text{Hamilton Equations}$$

Phase space (x, p)



$\mathcal{M}_u =$ unstable manifold

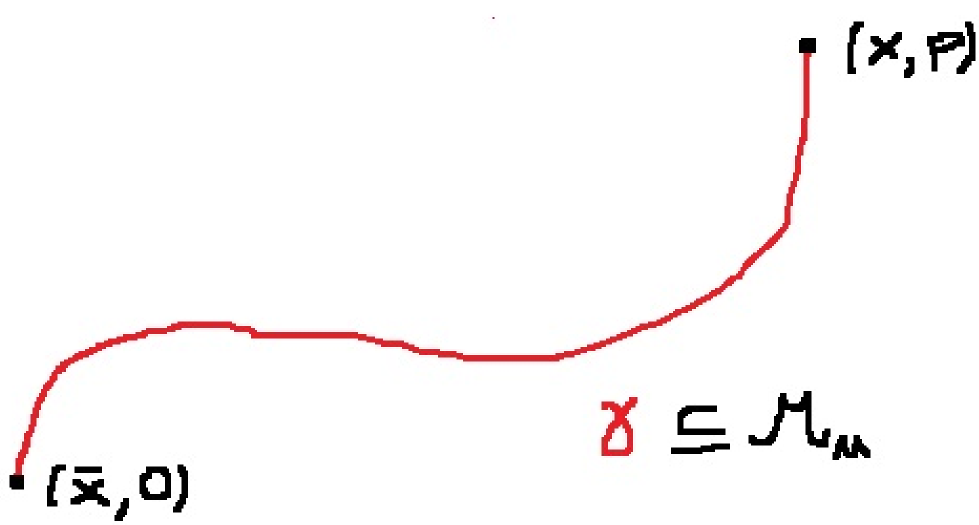
$\mathcal{M}_s =$ stable manifold

$\mathcal{M}_u, \mathcal{M}_s \subseteq \{\mathcal{H} = 0\}$ and are Lagrangian manifolds

$$\oint_\gamma p dq = 0 \qquad \gamma \subseteq \mathcal{M}_u, \mathcal{M}_u$$

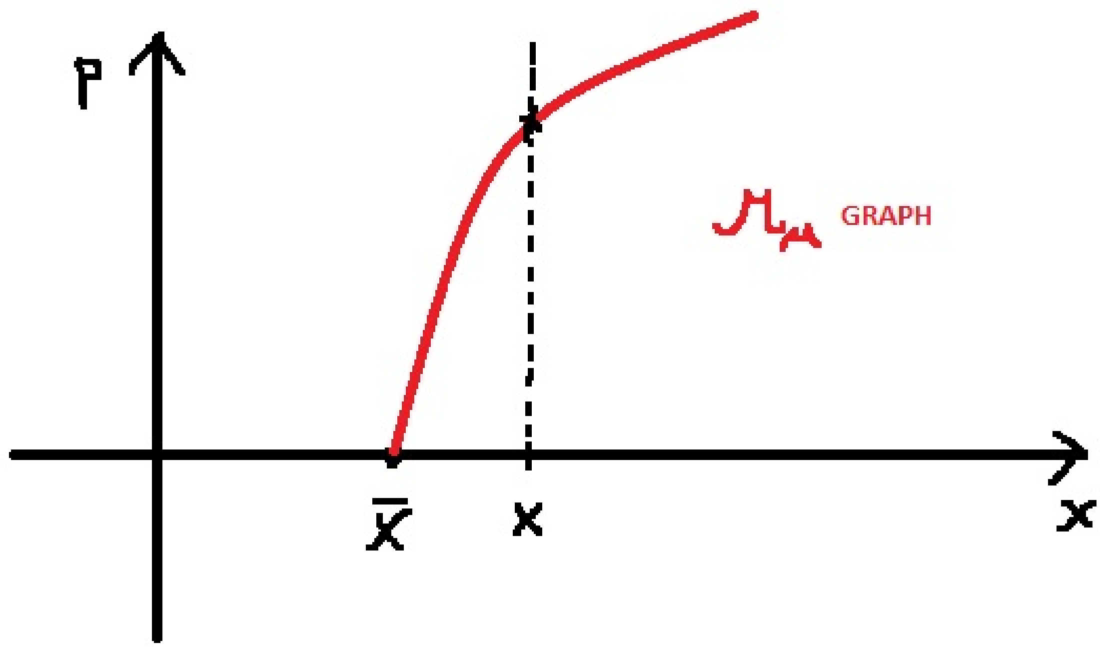
$(x,p) \in \mathcal{M}_u \Rightarrow \quad \widetilde{W}(x,p) =$ Prepotential (Day)

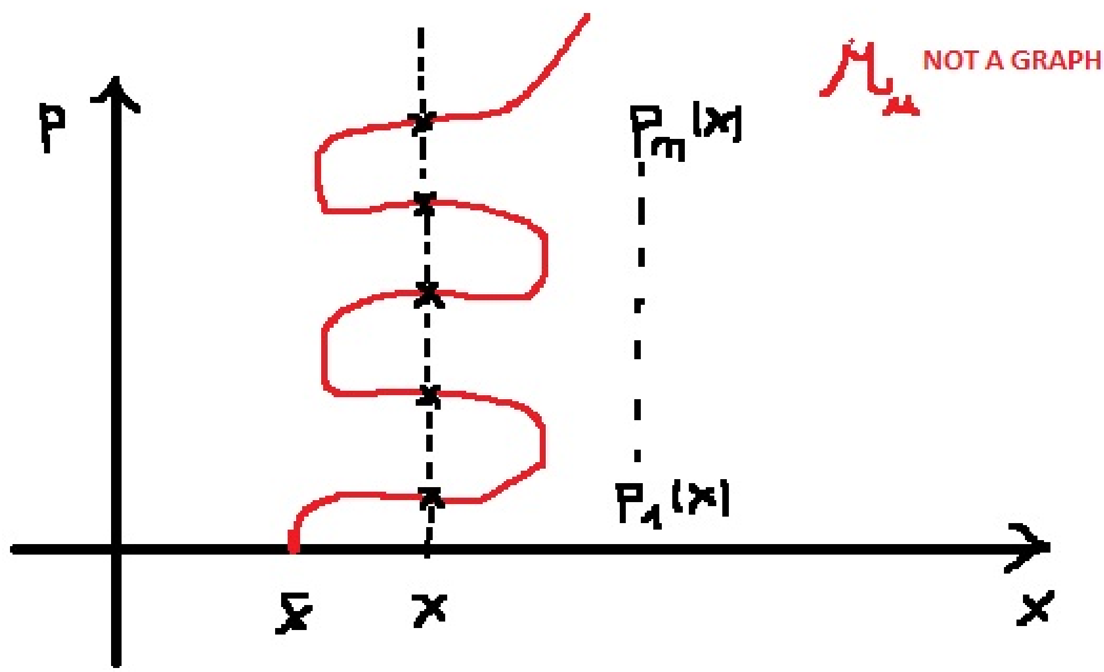
$$\widetilde{W}(x,p) := \int_\gamma p dq$$

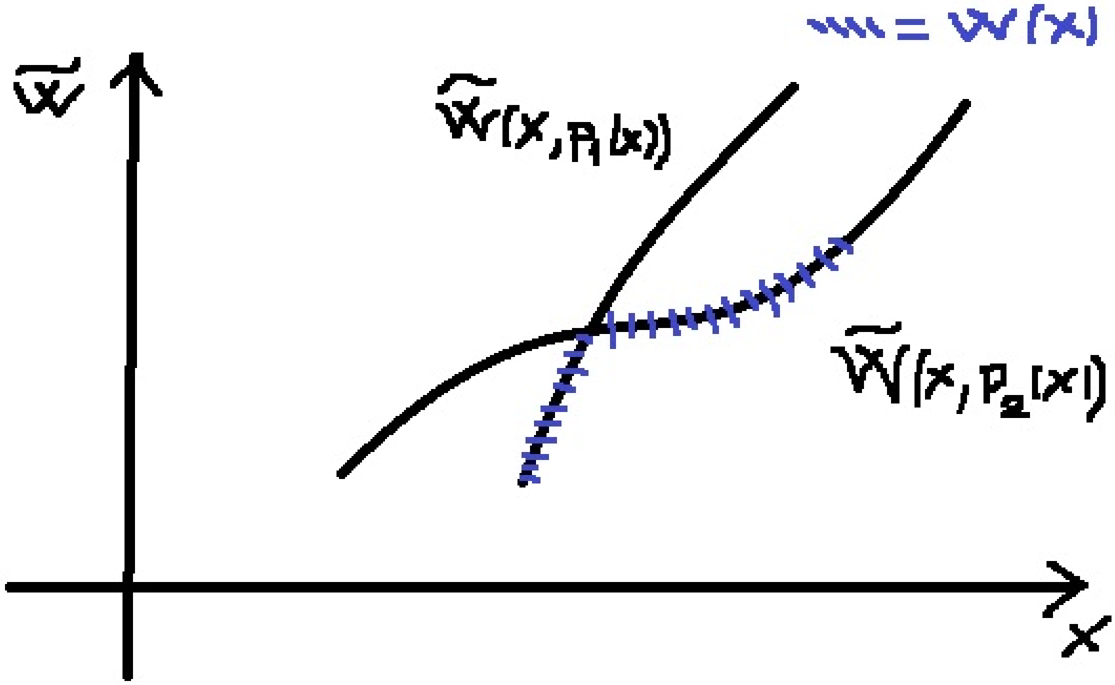


Quasipotential from prepotential
(Day)

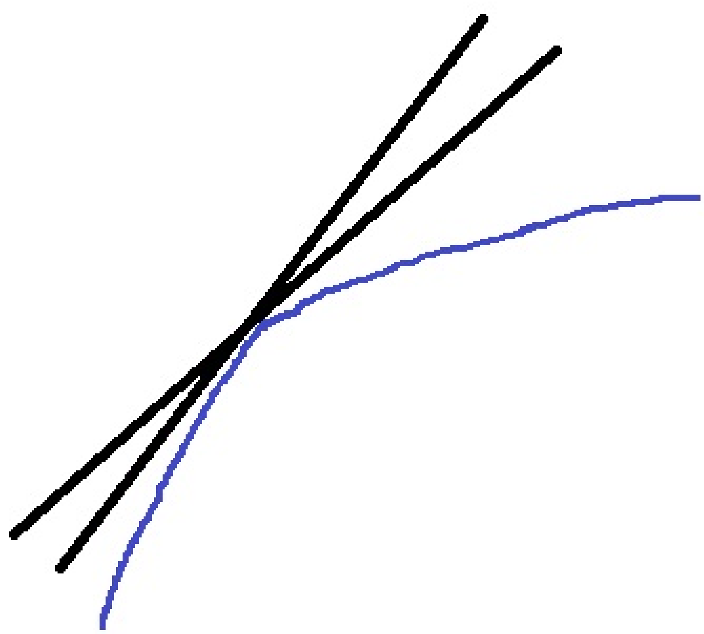
$$W(x) = \inf_{p:(x,p)\in\mathcal{M}_u} \widetilde{W}(x,p)$$


p
$\mathcal{M}_\mu$ GRAPH
$\bar{x}$
x
x


p
$\mathcal{M}_\mu$ NOT A GRAPH
$p_m(x)$
$p_1(x)$
$\bar{x}$
x
x



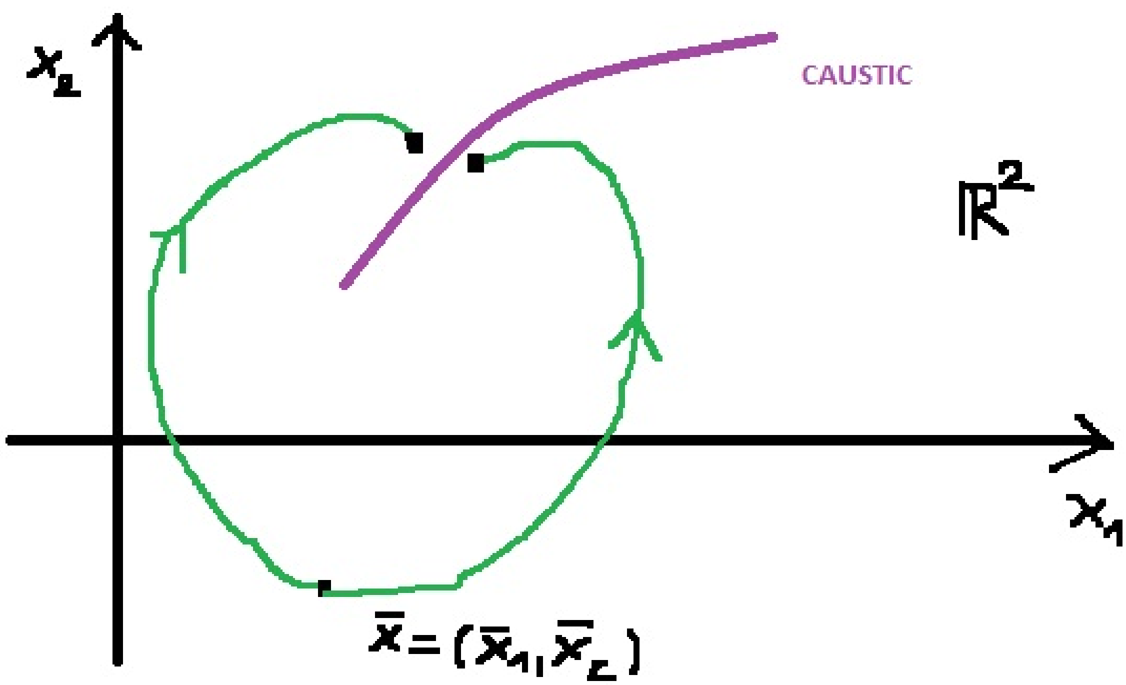
= W(x)
W̃
W̃(x, p1(x))
W̃(x, p2(x))
x



Superdifferential is not empty

$\Downarrow$

no convexity, instability



configuration space

Lagrangian phase transition on WASEP

1-d boundary driven exclusion weakly perturbed

$$c_{\{x,x+1\}} \to c^E_{\{x,x+1\}} := e^{\frac{E}{N}(\eta_x - \eta_{x+1})}$$

constant vector field

Diffusive rescaling, Hydrodynamic equation

$$\begin{cases} \rho_t = \rho_{uu} - E(\rho(1-\rho))_u \\ \rho(0,t) = \alpha\,,\; \rho(1,t) = \beta \end{cases}$$

$$\alpha < \beta \qquad \text{push } \Leftarrow$$

$$E > 0 \qquad \text{push } \Rightarrow$$

$$E > h'(\beta) - h'(\alpha)$$

recall

$$h(\rho) = \rho \log \rho + (1-\rho)\log(1-\rho)$$

Dynamic LDP $\Rightarrow$ Lagrangian structure

$\Rightarrow$ Hamiltonian structure $\Rightarrow$ prepotential

Hamilton equations

$$\begin{cases} \rho_t = \rho_{uu} - (\rho(1-\rho)(E+2\pi_u)_u) \\ \pi_t = -\pi_{uu} + (2\rho - 1)(\pi_u^2 - E\pi_u) \end{cases}$$

after a symplectic transformation

$$\mathcal{M}_u = \left\{ (\phi, \rho) \, : \, \frac{-\phi_{uu}}{\phi_u(E - \phi_u)} + \frac{1}{1+e^{\phi}} = \rho \right\}$$

where $0 < \phi_u < E$.

$$\widetilde{W}_E = \int_0^1 h(\rho) + h\left(\frac{\phi_u}{E}\right) + (1-\rho)\phi - \log(1+e^{\phi})\, du$$

$$S_E(\rho) = W_E(\rho) = \inf_{\phi:(\phi,\rho)\in\mathcal{M}_u} \widetilde{W}_E(\phi,\rho)$$

Limiting cases

$$E \downarrow h'(\beta) - h'(\alpha) := E_0$$

then

$$\begin{cases} 0 < \phi_u < h'(\beta) - h'(\alpha) \\ \phi(0) = h'(\alpha)\,, \phi(1) = h'(\beta) \end{cases}$$

$\exists\ !\ \bar{\phi}$ affine

$$\mathcal{M}_u = \left\{(\bar{\phi},\rho)\right\} \qquad \text{is a graph}$$

$W_{E_0}(\rho) = \widetilde{W}_{E_0}(\bar{\phi},\rho)$ product of Bernouilli

$E \sim E_0$ no Lagrange phase transition

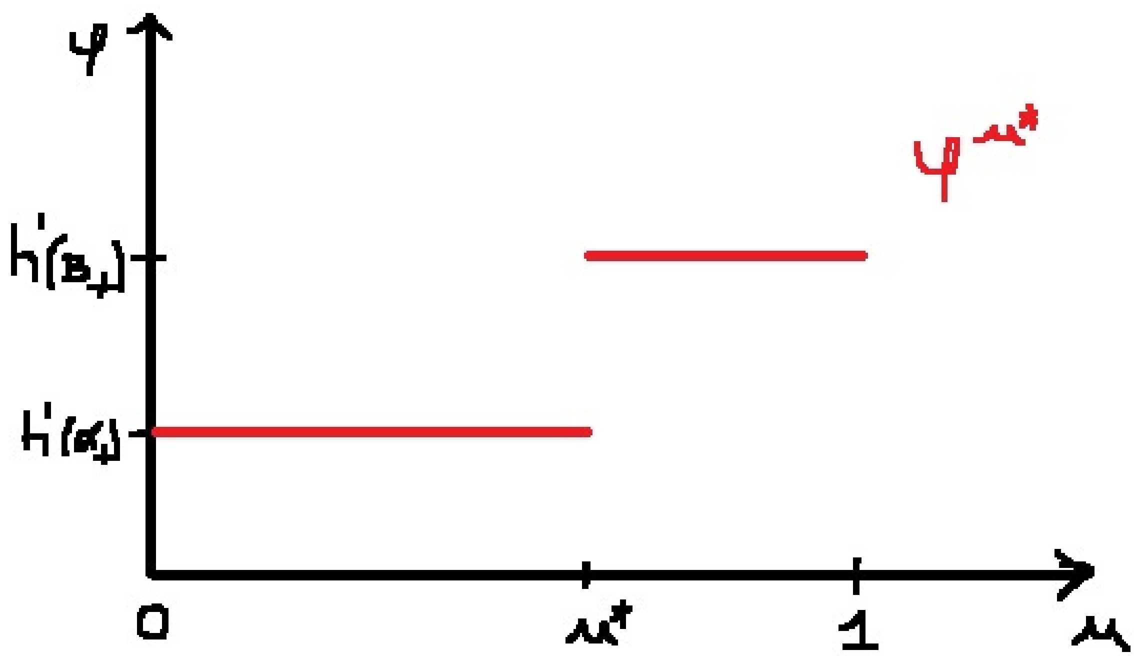
$$E \to +\infty \qquad \widetilde{W}_E \to \widetilde{W}_{+\infty}$$

where

$$\widetilde{W}_{+\infty}(\phi,\rho) := \int_0^1 h(\rho) + (1-\rho)\phi - \log(1+e^{\phi})\, du$$

ϕ CADLAG increasing + b.c.

$\widetilde{W}_{+\infty}$ is concave in ϕ $\Rightarrow$ infimum can be restricted to extremal elements (step functions)



$$\inf_{\phi} \widetilde{W}_{+\infty}(\phi, \rho) = \inf_{z \in [0,1]} \widetilde{W}_{+\infty}(\phi^z, \rho) =$$

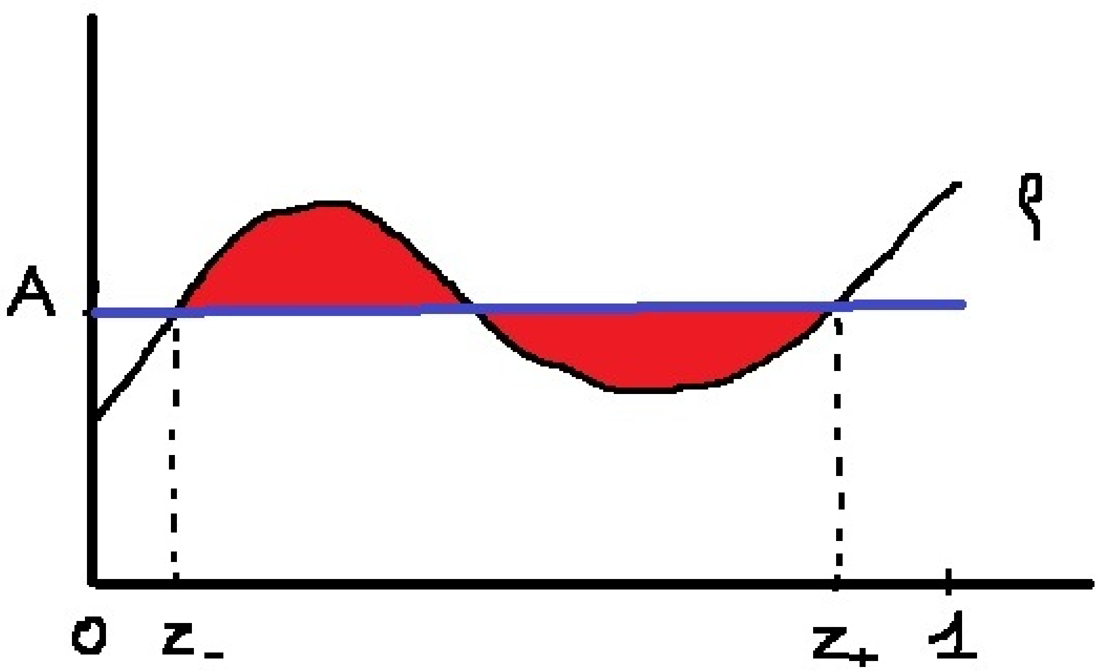
$$\inf_{z \in [0,1]} \left\{ \int_0^z h(\rho) + h'(\alpha)(1-\rho) - \log(1 + e^{h'(\alpha)}) \right.$$

$$\left. + \int_z^1 h(\rho) + h'(\beta)(1-\rho) - \log(1 + e^{h'(\beta)}) \right\}$$

LDP for TASEP (Derrida-Lebowitz-Speer)

$\alpha < \beta$, $E > 0$ not weak

For a density profile ρ such that exists A



$$\inf_z \widetilde{W}_{+\infty}(\phi^z, \rho) = \widetilde{W}_{+\infty}(\phi^{z-}, \rho) = \widetilde{W}_{+\infty}(\phi^{z+}, \rho)$$

Lagrangian phase transition, by a perturbative argument it happens also for E large but finite

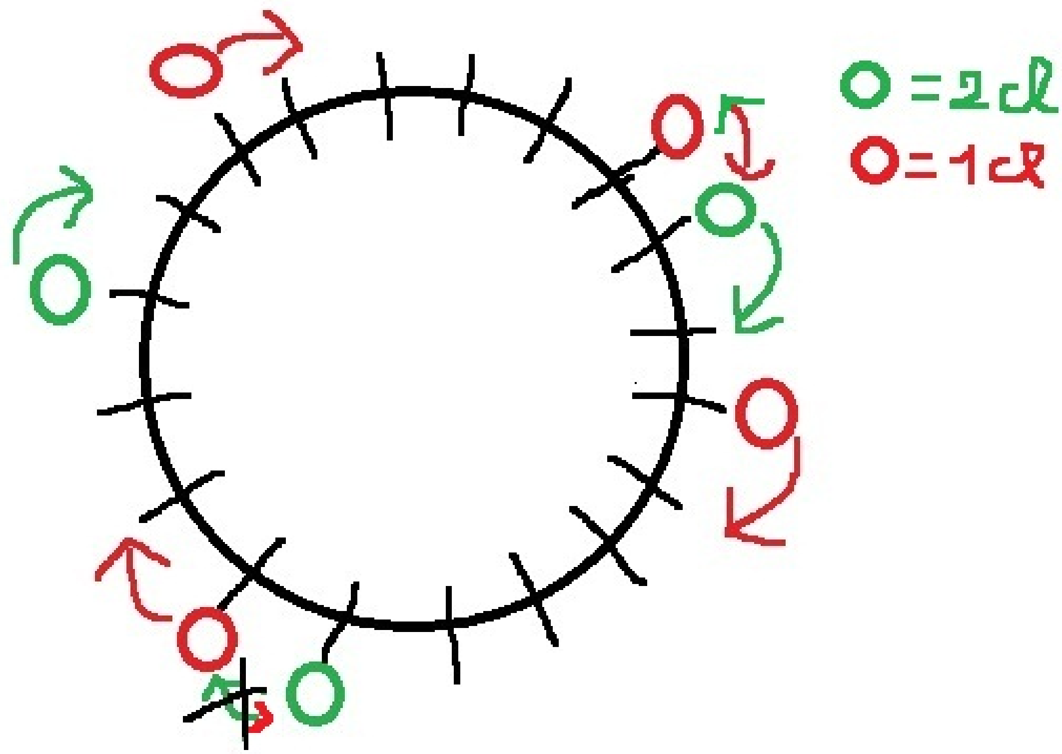
2-class TASEP

$\Lambda_N = \mathbb{T}_N$, $\eta_x \in \{0, 1, 2\}$

$$\eta_x = \begin{cases} 0 & \text{empty site} \\ 1 & \text{first class particle} \\ 2 & \text{second class particle} \end{cases}$$

two conserved quantities

Invariant measure $\rightarrow$ algorithmic combinatorial representation (O. Angel, Ferrari-Martin)


O = 2 cl
O = 1 cl

N_1 first class particles, N_2 second class particles

$$\text{configurations} \quad \eta \Leftrightarrow (\xi^1, \xi^2) \quad \xi^1 \leq \xi^2$$

$\xi_x^i \in \{0, 1\}$

$\xi_x^1 = 1$ if there is a first class particle at x

$\xi_x^2 = 1$ if there is a particle at x

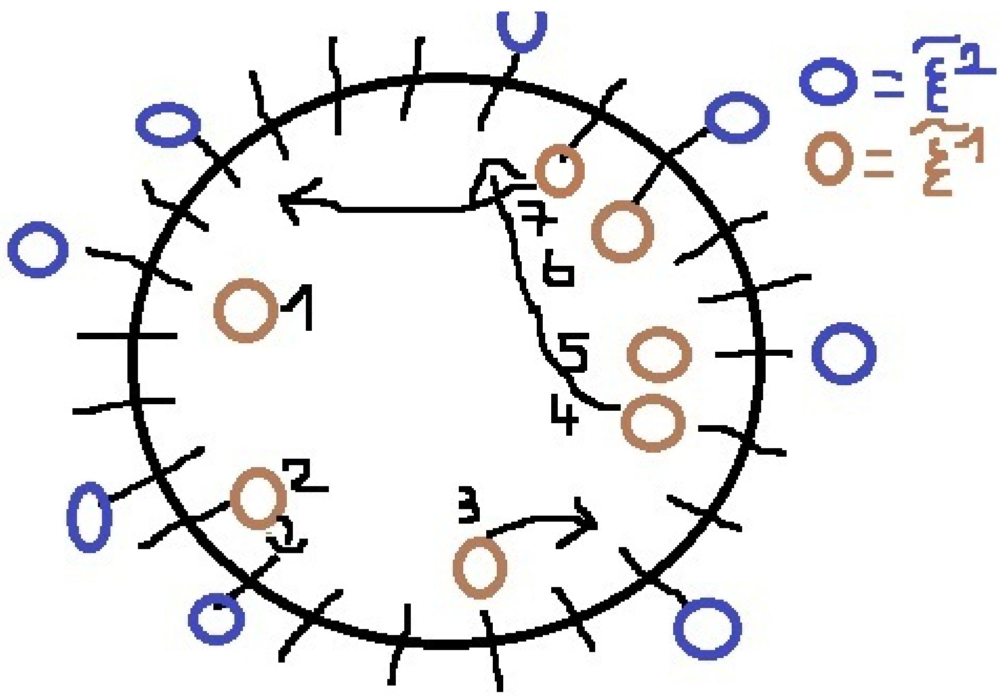
$$\sum_{x \in \mathbb{T}_N} \xi_x^1 = N_1 \qquad \sum_{x \in \mathbb{T}_N} \xi_x^2 = N_1 + N_2$$

The invariant measure

Generate $\tilde{\xi}^1$ with N_1 particles uniformly

Independently generate $\tilde{\xi}^2$ with $N_1 + N_2$ particles uniformly

$(\tilde{\xi}^1, \tilde{\xi}^2)$ collapsing procedure $\rightarrow (\xi^1, \xi^2)$ distributed according to μ_N


7
6
1
5
4
2
3

flux across $(x, x+1)$

$$J(x) = \sup_{y \in \mathbb{T}_N} \left[\sum_{z \in [y,x]} \left(\tilde{\xi}_z^1 - \tilde{\xi}_z^2 \right) \right]_+$$

Collapsed configuration $(\xi^1, \xi^2) = \mathcal{C}(\tilde{\xi}^1, \tilde{\xi}^2)$.

We have $\xi^2 = \mathcal{C}(\tilde{\xi}^2) = \tilde{\xi}^2$. $\xi^1 = \mathcal{C}(\tilde{\xi}^1)$ is defined by

$$\sum_{z \in [a,b]} \xi_z^1 = \sum_{z \in [a,b]} \tilde{\xi}_z^1 + J(a-1) - J(b)$$

continuity equation

Collapsing procedure for positive measures on $\mathbb{T}$

$\tilde{\rho_1}, \tilde{\rho_2} \in \mathcal{M}^+(\mathbb{T})$ such that:

$$\int_{[0,1]} d\tilde{\rho}_1 \leq \int_{[0,1]} d\tilde{\rho}_2$$

$$(\tilde{\rho}_1, \tilde{\rho}_2) \to \mathcal{C}(\tilde{\rho}_1, \tilde{\rho}_2) = (\rho_1, \rho_2)$$

Definition

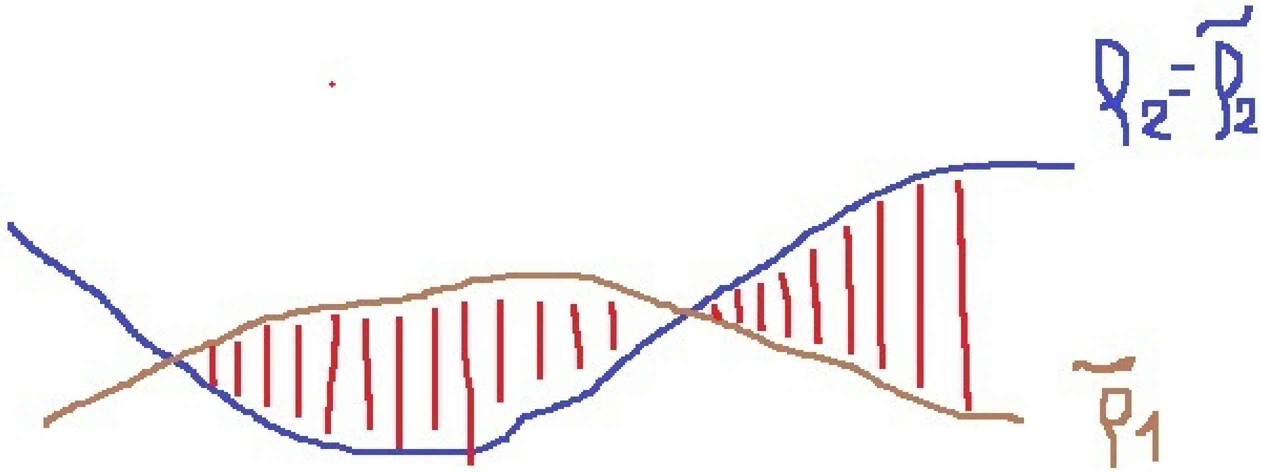
$$\int_{(a,b]} d\rho_1 = \int_{(a,b]} d\tilde{\rho}_1 + J(a) - J(b)$$

where

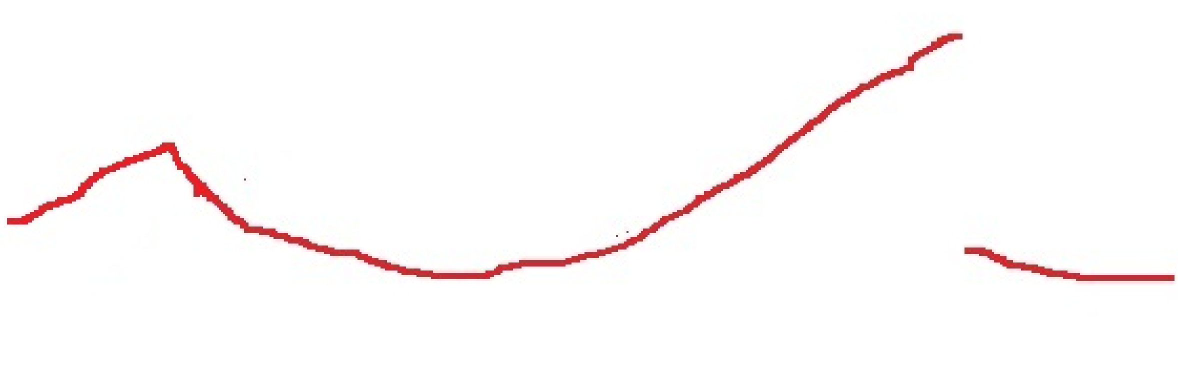
$$J(u) := \sup_v \left[\int_{(v,u]} d\tilde{\rho}_1 - \int_{(v,u]} d\tilde{\rho}_2 \right]_+$$

$$\rho_2 = \tilde{\rho}_2$$

Note that $\rho_1 \preceq \rho_2$



$\rho_1 = C(\tilde{\rho}_1)$



(G) when $N \to +\infty$ and $\frac{N_1}{N} \to \alpha$, $\frac{N_2}{N} \to \beta$

$\left(\pi_N(\tilde{\xi}^1), \pi_N(\tilde{\xi}^2)\right)$ satisfies LDP with rate

$$\widetilde{S}(\tilde{\rho}_1, \tilde{\rho}_2) = \int_{[0,1]} h_\alpha(\tilde{\rho}_1(u))du + \int_{[0,1]} h_{\alpha+\beta}(\tilde{\rho}_2(u))du$$

where

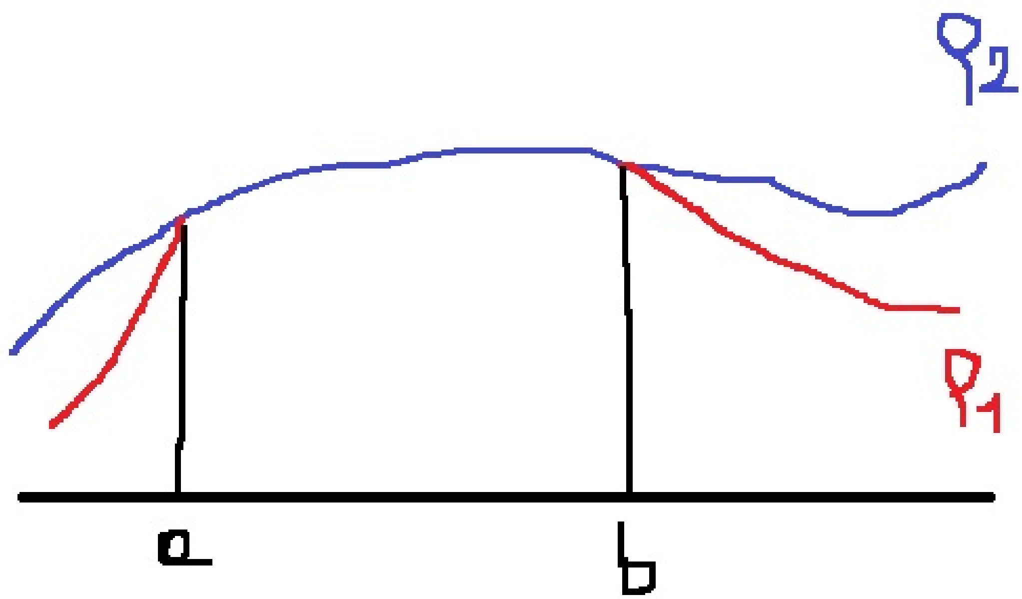
$$\int_{[0,1]} \tilde{\rho}_1(u)du = \alpha\,, \qquad \int_{[0,1]} \tilde{\rho}_2(u)du = \alpha + \beta$$

Generalized contraction principle

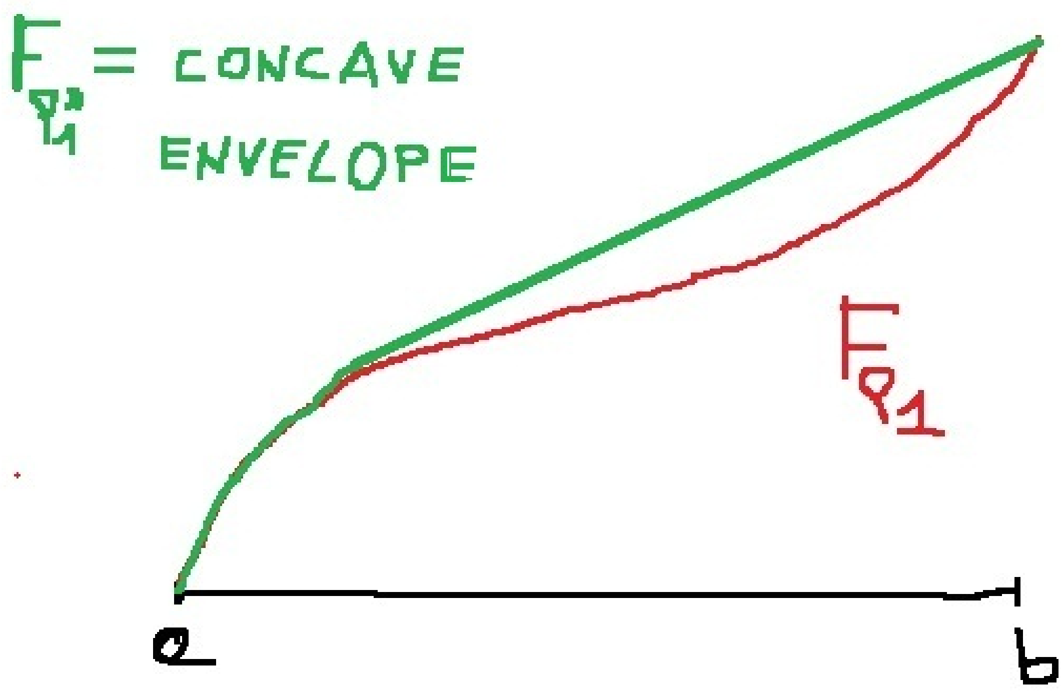
$\left(\pi_N(\xi^1), \pi_N(\xi^2)\right)$ satisfies LDP with rate

$$S(\rho_1, \rho_2) = \inf_{\{\tilde{\rho}_1 \,:\, \mathcal{C}(\tilde{\rho}_1) = \rho_1\}} \widetilde{S}(\tilde{\rho}_1, \rho_2) = \widetilde{S}(\rho_1^*, \rho_2)$$

Not convex !



ρ_2
ρ_1
a
b



$F_{\rho_1}^{**}$ = CONCAVE ENVELOPE
F_{ρ_1}
a
b