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Crumpled triangulations and critical points in 4D simplicial quantum gravity

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Abstract

We estimate analytically the critical coupling separating the weak and the strong coupling regime in 4D simplicial quantum gravity to be located at $k_2^{\rm crit} \simeq 1.3093$. By carrying out a detailed geometrical analysis of the strong coupling phase we argue that the distribution of dynamical triangulations with singular vertices and singular edges, dominating in such a regime, is characterized by distinct subdominating peaks. The presence of such peaks generates volume-dependent pseudo-critical points: $k_2^{\rm crit}(N_4 = 32\,000) \simeq 1.25795$, $k_2^{\rm crit}(N_4 = 48\,000) \simeq 1.26752$, $k_2^{\rm crit}(N_4 = 64\,000) \simeq 1.27466$, etc., which appear in good agreement with available Monte Carlo data. Under a certain scaling hypothesis we analytically characterize the (canonical) average value, $c_1(N_4; k_2) = \langle N_0 \rangle / N_4$, and the susceptibility, $c_2(N_4; k_2) = (\langle N_0^2 \rangle - \langle N_0 \rangle^2)/N_4$, associated with the vertex distribution of the 4D triangulations considered. Again, the resulting analytical expressions are found in quite a good agreement with their Monte Carlo counterparts. © 1999 Elsevier Science B.V.

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1. Introduction

In this article we shall characterize analytically the critical point separating the weak and the strong coupling regime in 4D-simplicial quantum gravity by locating it at $k_2^{\text{crit}} \simeq 1.3093$. The elementary techniques we develop here will allow us to get a rather detailed understanding of the geometry and the physics of the strong coupling phase of the theory. In particular we will show that the dynamics in such a phase is influenced by the presence of peaks in the distribution of singular triangulations. These latter are combinatorial manifolds characterized by the presence of vertices shared by a number of simplices diverging linearly with the volume of the triangulation, and possibly connected by a subsingular edge. The peaks in question are parameterized by the fraction of total volume which is allocated around such singular vertices. In order of decreasing entropic relevance, the peaks are found, according to a well-defined pattern, at $k_2 \simeq 1.24465$, $k_2 \simeq 1.2744$, $k_2 \simeq 1.2938$, $k_2 \simeq 1.30746$, $k_2 \simeq 1.31762$, $k_2 \simeq 1.32545$, etc.., asymptotically fading towards the weak coupling regime. By exploiting simple entropic arguments drawing from our recent work [1,2] and by making use of a certain scaling hypothesis, we show how such collection of subdominating sets of singular triangulations significantly affects the dynamics of the transition between weak and strong coupling. We hope that our work offers the possibility of making progress in understanding the nature of the transition, one of the major issues which controls the validity of dynamical triangulations as the basis of a regularization scheme for gravity. Before embarking on this analysis we offer some general motivation for such a study.

1.1. The Model

Let M be a closed n-dimensional ($n \geqslant 2$) manifold of given topology. Let Riem(M) and Diff(M) respectively denote the space of Riemannian metrics g on M, and the group of diffeomorphisms on M. In the continuum formulation of Euclidean quantum gravity one attempts to give meaning to a formal path integration over Diff(M) equivalence classes of metrics in Riem(M):

$$Z(\Lambda, G, M) = \int_{\text{Riem}(M)/\text{Diff}(M)} \mathcal{D}[g(M)]\, e^{-S_g[\Lambda,G,\Sigma]}, \tag{1}$$

where $S_g[\Lambda, G, \Sigma]$ is the Einstein–Hilbert action associated with the Riemannian manifold (M, g), viz.,

$$S_g[\Lambda, G, \Sigma] = \Lambda \int_M d^n\xi \sqrt{g} - \frac{1}{16\pi G} \int_M d^n\xi \sqrt{g}\, R \tag{2}$$

and $\mathcal{D}[g(M)]$ is some a priori distribution on Riem(M)/Diff(M) describing the strong coupling statistics ($\Lambda \to 0$, $G \to \infty$) of the set of Riemannian manifolds $\{(M, g)\}$ considered. We avoid here discussing well-known specific pathologies in dealing with (1) and about which the reader can find abundant literature, and simply recall that in the dynamical triangulations approach to quantum gravity one attempts to give meaning to (1) by replacing the continuum Riemannian manifold (M, g) with a piecewise-linear (PL) manifold (still denoted by M) endowed with a triangulation $T_a \to M$ generated by gluing a (large) number of equilateral n-simplices σ^n. One approximates Riemannian structures by means of such triangulated manifolds by using a representative metric where each simplex σ^n is a Euclidean equilateral simplex with sides of length a (typically we set $a = 1$). This metric is locally Euclidean everywhere on the PL manifold except near the $(n-2)$ subsimplices σ^{n-2} (the *bones*) where the sum of the dihedral angles, $\theta(\sigma^n)$, of the incident σ^n's is in excess (negative curvature) or in defect (positive curvature) with respect to the 2π flatness constraint, the corresponding deficit angle r being defined by $r = 2\pi - \sum_{\sigma^n} \theta(\sigma^n)$. If K^{n-2} denotes the $(n-2)$-skeleton of $T \to M^n$, then $M^n \backslash K^{n-2}$ is a flat Riemannian manifold, and any point in the interior of an r-simplex σ^r has a neighborhood homeomorphic to $B^r \times C(\mathrm{link}(\sigma^r))$, where B^r denotes the ball in $\mathbb{R}^n$ and $C(\mathrm{link}(\sigma^r))$ is the cone over the link $\mathrm{link}(\sigma^r)$ (the product $\mathrm{link}(\sigma^r) \times [0,1]$ with $\mathrm{link}(\sigma^r) \times \{1\}$ identified to a point). Note that for dynamical triangulations the deficit angles are generated by the string of integers, the *curvature assignments*, $\{q(k)\}_{k=0}^{N_{n-2}-1}$ providing the numbers of top-dimensional simplices incident on the N_{n-2} distinct bones, viz., $r(i) = 2\pi - q(i) \arccos(1/n)$.

By specializing to this setting the standard Regge calculus, the formal path integration (1) is replaced on a dynamically triangulated PL manifold M (of fixed topology), by the (grand-canonical) partition function [3,2,4]

$$Z[k_{n-2}, k_n] = \sum_{T \in \mathcal{T}(M)} \frac{1}{C_T} e^{-k_n N_n + k_{n-2} N_{n-2}}, \tag{3}$$

where k_{n-2} and k_n are two (running) couplings, the former proportional to the inverse gravitational coupling $1/G$, while the latter is a linear combination of $1/16\pi G$ and of the cosmological constant Λ. The summation in (3) is extended to the set $\{\mathcal{T}(M)\}$ of all distinct dynamical triangulations the PL manifold M can support, and it is weighted by the symmetry factor, C_T, of the triangulation: the order of the automorphism group of the graph associated with the triangulation T. Since symmetric triangulations are the exception rather then the rule, we shall assume $C_T = 1$ in the estimates of the partition functions below. Thus, in the following we will omit the symmetry factor when writing the partition function.

One can introduce also the *canonical* partition function defined by

$$W(k_{n-2})_{\mathrm{eff}} = \sum_{T \in \mathcal{T}(N_n)} e^{k_{n-2} N_{n-2}}, \tag{4}$$

where the summation is extended over all distinct dynamical triangulations with given N_n (i.e. at fixed volume) of a given PL manifold M. Finally, we shall consider the *micro-canonical* partition function

$$W[N_{n-2}, b(n, n-2)] = \sum_{T \in \mathcal{T}(N_n; N_{n-2})} 1, \tag{5}$$

where the summation is extended over all distinct dynamical triangulations with given N_n and N_{n-2}, i.e. at fixed volume and fixed *average incidence*

$$b(n,n-2) = \tfrac{1}{2}n(n+1)(N_n/N_{n-2}), \tag{6}$$

of a given PL manifold M. The micro-canonical partition function is simply the number of distinct dynamical triangulations with given volume ($\propto N_n$) and fixed average curvature ($\propto b(n,n-2)$), of a given PL manifold M. In other words, $W[N_{n-2}, b(n,n-2)]$ is the *entropy function* for the given set of dynamical triangulations: it provides the discretized counterpart of the a priori distribution $\mathcal{D}[g(M)]$ describing the strong coupling statistics ($k_n \to 0$, $k_{n-2} \to 0$) of the set of Riemannian manifolds $\{(M,g)\}$.

Recently there have been a number of significant advances in 3- and 4-dimensional simplicial quantum gravity that fit together in a coherent whole; roughly speaking, these results are related to (i) a deeper understanding of the geometry of $n \geqslant 3$-dimensional dynamical triangulations; (ii) the study of simpler models mimicking quite accurately the critical structure of simplicial quantum gravity, and (iii) more refined computer simulations of the phase structure of the 4-dimensional theory. These results imply that both in dimension $n = 3$ and $n = 4$, simplicial quantum gravity has two geometrically distinct phases parameterized by the value of the inverse gravitational coupling k_{n-2}. In the weak coupling phase (large values of k_{n-2}) we have a dominance of PL manifolds which collapse to branched polymer structures with an Hausdorff dimension $d_H = 2$ and an entropy exponent (analogous to the string susceptibility of the 2-dimensional theory) $\gamma = 1/2$. In this phase the theory has a well defined continuum limit which is independent of any fine tuning of the (inverse) gravitational constant k_{n-2}. We are not really interested in this continuum limit, even if it exists. The situation is not unusual from the point of view of lattice theories. For instance in compact $U(1)$ gauge theories one hits the trivial Coulomb phase for all $\beta > \beta_0$. In the strong coupling phase (small values of k_{n-2}), we have a dominance of crumpled manifolds: the typical configuration sampled by the computer simulations is that of a triangulation with a few vertices on which most of the top dimensional simplices are incident, such presence of *singular vertices*, typically connected by a subsingular edge, seems to be a signature of the strong coupling phase [5]. Note that the word *singular vertex* is used in DT theory with a meaning quite different from the accepted meaning adopted in PL geometry. What is actually meant is that a metric ball around any such a vertex, of radius equal to the given lattice spacing, has a volume that grows proportionally to the volume of the whole PL manifold. This behavior indicates that the Hausdorff dimension of the typical triangulation in the strong coupling phase is very large if not infinite. There is strong evidence that the transition between weak and strong coupling, marked by a critical value k^{crit}_{n-2}, is of a first-order nature in the $n = 3$-dimensional case. In dimension $n = 4$, the original numerical simulations seemed to indicate a second-order nature of the transition, a result that invited positive speculations on the possibility that simplicial quantum gravity could indeed provide a reliable regularization of Euclidean quantum gravity. However, recent and more accurate analyses [6] of the Monte Carlo simulations seem rather to point toward a first-order nature of the transition. These results are not definitive since the latent heat at the critical point is very small as compared to the 3D case, so that the question remains open whether any further increase in the sophistication of the simulations will definitively establish, within the limit of the reached accuracy, the nature of the transition. In any case, all recent numerical results [7] strongly indicates that the phase transition in 4-dimensional simplicial gravity is associated with the creation of singular geometries. These results have put to the fore the basic problems of theory, and in this sense important questions abound: What is the geometrical nature of the crumpled phase? What is the mechanism driving the transition from the polymer phase to the crumpled phase? Is there any geometrical intuition behind the first or higher order nature of the transition? Is it possible to take dynamical control over the occurrence of singular vertices and edges which otherwise would entropically dominate?

Some of these questions can be systematically addressed by a detailed but otherwise elementary discussion of the geometry of dynamical triangulations along the lines of [1,2]. This geometrical viewpoint has turned out to be useful and interesting in terms of providing an analytical framework within which discuss simplicial quantum gravity while at the same time maintaining a strong contact with computer simulations. In this paper we discuss in detail how this approach can be further exploited to estimate analytically the value of k_2^{crit}, and describing the geometrical properties of the strong coupling phase of the 4-dimensional theory.

2. Large volume asymptotics of the partition functions

For the convenience of the reader we recall in this section a few basic results of [1] that we are going to use later on.

The discretized distribution $W[N_{n-2}, b(n,n-2)]$ is one of the objects of main interest in simplicial quantum gravity, and for $n \geqslant 3$ an exact evaluation of $W[N_{n-2}, b(n,n-2)]$ is an open and very difficult problem. However, since in (3) and (4) one is interested in the large volume limit, what really matters, as far as the criticality properties of (3) are concerned, is the asymptotic behavior of $W[N_{n-2}, b(n,n-2)]$ for large N_n. This makes the analysis of $W[N_{n-2}, b(n,n-2)]$ somewhat technically simpler, and according to [1] one can actually estimate its leading asymptotics with the relevant subleading corrections. If we consider an n-dimensional ($n \geqslant 2$) PL manifold M of given fundamental group $\pi_1(M)$), then the distribution $W[N_{n-2}, b(n,n-2)]$ of distinct dynamical triangulations, with given N_{n-2} bones and average curvature $b \equiv b(n,n-2)$, factorizes according to

$$W[N_{n-2}, b(n,n-2)] = p^{\text{curv}}_{N_{n-2}} \langle \text{Card}\{T_a^{(i)}\}_{\text{curv}} \rangle, \tag{7}$$

where $p^{\text{curv}}_{N_{n-2}}$ is the number of possible distinct curvature assignments $\{q(\alpha)\}_{\alpha=0}^{N_{n-2}}$ for triangulations $\{T_a\}$ with N_{n-2} bones and given average incidence $b(n,n-2)$, viz.,

$$\{q(\alpha)\}_{\alpha=0}^{N_{n-2}} \neq \{q(\beta)\}_{\beta=0}^{N_{n-2}} \neq \{q(\gamma)\}_{\gamma=0}^{N_{n-2}} \neq \dots, \tag{8}$$

while $\langle \text{Card}\{T_a^{(i)}\}_{\text{curv}} \rangle$ is the average (with respect to the distinct curvature assignments) of the number of distinct triangulations sharing a common set of curvature assignments, (for details, see Section 5.2, pp. 102 of Ref. [1]). This factorization allows for a rather straightforward asymptotic analysis of $W[N_{n-2}, b(n,n-2)]$, and in the limit of large N_n we get [1] (Theorem 6.4.2, pp. 131 with the expression $\langle \mathrm{Card}\{T_a^{(i)}\}_{\mathrm{curv}}\rangle$ provided by Theorem 5.2.1, pp. 106)

$$W[N_{n-2}, b(n,n-2)] \simeq \frac{W_\pi}{\sqrt{2\pi}} e^{(\alpha_n b(n,n-2)+\alpha_{n-2})N_{n-2}} \times \sqrt{\frac{(b-\hat{q}+1)^{1-2n}}{(b-\hat{q})^3}} \left[\frac{(b-\hat{q}+1)^{b-\hat{q}+1}}{(b-\hat{q})^{b-\hat{q}}}\right]^{N_{n-2}} \times e^{[-m(b(n,n-2))N_n^{1/n_H}]} \left(\frac{b(n,n-2)}{n(n+1)} N_{n-2}\right)^{D/2} N_{n-2}{}^{\tau(b)-\frac{2n+3}{2}}. \tag{9}$$

The notation here is the following. W_π is a topology-dependent parameter of no importance for our present purposes (see Ref. [1] for its explicit expression), α_{n-2} and α_n are two constants depending on the dimension n (for instance, for $n = 4$, $\alpha_n = -\arccos(1/n)|_{n=4}$, $\alpha_{n-2} = 0$); $\hat{q}$ is the minimum incidence order over the bones (typically $\hat{q} = 3$); $D \doteq \dim[\mathrm{Hom}(\pi_1(M), G)]$ is the topological dimension of the representation variety parameterizing the set of distinct dynamical triangulations approximating locally homogeneous G-geometries, $(G \subset SO(n))$. Finally, $m(b) \geqslant 0$ and $\tau(b) \geqslant 0$ are two parameters depending on $b(n,n-2)$ which, together with $n_H > 1$, characterize the subleading asymptotics of $W[N_{n-2}, b(n,n-2)]$. In particular, note that

$$e^{[-m(b(n,n-2))N_n^{1/n_H}]} \left(\frac{b(n,n-2)}{n(n+1)} N_{n-2}\right)^{D/2} N_{n-2}{}^{\tau(b)} \tag{10}$$

is the asymptotics associated with $\langle \mathrm{Card}\{T^{(i)}\}_{\mathrm{curv}}\rangle$. The remaining part of (9) is the leading exponential contribution coming from the large N_n behavior of the distribution $p_{N_{n-2}}^{\mathrm{curv}}$ of the possible curvature assignments. This latter term provides the correct behavior of the large volume limit of dynamically triangulated manifolds, an asymptotics that matches nicely with the existing Monte Carlo simulations (see, e.g., Ref. [1], Section 7.1, pp. 160). While the exponential asymptotics is basically under control, it must be stressed that some of the most delicate aspects of the theory are actually contained in $\langle \mathrm{Card}\{T^{(i)}\}_{\mathrm{curv}}\rangle$. Roughly speaking, the set of triangulations $\{T^{(i)}\}_{\mathrm{curv}}$ can be rather directly interpreted as a finite-dimensional approximation of the *moduli space* of constant curvature metrics in smooth Riemannian geometry (the standard example being the parameterization of the moduli space of surfaces of genus $\geqslant 2$ with the set of inequivalent constant curvature ($= -1$) metrics admitted by such surfaces; due to rigidity phenomena, such example is to be taken with care for $n \geqslant 3$). With this geometric interpretation in mind we can consider

$$\frac{\ln\langle \mathrm{Card}\{T^{(i)}\}_{\mathrm{curv}}\rangle}{\ln N_n} \simeq -\frac{m(b(n,n-2))N_n^{1/n_H}}{\ln N_n} + \frac{D}{2} + \tau(n) \quad \text{for } N_n \to \infty \tag{11}$$

as the (formal) covering dimension of such a moduli space. Clearly, whenever $m(b) > 0$ (and n_H finite or $O(\ln N_n)$), such covering dimension is singular. This signals the fact that in the corresponding range of the parameter b, dynamical triangulations fail to approximate in the large volume limit any smooth Riemannian manifold.

2.1. The critical incidence

The parameters $m(b)$, $\tau(b)$, and n_H, characterizing the covering dimension and the large volume asymptotics (10) are not yet explicitly provided by the analytical results of [1]. By exploiting geometrical arguments, one can only prove [1] (Theorem 5.2.1, pp. 106) an existence result to the effect that if $n \geqslant 3$, there is a *critical value* $b_0(n)$, of the average incidence $b(n, n-2)$, to which we can associate a critical value k^{crit}_{n-2} of the inverse gravitational coupling, such that

$$m(b) = 0, \tag{12}$$

for $b(n, n-2) \leqslant b_0(n)$; whereas

$$m(b) > 0, \tag{13}$$

for $b_0(n) < b(n, n-2)$. In other words, for $b < b_0(n)$ the subleading asymptotics in (9) is at most polynomial, whereas for $b > b_0(n)$ this asymptotics becomes subexponential as N_n goes to infinity, (note that in the 2-dimensional case (9) has always a subleading polynomial asymptotics).

This change in the subleading asymptotics qualitatively accounts for the jump from the strong to the weak coupling phase observed in the real system during Monte Carlo simulations. However, the lack of an explicit expression for $m(b)$ hampers a deeper analysis of the nature of this transition. In particular, one is interested in the way the parameter $m(b(n, n-2))$ approaches 0 as $b(n, n-2) \to b_0(n)$, since adequate knowledge in this direction would provide the order of the phase transition. It is clear that a first necessary step in order to discuss the properties of $m(b(n, n-2))$ is to provide a constructive geometrical characterization of the critical average incidence $b_0(n)$, and not just an existence result.

As far as the other parameter $\tau(n)$ is concerned, the situation is on more firm ground. $\tau(n)$ characterizes the subleading polynomial asymptotics in the weak coupling phase, and recently [8], an analysis of the geometry of dynamical triangulations in this phase has provided convincing analytical evidence that $\tau(n) - (2n+3)/2 + 3 = 1/2$. As expected, this corresponds to a dominance, in the weak coupling phase, of branched polymers structures.

2.2. Canonical averages and the curvature susceptibility

If we consider the weighted distribution $W[N_{n-2}, b(n, n-2)] \exp[k_{n-2}N_{n-2}]$, characterizing the canonical partition function (4), then it is straightforward to check that, as $N_n \to \infty$, this distribution is strongly peaked around triangulations with an average incidence $b^*(n, n-2; k_{n-2})$ given by (see Ref. [1], Eq. (6.39), pp. 134)

$$b^*(n, n-2; k_{n-2}) = 3\left(\frac{A(k_{n-2})}{A(k_{n-2}) - 1}\right), \tag{14}$$

where for notational convenience we have set

$$A(k_{n-2}) \doteq \left[\frac{27}{2}e^{k_{n-2}} + 1 + \sqrt{\left(\frac{27}{2}e^{k_{n-2}} + 1\right)^2 - 1}\right]^{1/3} + \left[\frac{27}{2}e^{k_{n-2}} + 1 - \sqrt{\left(\frac{27}{2}e^{k_{n-2}} + 1\right)^2 - 1}\right]^{1/3} - 1. \tag{15}$$

This remark allows us to compute, via a uniform Laplace estimation, the large volume asymptotics of the canonical partition function $W(k_{n-2})_{\mathrm{eff}} = \sum_{T \in \mathcal{T}(N_n)} e^{k_{n-2} N_{n-2}}$. A discussion of this asymptotics at the various orders is rather delicate and the reader can find the details in Ref. [1] (Ch. 6, Theorem 6.6.1). For our purposes it is sufficient to consider the leading order expression which can be readily obtained, starting from the micro-canonical partition function, by means of a standard saddle point evaluation, viz.

$$W(N_n, k_{n-2})_{\mathrm{eff}} = c_n \left(\frac{(A(k_{n-2}) + 2)}{3A(k_{n-2})}\right)^{-n} N_n^{\tau(n) + D/2 - n - 1} e^{[-m(b^*(n,n-2;k_{n-2}))N_n^{1/nH}]} \times e^{\left[\left[\frac{1}{2}n(n+1)\ln\frac{A(k_{n-2})+2}{3}\right]N_n\right]}\left(1 + O(N_n^{-3/2})\right), \tag{16}$$

where c_n is a scaling factor not depending on k_2.

As the inverse gravitational coupling k_{n-2} varies, the average curvature correspondingly changes according to (14). It follows that there is a well-defined critical value, k_{n-2}^{crit}, solution of the equation

$$b_0(n) = 3\left(\frac{A(k_{n-2})}{A(k_{n-2}) - 1}\right), \tag{17}$$

for which $b^*(n, n-2; k_{n-2}) = b_0(n)$, where $b_0(n)$ is the critical average incidence (see (12) and (13)). This k_{n-2}^{crit} describes the transition between the strong coupling phase ($k_{n-2} < k_{n-2}^{\mathrm{crit}}$) and the weak coupling phase ($k_{n-2} > k_{n-2}^{\mathrm{crit}}$) associated with the two distinct subleading asymptotics regimes of (16). The explicit geometric characterization of $b_0(n)$ and the evaluation of the corresponding critical value k_2^{crit} in the 4-dimensional case are among the most important issues that we discuss in this paper. In order to compare the geometrical results we obtain with the data coming from recent Monte Carlo simulation it will be useful to have at hand the expressions of the free energy $\ln W_{\mathrm{eff}}(N_4, k_2)$, of the (canonical) average of the number of bones $\langle N_2 \rangle = \partial \ln W_{\mathrm{eff}}(N_4, k_2)/\partial k_2$, and of the associated *curvature-curvature* correlator $[\langle N_2^2 \rangle - \langle N_2 \rangle^2] = \partial^2 \ln W_{\mathrm{eff}}(N_4, k_2)/\partial k_2{}^2$. The large volume asymptotics of the (canonical) free energy is readily obtained from (16); by setting $n = 4$ and discarding the unessential constant terms we get (in the saddle-point approximation used above)

$$\ln W(N, k_2)_{\text{eff}} = 10N_4 \ln \frac{A(k_2)+2}{3} - m(k_2)N^{1/n_H}. \tag{18}$$

The canonical average $\langle N_2 \rangle$ follows from differentiating (18) with respect to the inverse gravitational coupling k_2,

$$\langle N_2 \rangle = 10\frac{A_1(k_2)}{A(k_2)+2}N_4 - N_4^{1/n_H}\frac{\partial m(k_2)}{\partial k_2}, \tag{19}$$

where we have set

$$\begin{aligned} A_1(k_2) &\doteq \frac{\partial A(k_2)}{\partial k_2} \\ &= \frac{9}{2}e^{k_2}\left[1+\frac{\frac{27}{2}e^{k_2}+1}{\sqrt{\left(\frac{27}{2}e^{k_2}+1\right)^2-1}}\right]\left[\frac{27}{2}e^{k_2}+1+\sqrt{\left(\frac{27}{2}e^{k_2}+1\right)^2-1}\right]^{-2/3} \\ &\quad +\frac{9}{2}e^{k_2}\left[1-\frac{\frac{27}{2}e^{k_2}+1}{\sqrt{(\frac{27}{2}e^{k_2}+1)^2-1}}\right]\left[\frac{27}{2}e^{k_2}+1-\sqrt{(\frac{27}{2}e^{k_2}+1)^2-1}\right]^{-2/3}. \end{aligned} \tag{20}$$

Note that for a 4-dimensional PL manifold M of Euler characteristic $\chi(M)$, we have $N_0 = \frac{N_2}{2} - N_4 + \chi$. Thus, from $\langle N_2 \rangle$ we immediately get the expression for the first normalized cumulant of the distribution of the number of vertices of the triangulation, viz.

$$c_1(N_4; k_2) \doteq \frac{\langle N_0 \rangle}{N_4} = \frac{5A_1(k_2)}{A(k_2)+2} - 1 - N_4^{1/n_H-1}\frac{\partial m(k_2)}{\partial k_2}. \tag{21}$$

This is a typical quantity monitored in Monte Carlo simulation, and later on we will discuss how (21) actually compares with respect to existing numerical data.

Finally, the curvature susceptibility

$$\frac{\langle N_2^2 \rangle - \langle N_2 \rangle^2}{N_4} = \frac{1}{N_4}\frac{\partial^2 \ln W_{\text{eff}}(N_4, k_2)}{\partial {k_2}^2} \tag{22}$$

is explicitly computed as

$$\begin{aligned} 4c_2(N_4; k_2) &= \frac{\langle N_2^2 \rangle - \langle N_2 \rangle^2}{N_4} \\ &= 10\frac{(A(k_2)+2)A_2(k_2) - A_1(k_2)^2}{(A(k_2)+2)^2} - N_4^{1/n_H-1}\frac{\partial^2 m(k_2)}{\partial {k_2}^2}, \end{aligned} \tag{23}$$

where $c_2(N_4; k_2) \doteq (\langle N_0^2 \rangle - \langle N_0 \rangle^2)/N_4$ is the second normalized cumulant of the distribution of the number of vertices of the triangulation, and where we have set

$$
\begin{aligned}
A_2(k_2) \doteq \frac{\partial^2 A(k_2)}{\partial {k_2}^2} &= A_1(k_2) \\
&+ \frac{\frac{243}{4}e^{2k_2}}{[(\frac{27}{2}e^{k_2}+1)^2-1]^{3/2}}\left[\frac{27}{2}e^{k_2}+1-\sqrt{(\frac{27}{2}e^{k_2}+1)^2-1}\right]^{-2/3} \\
&- \frac{\frac{243}{4}e^{2k_2}}{[(\frac{27}{2}e^{k_2}+1)^2-1]^{3/2}}\left[\frac{27}{2}e^{k_2}+1+\sqrt{(\frac{27}{2}e^{k_2}+1)^2-1}\right]^{-2/3} \\
&- \frac{81}{2}e^{2k_2}\left[\frac{27}{2}e^{k_2}+1+\sqrt{(\frac{27}{2}e^{k_2}+1)^2-1}\right]^{-5/3}\left[1+\frac{\frac{27}{2}e^{k_2}+1}{\sqrt{(\frac{27}{2}e^{k_2}+1)^2-1}}\right]^2 \\
&- \frac{81}{2}e^{2k_2}\left[\frac{27}{2}e^{k_2}+1-\sqrt{(\frac{27}{2}e^{k_2}+1)^2-1}\right]^{-5/3}\left[1-\frac{\frac{27}{2}e^{k_2}+1}{\sqrt{(\frac{27}{2}e^{k_2}+1)^2-1}}\right]^2 .
\end{aligned}
\tag{24}
$$

2.3. A scaling hypothesis

Clearly the above expressions for $c_1(N_4;k_2)$ and $c_2(N_4;k_2)$ are useless if we do not specify how $m(k_2)$ depends on the inverse gravitational coupling k_2. Since according to Theorem 5.2.1 of Ref. [1], $m(k_2) \to 0$ as $b(k_2)$ approaches a critical incidence $b_0(4)$ (henceforth denoted by b_0) the simplest *hypothesis* we can make is that, for $(b(k_2) - b_0) \to 0^+$, $m(k_2)$ scales to zero according to a power law given by

$$
m(k_2) = \frac{1}{\nu}\left(\frac{1}{b(k_2)} - \frac{1}{b_0}\right)^{\nu}, \tag{25}
$$

where $0 < \nu < 1$ is a critical exponent to be determined (the factor $1/\nu$ is inserted for later convenience).

The expression (16) for the canonical partition function $W[N_{n-2}, b(n, n-2)]$ is a large volume (N_4) asymptotics evaluated at fixed volume. It contains a non-trivial subleading asymptotics governed by $m(k_2)$. The net effect of this subleading term is clearly visible in the expressions (21) and (23) of the two cumulants, and shows that in order to capture the behavior of $c_1(N_4;k_2)$ and $c_2(N_4;k_2)$ as $k_2 \to {k_2}^{\rm crit}$, (25) is not sufficient. It must be combined with a finite scaling hypothesis telling us how $m(k_2)$ scales with the volume N_4, as $b(k_2) \to b_0$. From the asymptotics (16), and the expression (21) for the first cumulant, it easily follows that the simplest, if not the most natural, hypothesis we can make is to assume that $m(k_2)$ scales asymptotically with the volume according to

$$
\lim_{\substack{N_4\to\infty \\ (k_2-k_2^{\rm crit})\to 0^-}} \left|\frac{1}{b(k_2)} - \frac{1}{b_0}\right|^{\nu-1} \cdot N_4^{\frac{1}{\nu_H}-1} = 1, \tag{26}
$$

where according to Theorem 5.2.1 of Ref. [1] $n_H > 1$. This implies that, when $b(k_2) \to b_0$, $m(k_2)$ scales as

$$m(k_2) \simeq N_4^{-\nu(n_H-1)/n_H(1-\nu)}. \tag{27}$$

Together with $m(k_2)N_4^{1/n_H} \to 0$ as $b(k_2) \to b_0$, (26) yields $(1/n_H) < \nu < 1$, a finite size scaling relation connecting the critical exponents ν and n_H.

Introducing this ansatz in (21) and (23) we explicitly obtain

$$c_1(N_4;k_2) \doteq \frac{\langle N_0\rangle}{N_4} = \frac{5A_1(k_2)}{A(k_2)+2} - \frac{1}{3}\frac{A_1(k_2)}{A^2(k_2)} - 1 \tag{28}$$

and

$$\begin{aligned} c_2(N_4;k_2) &= \frac{\langle N_0^2\rangle - \langle N_0\rangle^2}{N_4} \\ &= \frac{5}{2}\frac{(A(k_2)+2)A_2(k_2) - A_1(k_2)^2}{(A(k_2)+2)^2} - \frac{1}{12}\frac{A_2(k_2)A(k_2) - 2A_1(k_2)}{A^3(k_2)} \\ &+ \frac{|\nu-1|}{36}\frac{A_1^2(k_2)}{A^4(k_2)}\left|\frac{A(k_2)-1}{3A(k_2)} - \frac{1}{b_0}\right|^{-1}. \end{aligned} \tag{29}$$

Note that in this latter expression the only undetermined parameters are the critical exponent ν and the critical incidence b_0. In the following paragraphs we provide an explicit geometric characterization of such b_0. The only remaining unknown quantity is then ν. Anticipating the conclusion of the paper, it turns out that it is possible the choose a value of ν (≈ 0.94) which leads to quite a good agreement between (28), (29) and the available Monte Carlo data for the distributions of these two cumulants.

3. The geometry of the strong coupling phase

A PL manifold endowed with a dynamical triangulation is a particular example of an Alexandrov space, i.e. a finite dimensional, inner metric space with a lower curvature bound in distance-comparison sense (a brief introduction with the relevant references can be found in Ref. [1], Section 3.2). The natural topology specifying in which sense dynamical triangulations approximate Riemannian manifolds is associated with an Hausdorff-like distance introduced by Gromov [9], and which is a direct generalization of the classical Hausdorff distance between compact subsets of a metric space. The role of this topology stems from the fact there are many geometric constructions in dynamical triangulations theory that are close in Gromov–Hausdorff topology, but not in smooth Riemannian geometry. In Ref. [1] we proved that every Riemannian manifold (of bounded geometry) can be uniformly approximated in this topology by dynamical triangulations (see Ref. [1], Section 3.3, Theorem 3.3.1); the converse result, namely if every dynamical triangulation approximates, as the number of simplices goes to ∞, an n-dimensional Riemannian manifold, is deeply tied to understanding the structure of the thermodynamical behavior of the large volume limit of the set of possible dynamical triangulations. It is interesting to note that the geometry of the set of all possible dynamical triangulations of a manifold of given topology is a subject of which little is actually known, even in dimension two. Recently, in a remarkable paper [10], Thurston has shed some light in the two-dimensional case by showing that the space of triangulations (of positive curvature) has the rich geometric structure of a complex hyperbolic manifold. We do not need to reach, in this work, such a level of sophistication and in order to determine the critical average incidence, b_0, we discuss mostly the kinematical properties of the space of all possible dynamical triangulations admitted by an n-dimensional PL manifold M of given topology. Let (M, T_a) be a dynamically triangulated manifold, then the f-vector of the triangulation is the string of integers $(N_0(T_a), N_1(T_a), \ldots, N_n(T_a))$, where $N_i(T_a) \in \mathbb{N}$ is the number of i-dimensional subsimplices σ^i of T_a. This vector is constrained by the Dehn–Sommerville relations

$$\sum_{i=0}^{n} (-1)^i N_i(T) = \chi(T), \tag{30}$$

$$\sum_{i=2k-1}^{n} (-1)^i \frac{(i+1)!}{(i-2k+2)!(2k-1)!} N_i(T) = 0, \tag{31}$$

if n is even, and $1 \leqslant k \leqslant n/2$, whereas if n is odd

$$\sum_{i=2k}^{n} (-1)^i \frac{(i+1)!}{(i-2k+1)!2k!} N_i(T) = 0, \tag{32}$$

with $1 \leqslant k \leqslant (n-1)/2$, and where $\chi(T)$ is the Euler–Poincaré characteristic of T. It is easily verified that the relations (30), (31), (32) leave $\frac{1}{2}n - 1$ (n even) or $\frac{1}{2}(n-1)$ unknown quantities among the n ratios $N_1/N_0, \ldots, N_n/N_0$ [11]. Thus, in dimension $n = 2, 3, 4$, the datum of N_n, and of the number of bones N_{n-2}, fixes through the Dehn–Sommerville relations all the remaining $N_i(T)$. These extremely simple and perhaps even naive-sounding remarks turn out to be quite powerful in providing information on the global metrical properties of the underlying PL manifold. Not only, as is obvious, on the volume ($\propto N_n(T_a)$), and on the average curvature ($\propto \frac{1}{2}n(n+1)N_n(T_a)/N_{n-2}(T_a)$), but, corroborated by a few more elementary facts, also on the genesis of singular vertices and edges. An elementary but geometrically significant result of this type is provided by the range of variation of the possible average incidence $b(n, n-2)$. One gets (see Ref. [1] (Lemma 2.1.1))

Lemma 1. Let $T_a \to M^n$ a triangulation of a closed n-dimensional PL manifold M, with $2 \leqslant n \leqslant 4$, then for $N_n(T_a) \to \infty$, we get
(i) For $n = 2$:

$$b(2,0) = 6; \tag{33}$$

(ii) for $n = 3$:

$$\tfrac{9}{2} \leqslant b(3,1) \leqslant 6; \tag{34}$$

(iii) for $n = 4$:

$$4 \leqslant b(4,2) \leqslant 5. \tag{35}$$

The 2-dimensional case as well as the upper bounds for $n = 3$ and $n = 4$ are well-known trivial consequences of the Dehn–Sommerville relations. The lower bounds $b(3,1) \geqslant 9/2$ and $b(4,2) \geqslant 4$ are instead related to a rather sophisticated set of results proved by Walkup [12] in the sixties concerning the proof of some conjectures for 3-and 4-dimensional PL manifolds (apparently, these results went unnoticed by researchers in simplicial quantum gravity). Walkup's theorems have important implications for understanding the geometry both of the strong and of the weak coupling phase of simplicial gravity. In dimension $n = 3$, we have [12]

Theorem 1. There exists a triangulation $T \to \mathbb{S}^3$ of the 3-sphere $\mathbb{S}^3$ with N_0 vertices and N_1 edges if and only if $N_0 \geqslant 5$ and

$$4N_0 - 10 \geqslant N_1 \geqslant \frac{N_0(N_0-1)}{2}. \tag{36}$$

Moreover T is a triangulation of $\mathbb{S}^3$ satisfying $N_1 = 4N_0 - 10$ if and only if T is a stacked sphere, whereas T is a triangulation of $\mathbb{S}^3$ satisfying $N_1 = N_0(N_0-1)/2$ if and only if T is a 2-neighborly triangulation, namely if every pair of vertices is connected by an edge.

A stacked sphere $(\mathbb{S}^n, T)$ is a triangulation $T \to \mathbb{S}^n$ of a sphere which can be constructed from the boundary $\partial\sigma^{n+1} \simeq \mathbb{S}^n$ of a simplex σ^{n+1} by successive adding of pyramids over some facets. More explicitly, the boundary complex of any abstract $(n+1)$-simplex σ^{n+1} is by definition a stacked sphere, and if T is a stacked sphere and σ^n is any n-simplex of T, then $\hat{T}$ is a stacked sphere if $\hat{T}$ is any complex obtained by T by removing σ^n and adding the join of the boundary $\partial\sigma^n$ with a new vertex distinct from the vertices of T. Note also that a triangulated PL manifold is called k-neighborly if

$$N_{k-1}(T) = \frac{N_0!}{k!(N_0-k)!}. \tag{37}$$

We are referring explicitly to 3- and 4-spheres $\mathbb{S}^n$, because the majority of Monte Carlo simulations have been carried out in these cases (for a recent discussion of more general topologies, see Ref. [13]). However, it must be stressed that the above definitions, as well as Walkup's theorems, can be naturally extended (with suitable modifications [12]) to any n-dimensional PL manifolds M. Note in particular that every triangulable 3-manifold M can be triangulated so that the closed star of some edge contains all the vertices and every pair of vertices is connected by an edge.

In dimension $n = 4$ we have a somewhat weaker characterization of the possible set of triangulations:

Theorem 2. If $T \to M$ is a triangulation of a closed connected 4-manifold, then

$$N_1(T) \geqslant 5N_0(T) - \tfrac{15}{2}\chi(T), \tag{38}$$

and equality holds if and only if (M,T) is a stacked sphere.

Note that actually one has a stronger statement in the sense that equality in (38) holds if and only is all vertex links in the 4-manifold M are stacked 3-spheres.

Contrary to what happens for 3-manifolds, 2-neighborly triangulations (i.e. triangulations where every pair of vertices is connected by an edge), are not *generic* for 4-dimensional PL manifolds, and as matter of fact, the above theorem immediately implies [14] that for any such (M,T)

$$N_0(T)(N_0(T) - 11) \geqslant -15\chi(M), \tag{39}$$

where the equality implies that (M,T) is 2-neighborly. Thus, the equality is not possible for large and arbitrary values of $N_0(T)$, but (depending on topology) [14] only in the cases $N_0(T) = 0$, $N_0 = 5$, $N_0 = 6$, or $N_0 = 11 \bmod 15$.

Even if 2-neighborly triangulations are not generic, one can easily construct voluminous (i.e. with $N_4(T)$ arbitrarily large) triangulations of the 4-sphere where all vertices *but two* are connected by an edge. In order to realize such triangulations, consider a 2-neighborly triangulation $T(3)$ of the 3-sphere $\mathbb{S}^3$ with f-vector $[N_0(T(3)), N_1(T(3)), N_2(T(3)), N_3(T(3))]$. If we take the *Cone*, $C(\mathbb{S}^3)$, on such $(\mathbb{S}^3, T(3))$, viz., the product $\mathbb{S}^3 \times [0,a]$ with $\mathbb{S}^3 \times \{a\}$ identified to a point, then we get a triangulation of a 4-dimensional ball B^4 with f-vector given by

$$\begin{aligned} f(B^4) = (&N_0(T(3)) + 1, N_1(T(3)) + N_0(T(3)),\\ &N_2(T(3)) + N_1(T(3)), N_3(T(3)) + N_2(T(3)), N_3(T(3))). \end{aligned} \tag{40}$$

By gluing two copies of such a cone $C(\mathbb{S}^3)$ along their isometric boundary $\partial C(\mathbb{S}^3) \simeq \mathbb{S}^3$, we get a triangulation of the 4-sphere $\mathbb{S}^4$ with f-vector

$$\begin{aligned} f(\mathbb{S}^4) = (&N_0(T(3)) + 2, N_1(T(3)) + 2N_0(T(3)),\\ &N_2(T(3)) + 2N_1(T(3)), N_3(T(3)) + 2N_2(T(3)), 2N_3(T(3))). \end{aligned} \tag{41}$$

It is trivially checked that corresponding to such a triangulation we get

$$N_1(\mathbb{S}^4) = \frac{N_0(\mathbb{S}^4)(N_0(\mathbb{S}^4) - 1)}{2} - 1, \tag{42}$$

where the -1 accounts for the missing edge between the two cone vertices in $C(\mathbb{S}^3) \cup_{S^3} C(\mathbb{S}^3)$. When applied to simplicial quantum gravity, the existence of such 2-neighborly (or almost 2-neighborly) triangulations implies that there are dynamical triangulations of the n-sphere $\mathbb{S}^n$, $n = 3$, $n = 4$, where all vertices are singular. Corresponding to such configurations we have that $b(n,n-2)|_{n=3} = 6$, and $b(n,n-2)|_{n=4} = 5$. Thus, not surprisingly, for such triangulations the kinematical upper bound for the average incidence $b(n,n-2)$ is attained. However, it is important to stress that such extremely singular configurations *do not saturate* the set of possible configurations for which $b_{\max}(n,n-2)$ is reached. From the Dehn–Sommerville relations one immediately gets

$$b(n,n-2)|_{n=3} = 6 \cdot \frac{N_3}{N_3 + N_0} \tag{43}$$

and

$$b(n,n-2)|_{n=4} = 10 \cdot \frac{N_4}{2N_4 + 2N_0 - 2\chi(T)}, \tag{44}$$

which, together with the obvious relation $N_1 \leqslant N_0(N_0-1)/2$, implies that in order to attain the upper kinematical bounds $b_{\max}(n,n-2)|_{n=3} = 6$ and $b_{\max}(n,n-2)|_{n=4} = 5$ it is sufficient that

$$N_0(T) = O[N_n(T)^{\alpha}], \tag{45}$$

with $1/2 \leqslant \alpha < 1$. Note that 2-neighborly or almost 2-neighborly triangulations correspond to $\alpha = 1/2$.

3.1. Singular stacked spheres

It should be stressed that the presence of singular vertices can occur also for $b(n,n-2) = b_{\min}(n,n-2)$, i.e. for stacked spheres. In other words, singular vertices are *not kinematically* forbidden by the geometry of the triangulations. Their suppression or enhancement in the different phases of simplicial quantum gravity is rather related to the relative abundance, with respect to the totality of possible triangulations, of the number of distinct triangulations with singular vertices as $b(n,n-2)$ varies. In other words, it is an entropic phenomenon as clearly suggested by Catterall, Thorleifsson, Kogut and Renken [5]. For definiteness, we can describe a concrete construction of a singular stacked sphere. It amounts to gluing a 4-dimensional ball B^4 bounded by a stacked 3-sphere $\mathbb{S}^3$ with a cone over such an $\mathbb{S}^3$.

Consider a 3-dimensional stacked sphere $\mathbb{S}^3$. According to one of Walkup's theorems, such an $\mathbb{S}^3$ is the boundary of a 4-dimensional ball B^4 with a tree-like structure and corresponds to a triangulation with f-vector

$$f(B^4) = (N_0(S^3), N_1(S^3), N_2(S^3), N_3(S^3) + N_3(\hat{B}^4), N_4(B^4)), \tag{46}$$

where $N_i(S^3)$, $i = 0,1,2,3$ is the f-vector of the boundary stacked sphere and $N_3(\hat{B}^4)$ is the number of σ^3 in the interior, $\hat{B}^4$, of B^4. Note that if we take the cone, $C(S^3)$, over the boundary stacked sphere, we get another triangulation of the 4-dimensional ball,

B^4_{sing}, whose boundary is again isometric to the given $\mathbb{S}^3$, but whose interior contains a (unique) singular vertex. The f-vector of such a triangulation is

$$f(B^4_{\text{sing}}) = (1 + N_0(S^3), N_1(S^3) + N_0(S^3), N_2(S^3) + N_1(S^3),$$
$$N_3(S^3) + N_2(S^3), N_3(S^3)). \tag{47}$$

Gluing these two triangulated balls B^4 to B^4_{sing} through their common boundary $\mathbb{S}^3$, we get a triangulation of the 4-sphere $\mathbb{S}^4 \simeq B^4 \cup_{S^3} C(\partial B^4)$, with f-vector

$$\begin{aligned}
N_0 &= N_0(S^3) + 1,\\
N_1 &= N_1(S^3) + N_0(S^3),\\
N_2 &= N_2(S^3) + N_1(S^3),\\
N_3 &= N_3(S^3) + N_3(\hat{B}^4) + N_2(S^3),\\
N_4 &= N_4(B^4) + N_3(S^3).
\end{aligned} \tag{48}$$

Since $\mathbb{S}^3$ is a stacked sphere, we have $4N_3(S^3) = 3N_1(S^3)$ which, together with $N_2(S^3) = 2N_3(S^3)$, implies

$$N_2 = \frac{10}{3}N_3(S^3). \tag{49}$$

From $2N_3 = 5N_4$ and the Euler relation for the 4-dimensional ball B^4 (with $\chi(B^4) = 1$) we immediately get $N_4(B^4) = \frac{1}{3}N_3(S^3) - \frac{2}{3}$, which implies

$$N_4 = \frac{4}{3}N_3(S^3) - \frac{2}{3}. \tag{50}$$

Thus, for $N_3(S^3) \to \infty$ we get a voluminous triangulation ($N_4 \to \infty$) of $\mathbb{S}^4$ with average incidence

$$b(n, n-2)|_{n=4} = 10 \cdot \frac{N_4 = \frac{4}{3}N_3(S^3) - \frac{2}{3}}{N_2 = \frac{10}{3}N_3(S^3)} \to_{N_3\to\infty} 4, \tag{51}$$

which shows that $\mathbb{S}^4 \simeq B^4 \cup_{S^3} C(\partial B^4)$ is a stacked sphere with a singular vertex (the apex of the cone $C(\partial B^4)$). An even simpler construction suffices to prove an analogous result in the 3-dimensional case. As mentioned in the introductory remarks, stacked spheres are relevant in providing the geometrical rationale for the prevalence of branched polymer structures in the weak coupling phase of simplicial quantum gravity. As a matter of fact [8], it is their tree-like structure that accounts for the *kinematical* possibility of polymerization. However, the existence of stacked spheres with singular vertices, shows that the dynamical onset of polymerization is not just a consequence of the geometry of triangulated manifolds as $b(n, n-2) \to b_{\min}(n, n-2)$. On the kinematical side we may have, in the configuration space, extremal cases such as the 2-neighborly triangulations occurring for $b(n, n-2) \to b_{\max}(n, n-2)$ or the singular stacked spheres for $b(n, n-2) \to b_{\min}(n, n-2)$. Monte Carlo simulations do confirm that such configurations are not generic. Near $b_{\max}(n, n-2)$ we generically sample singular triangulations with just a few singular vertices [5]. Similarly, as $b(n,n-2) \to b_{\min}(n,n-2)$ the dominant configurations sampled correspond to stacked spheres without singular vertices. The mechanism for understanding the dynamical prevalence of such configurations over the other configurations which are kinematically possible is simply related to the fact that, with respect to the counting measure, distinct dynamical triangulations are not equally probable as a function of the average incidence $b(n,n-2)$. In order to discuss this point we need to exploit a few elementary facts related to the geometry of the ergodic moves used in simplicial quantum gravity.

3.2. Ergodic moves and the onset of criticality

The (k,l) moves [15] in 3 and 4 dimensions are a well-known set of elementary surgery operations (related to the Pachner moves [16]) which allow us to construct all triangulations of a PL manifolds starting from a given triangulation. Roughly speaking, the generic (k,l) move consists in cutting out a subcomplex made up of k-dimensional simplices σ^k and replacing it with a complex of l-dimensional simplices σ^l with the same boundary. Note that $k+l=n+2$. We are interested in discussing how a finite set of such moves generate the f-vector of voluminous triangulations of the n-sphere $\mathbb{S}^n$ $(n=3,4)$ starting from the standard f-vector of the simplex $\partial\sigma^{n+1} \simeq \mathbb{S}^n$. For $n=3$, the relevant moves are the $(1,4)$ move (barycentric subdivision), the $(2,3)$ move (triangle to link exchange) and their inverses. For $n=4$, since the *flip* move $(3,3)$ does not alter the distribution of the number $N_i(T)$ of simplices, the f-vector of the sphere is generated by the moves $(1,5)$ (barycentric subdivision) and $(2,4)$ (two–four exchange) and their inverses. Following Ref. [8] (with a slight change in notation), we denote by $P_{k,l}(n)$ the number of moves of type (k,l) in dimension n, and introduce the balance variables $(n=3)$: $x_1 \doteq (P_{1,4}-P_{4,1})$, $x_2 \doteq (P_{2,3}-P_{3,2})$; and $(n=4)$: $y_1 \doteq (P_{1,5}-P_{5,1})$, $y_2 \doteq (P_{2,4}-P_{4,2})$. In terms of such quantities we can easily characterize the string of integers $\{N_i\}$, $i=0,\ldots,n$, which are *possible* f-vectors of triangulated $\mathbb{S}^n$.

For $n=3$, we get for $f(\mathbb{S}^3)=(N_0(S^3),N_1(S^3),N_2(S^3),N_3(S^3))$

$$
\begin{aligned}
N_0(S^3) &= 5+x_1,\\
N_1(S^3) &= 10+4x_1+x_2,\\
N_2(S^3) &= 10+6x_1+2x_2,\\
N_3(S^3) &= 5+3x_1+x_2,
\end{aligned}
\tag{52}
$$

whereas for $n=4$ we have for $f(\mathbb{S}^4)=(N_0(S^4),N_1(S^4),N_2(S^4),N_3(S^4),N_4(S^4))$

$$
\begin{aligned}
N_0(S^4) &= 6+y_1,\\
N_1(S^4) &= 15+5y_1+y_2,\\
N_2(S^4) &= 20+10y_1+4y_2,\\
N_3(S^4) &= 15+10y_1+5y_2,
\end{aligned}
$$

$$N_4(S^4) = 6 + 4y_1 + 2y_2. \tag{53}$$

Note that not all $f(\mathbb{S}^n)$ obtained in this way are actual f-vectors of triangulated $\mathbb{S}^n$. This is a consequence of the fact that the above relations between the $\{N_i\}$ and the variables $P_{k,l}(n)$ are equivalent to the Dehn–Sommerville constraints. And these latter are known to be necessary but not sufficient conditions in characterizing the possible f-vectors of a triangulated manifold (sufficient conditions have been conjectured by Stanley [17], see Ref. [1] for a brief discussion of this point).

Walkup's theorems imply the following kinematical bounds on the variables x_i, y_i $(i = 1, 2)$:

$$\begin{aligned} &x_1 \geqslant 0,\\ &y_1 \geqslant 0, \end{aligned} \tag{54}$$

(both from the obvious condition $N_0(S^n) \geqslant n + 2$);

$$\begin{aligned} &x_2 \geqslant 0,\\ &y_2 \geqslant 0, \end{aligned} \tag{55}$$

(the former from $N_1(S^3) \geqslant 4N_0(S^3) - 10$; the latter from $N_1(S^4) \geqslant 5N_0(S^4) - \frac{15}{2}\chi(S^4)$, with $\chi(S^4) = 2$);

$$\begin{aligned} &x_1^2 + x_1 - 2x_2 \geqslant 0,\\ &y_1^2 + y_1 - 2y_2 \geqslant 0, \end{aligned} \tag{56}$$

(both from $N_1(S^n) \leqslant N_0(S^n)(N_0(S^n) - 1)/2$). Finally, one can express the average incidence $b(n, n-2)$ as a function of x_i and y_i, so as to obtain

$$b(n, n-2)|_{n=3} = 6 \cdot \frac{5 + 3x_1 + x_2}{10 + 4x_1 + x_2} \tag{57}$$

and

$$b(n, n-2)|_{n=4} = 10 \cdot \frac{6 + 4y_1 + 2y_2}{20 + 10y_1 + 4y_2}. \tag{58}$$

It is also interesting to discuss in terms of the variables x_i and y_i, the average incidence of the top-dimensional simplices σ^n on the vertices σ^0 of the triangulations considered. A straightforward computation provides

$$Q(n) \doteq \frac{1}{N_0} \sum_{\{\sigma^0\}} q(\sigma^0) = (n+1)\frac{N_n}{N_0}, \tag{59}$$

yielding

$$Q(n)|_{n=3} = 4 \cdot \frac{5 + 3x_1 + x_2}{5 + x_1} \tag{60}$$

and

$$Q(n)|_{n=4} = 5 \cdot \frac{6+4y_1+2y_2}{6+y_1}. \tag{61}$$

As expected, $Q(n)$ is not bounded above: when the move $(2,3)$ (for $n=3$), or $(2,4)$ (for $n=4$) dominates, i.e. near the $b_{\max}(n,n-2)$ kinematical boundary, $Q(n) \to \infty$ as $N_n \to \infty$. One may wonder if this unboundedness is related to the unboundedness of the Einstein–Hilbert action, the answer is most likely no. It is certainly reasonable to put restrictions on $Q(n)$ in the search of a continuum limit of the theory, and this may change the phase structure of the theory. But if it does, it is just an illustration of the fact that this particular part of the phase diagram has no relevance for a genuine continuum limit. There should be a reasonable universality. This is nicely illustrated in 2D dynamical triangulation theory where any restriction (except the strict flatness constraint $q(i)=6$) leads to 2D gravity.

The above elementary remarks are a trivial restatement of the well known fact that the moves $(1,4)$ and $(1,5)$ (the barycentric subdivision) drive the system into the elongated phase, whereas the moves $(2,3)$ and $(2,4)$ drive to the crumpled phase. The crumpling transition occur as soon as singular vertices are statistically enhanced by the presence of enough $(2,4)$ moves with respect to $(1,5)$ (for $n=3$ this enhancement is generated by the dominance of $(2,3)$ moves with respect to $(1,4)$ moves).

3.3. *The genesis of singular vertices:* $\mathbb{S}^4_{\mathrm{sv}}$

In order to characterize the onset of crumpling we describe the f-vector of the generic triangulation of $\mathbb{S}^n$ in a way that clearly shows the mechanism of formation of singular vertices. Such a description is obtained by gluing a triangulated ball B^n to the cone over its boundary $\partial B^n \simeq \mathbb{S}^{n-1}$. Thus, by referring to the 4-dimensional case for definiteness, we consider $\mathbb{S}^4_{\mathrm{sv}} \simeq B^4 \cup_{S^3} C(\partial B^4)$ (sv for singular vertex). Note that any triangulation of $\mathbb{S}^4$ can be factorized in this way (since $C(\partial B^4)$ and ∂B^4 are the star and the link of a vertex, respectively), and we have

$$N_4 = N_4(B^4) + N_4(C(\partial B^4)). \tag{62}$$

The triangulation is singular as soon as we have

$$N_4(B^4) \propto N_4(C(\partial B^4)), \tag{63}$$

namely when the cone $C(\partial B^4)$ contains a number of top-dimensional simplices growing linearly with the volume of the whole manifold.

It is easily checked that the f-vector of $\mathbb{S}^4_{\mathrm{es}} \simeq B^4 \cup_{S^3} C(\partial B^4)$ is given by

$$\begin{aligned}
N_0 &= N_0(S^3) + 1 + N_0(\hat{B}^4),\\
N_1 &= N_1(S^3) + N_0(S^3) + N_1(\hat{B}^4),\\
N_2 &= N_2(S^3) + N_1(S^3) + N_2(\hat{B}^4),\\
N_3 &= N_3(S^3) + N_2(S^3) + N_3(\hat{B}^4),\\
N_4 &= N_3(S^3) + N_4(B^4),
\end{aligned} \tag{64}$$

where $N_i(S^3)$ denotes the f-vector of the boundary $\partial(B^4) \simeq \mathbb{S}^3$ of the triangulated ball B^4, and $N_i(\hat{B}^4)$ is the f-vector of the interior of B^4. The Dehn–Sommerville relations for $\mathbb{S}^4$ and for $\mathbb{S}^3$ constrain $N_i(\hat{B}^4)$ and $N_k(S^3)$ according to

$$\begin{aligned} N_0(\hat{B}^4) - N_1(\hat{B}^4) + N_2(\hat{B}^4) - N_3(\hat{B}^4) + N_4(B^4) &= 1, \\ 2N_1(\hat{B}^4) - 3N_2(\hat{B}^4) + 4N_3(\hat{B}^4) - 5N_4(B^4) + N_0(S^3) &= 0, \\ 2N_3(\hat{B}^4) + N_3(S^3) &= 5N_4(\hat{B}^4). \end{aligned} \tag{65}$$

The average incidence $b(4,2)$ of such triangulated $\mathbb{S}^4$ can be easily computed in terms of the f-vectors $N_i(\hat{B}^4)$ and $N_k(S^3)$ according to

$$b(4,2) = 10\frac{4b(3,1)N_3(S^3) + 2b(3,1)[N_0(\hat{B}^4) - N_1(\hat{B}^4) + N_2(\hat{B}^4)] - 2b(3,1)}{[6b(3,1) + 18]N_3(S^3) + 3b(3,1)N_2(\hat{B}^4)}, \tag{66}$$

where $b(3,1) \doteq 6[N_3(S^3)/N_1(S^3)]$ is the average incidence of $\partial B^4 \simeq \mathbb{S}^3$. The presence of a singular vertex corresponds to

$$\frac{N_4(\hat{B}^4)}{N_3(S^3)} = O(1), \tag{67}$$

and it is easily verified that under such condition $b(4,2)$ is an increasing function of $b(3,1)$. *This remark implies that singular triangulations with the smallest possible* $b(4,2)$ *are to be found corresponding to* $b(3,1) = b(3,1)_{\min} = 9/2$.

We have already seen an example of such a triangulation in the previous section, one for which the lowest kinematically possible incidence, $b(2,4) = 4$, is attained. However, such examples are not generic. They correspond to assuming $y_2 = 0$ (or more generally, they still occur if one interprets the right-hand side as $y_2 = O(1)$), and the singular vertex is not stable under $(1,5)$ moves. Eventually by performing enough barycentric subdivisions the initial singular vertex is smoothed out. Explicitly, assume that we start our chain of barycentric subdivisions on an a $\mathbb{S}^4_{sv} \simeq B^4 \cup_{S^3} C(\partial B^4)$ with a given value of N_4, say $N_4(0)$. Denote by $\mathbb{S}^4_{sv}(0)$ this initial triangulation. Note that at this initial step

$$N_4(B^4(0)) = \tfrac{1}{3}N_4(C(\partial B^4(0))) \tag{68}$$

(see the previous section). If we carry out a $(1,5)$ move on each 4-simplex of $\mathbb{S}^4_{sv}(0)$, we get a triangulation of $\mathbb{S}^4$ still of the form $\mathbb{S}^4_{sv} \simeq B^4 \cup_{S^3} C(\partial B^4)$, which we denote $\mathbb{S}^4_{sv}(1)$. For such triangulation we have

$$N_4(C(\partial B^4(1))) = 4 \cdot N_4(C(\partial B^4(0))), \tag{69}$$

$$N_4(B^4(1)) = 5 \cdot N_4(B^4(0)) + N_4(C(\partial B^4(0))). \tag{70}$$

Now proceed by induction, noticing that if at each step we carry out a barycentric subdivision of each 4-simplex of the $\mathbb{S}^4_{sv}$ generated at the previous step, we still get a

4-sphere, $\mathbb{S}^4_{sv}(k)$, triangulated according to $\mathbb{S}^4_{sv} \simeq B^4 \cup_{S^3} C(\partial B^4)$, and such that, at the kth step,

$$N_4(C(\partial B^4(k))) = 4^k \cdot N_4(C(\partial B^4(0))), \tag{71}$$

$$N_4(B^4(k)) = 5^k \cdot N_4(B^4(0)) + N_4(C(\partial B^4(0))) \sum_{i=1}^{k-1} 5^{i-1} \cdot 4^{k-1}. \tag{72}$$

Thus, as k grows (corresponding to $y_1 \to +\infty$, $y_2 = O(1)$), $N_4(B^4(k))$ largely dominates over $N_4(C(\partial B^4(k)))$:

$$N_4(B^4(k)) \gg_{y_1 \to +\infty} N_4(C(\partial B^4(k))), \tag{73}$$

and the resulting triangulation of $\mathbb{S}^4$ is no longer singular. In this sense, the dominance of the $(1,5)$ move naturally yields regular stacked sphere and thus for branched polymers. Discarding these particular examples of unstable triangulated spheres with a singular vertex, we can easily characterize the smallest $b(4,2)$ corresponding to generic singular triangulations, namely triangulations generated in the large volume limit as $y_1 \to \infty$ and $y_2 \to \infty$, and whose singular vertices are stable under the action of the (k,l) moves (at a fixed ratio y_1/y_2). Let us start by noticing that corresponding to $b(3,1) = 9/2$, the expression (66) for the average incidence reduces to

$$b(4,2) = 10 \cdot \frac{6N_1(S^3) + 4[N_0(\hat{B}^4) - N_1(\hat{B}^4) + N_2(\hat{B}^4)]}{15N_1(S^3) + 6N_2(\hat{B}^4)}, \tag{74}$$

where, in the numerator, we have discarded terms which are $o(1)$, thus irrelevant in the large volume limit. Since $\mathbb{S}^3$ is *stacked*, the integers $N_3(S^3)$ and $N_1(S^3)$ are related by $4N_3(S^3) = 3N_1(S^3) - 10$, which implies that $3N_1(S^3) \equiv 10(mod\ 4)$. Thus, $N_1(S^3)$ must be an integer multiple of 4 up to an error term which goes to zero, with increasing $N_1(S^3)$, as $10/N_1(S^3)$. More explicitly, and referring to the expression of the f-vector of $\mathbb{S}^3$ in terms of the balance variables $x_1 \in \mathbb{N}$ and $x_2 \in \mathbb{N}$ introduced in Section 3.2, we get the following components:

$$\begin{aligned} N_0(S^3) &= 5 + x_1, \\ N_1(S^3) &= 10 + 4x_1, \\ N_2(S^3) &= 10 + 6x_1, \\ N_3(S^3) &= 5 + 3x_1, \end{aligned} \tag{75}$$

since corresponding to a stacked $\mathbb{S}^3$ we have $x_2 = 0$ (see (52)). The congruence properties just established for the f-vector of a stacked 3-sphere suggest to parameterize both $N_2(\hat{B}^4)$ and $N_0(\hat{B}^4) - N_1(\hat{B}^4) + N_2(\hat{B}^4)$, appearing in (74), in terms of $N_1(S^3)$ by setting

$$N_2(\hat{B}^4) = \tilde{\beta} N_1(S^3) \tag{76}$$

and

$$N_0(\hat{B}^4) - N_1(\hat{B}^4) + N_2(\hat{B}^4)] = \tilde{\alpha} N_1(S^3). \tag{77}$$

According to the above remarks, $N_1(S^3)$ is asymptotically an integer multiple of 4, thus if we are interested to triangulations for which $N_1(S^3)$ can grow arbitrarily large, it follows that the two parameters $\tilde{\alpha}$ and $\tilde{\beta}$ necessarily are rational numbers of the form $\tilde{\beta} = \frac{\beta}{4}$ and $\tilde{\alpha} = \frac{\alpha}{4}$ with β and α integers. In other words, the generic triangulations of $\mathbb{S}^4_{sv} \simeq B^4 \cup_{S^3} C(\partial B^4)$ with the joining $\mathbb{S}^3$ stacked $(b(3,1) = 9/2)$, can be conveniently parameterized by setting

$$\frac{N_2(\hat{B}^4)}{N_1(S^3)} \doteq \frac{\beta}{4} \tag{78}$$

and

$$\frac{N_0(\hat{B}^4) - N_1(\hat{B}^4) + N_2(\hat{B}^4)}{N_1(S^3)} \doteq \frac{\alpha}{4}, \tag{79}$$

where α and β are integers. Note that while $\beta \geqslant 0$, α can possibly take also negative values. However, if we rewrite (74) in terms of such parameters

$$b(4,2) = 10 \cdot \frac{12 + 2\alpha}{30 + 3\beta}, \tag{80}$$

the kinematical bound $b(4,2) \geqslant 4$ implies $5\alpha \geqslant 3\beta$, and thus α is non-negative as well.

The parameters α and β so introduced are completely equivalent to the balance variables y_1 and y_2 related to the cumulant action of the (k,l) moves. Explicitly, we obtain

$$2y_1 = \left(\frac{1}{2} + \frac{1}{4}\beta - \frac{1}{3}\alpha\right) N_1(S^3) - \frac{20}{3},$$
$$2y_2 = \left(\frac{5}{6}\alpha - \frac{1}{2}\beta\right) N_1(S^3) + \frac{20}{3}. \tag{81}$$

The Dehn–Sommerville relations for the f-vector $N_i(\hat{B}^4)$ allow us to express also its components in terms of α and β according to

$$3N_0(\hat{B}^4) = \left[\frac{3\beta - 4\alpha}{8}\right] N_1(S^3) + 10,$$
$$3N_1(\hat{B}^4) = \left[\frac{9\beta - 10\alpha}{8}\right] N_1(S^3) + 10,$$
$$N_2(\hat{B}^4) = \frac{1}{4}\beta N_1(S^3),$$
$$3N_3(\hat{B}^4) = \left[\frac{3 + 5\alpha}{4}\right] N_1(S^3) - 5,$$
$$3N_4(\hat{B}^4) = \left[\frac{3 + 2\alpha}{4}\right] N_1(S^3) - 2. \tag{82}$$

The generic conditions $y_1 > 0$ and $y_2 > 0$ (and both approaching $+\infty$), together with $N_0(\hat{B}^4) > 0$, imply that the parameters α and β are related by

$$\tfrac{3}{5}\beta < \alpha < \tfrac{3}{4}\beta, \tag{83}$$

with $(\alpha,\beta) \in \mathbb{N}^+ \times \mathbb{N}^+$. From these remarks it follows that, as y_1 and y_2 go to $+\infty$, there are two distinct regimes for the set of triangulations considered:

(i) If we *constrain* the f-vector $N_i(S^3)$ of the connecting ∂B^4 to be $O(1)$, then according to (81), α and β go to ∞ as $y_1, y_2 \to +\infty$. From (82) we get that in this regime

$$\begin{aligned} N_4(S^4) &\simeq N_4(B^4) \simeq (\alpha/6)N_1(S^3), \\ N_2(S^4) &\simeq N_2(B^4) = (\beta/4)N_1(S^3), \end{aligned} \tag{84}$$

where $N_1(S^3)$ is a constant. The geometrical bounds (83) simply imply that as $\alpha,\beta \to \infty$, the corresponding average incidence $b(4,2)$ varies between the kinematical bounds $4 \leqslant b(4,2) \leqslant 5$, as required.

(ii) Conversely, if we do not constrain $N_i(S^3)$ to be $O(1)$, then according to (81), $N_1(S^3)$ (and hence $N_3(S^3)$) is allowed to grow unboundedly large as $y_1, y_2 \to +\infty$. This growth, which corresponds to the generation of singular vertices, is possible for any finite value of the parameters α and β compatible with (83). Note that if kinematically possible, according to (83), such singular triangulations *entropically dominate* over the regular ones since these latter are generated by the constrained configurations forcing $N_i(S^3)$ to be $O(1)$, while the former are unconstrained. More specifically, since the number of distinct triangulations of a 3-sphere $\mathbb{S}^3$ grows exponentially with $N_3(S^3)$, configurations with $N_3(S^3)$ as large as possible, if kinematically allowed, will dominate over configurations with $N_3(S^3) = O(1)$.

The kinematical bound (83) for the occurrence of singular triangulations is not trivial. In order to discuss its implications, let us consider the ratio between the total volume of the triangulated $\mathbb{S}^4_{sv}$ and the volume of the ball around the singular vertex σ^0, viz.

$$\frac{\mathrm{Vol}(S^4)}{\mathrm{Vol}_{\mathrm{sing}}(\sigma^0)} = \frac{N_4}{N_3(S^3)} = \frac{12+2\alpha}{9}. \tag{85}$$

A direct computation of the average incidence $b(4,2)$ (see 80) together with (83) immediately shows that the *smallest* $b(4,2)$'s for which we may have singular triangulations occur for

$$\begin{aligned} \alpha &= 5+3h, \\ \beta &= 8+5h, \end{aligned} \tag{86}$$

with $h = 0,1,2,\ldots$ As h varies, the average incidence $b(4,2)_h$ and the volume ratio (85) respectively take the values

$$b_h(4,2) = 10 \cdot \frac{22+6h}{54+15h}, \tag{87}$$

$$\left.\frac{\mathrm{Vol}(S^4)}{\mathrm{Vol}_{\mathrm{sing}}(\sigma^0)}\right|_h = \frac{22+6h}{9}. \tag{88}$$

Table 1
The smallest incidence numbers $b(4,2)|_h$ and the associated singular volume fraction $\mathrm{Vol}(S^4)/\mathrm{Vol}_{\mathrm{sing}}(\sigma^0)$ as a function of the parameters α and β

h	β	α	$b(2,4)$	$\mathrm{Vol}(S^4)/\mathrm{Vol}_{\mathrm{sing}}(\sigma^0)\|_h$
0	8	5	$\frac{110}{27} \simeq 4.07407$	$\frac{22}{9} \simeq 2.444$
1	13	8	$\frac{280}{69} \simeq 4.0579$	$\frac{28}{9} \simeq 3.111$
2	18	11	$\frac{340}{84} \simeq 4.04761$	$\frac{34}{9} \simeq 3.777$
3	23	14	$\frac{400}{99} \simeq 4.0404$	$\frac{40}{9} \simeq 4.444$
4	28	17	$\frac{460}{114} \simeq 4.03508$	$\frac{46}{9} \simeq 5.111$
5	33	20	$\frac{520}{129} \simeq 4.03100$	$\frac{52}{9} \simeq 5.777$

Since the singular triangulations that entropically dominate are those for which the ration $\mathrm{Vol}(S^4)/\mathrm{Vol}_{\mathrm{sing}}(\sigma^0)|_h$ is as low as possible, the smallest $b(4,2)$ for which we may have *generic* singular triangulations with the largest $\mathrm{Vol}_{\mathrm{sing}}(\sigma^0)$ is

$$b(2,4)_{\mathrm{sing}} = \frac{110}{27} \simeq 4.07407\ldots \tag{89}$$

(corresponding to $h = 0$ and $\mathrm{Vol}(S^4)/\mathrm{Vol}_{\mathrm{sing}}(\sigma^0) = 22/9 \simeq 2.444\ldots$). It is easily verified that such an average incidence is associated to a relative concentration of $(1,5)$ moves versus $(2,4)$ moves given by

$$y_1 = 5y_2. \tag{90}$$

It is clear from (87) that singular triangulations may appear also for smaller values of $b(4,2)$. A list of the first possible values of $b_h(4,2)$ is provided by Table 1. These triangulations are less singular than the ones associated with $b(4,2) = \frac{110}{27}$ since they correspond to larger values of the ratio $\mathrm{Vol}(S^4)/\mathrm{Vol}_{\mathrm{sing}}(\sigma^0)$, and for this reason we may be tempted to consider them as entropically *sub dominating* at least in the large-volume limit. Yet, this is mere appearance since their presence is particularly relevant for locating the critical incidence b_0 and for understanding the present status of the Monte Carlo simulations. Moreover, as we see in the next section, these triangulations have a subtle interplay with the particular singular geometry dominating in the strong coupling phase of 4D simplicial gravity: PL manifolds with a single singular edge connecting two singular vertices. In order to get the complete geometrical picture, one has to note that for $\alpha = 2 + 8h$ and $\beta = 3 + 13h$, we also get a highly degenerate configuration for which

$$b_h(4,2) = \frac{160}{39} \simeq 4.102564 \tag{91}$$

is a constant average incidence as h varies, whereas $\mathrm{Vol}(S^4)/\mathrm{Vol}_{\mathrm{sing}}(\sigma^0)|_h = \frac{16+16h}{9}$, $h = 0, 1, 2, \ldots$

In other words, corresponding to such value of $b(4,2)$ we have distinct triangulations with distinct ratios $\mathrm{Vol}(S^4)/\mathrm{Vol}_{\mathrm{sing}}(\sigma^0)|_h$ but with $b(4,2)$ fixed. Even if this set of triangulations contains configurations for which $\mathrm{Vol}(S^4)/\mathrm{Vol}_{\mathrm{sing}}(\sigma^0)|_h \simeq 1.777$, such a degeneration makes any particular configuration at fixed $\mathrm{Vol}(S^4)/\mathrm{Vol}_{\mathrm{sing}}(\sigma^0)|_h$ entropically subdominating with respect to the generic configurations described by (87), at least as $N_4 \to \infty$.

3.4. The development of singular edges: $\mathbb{S}^4_{\mathrm{es}}$

The explicit construction of the previous section may suggest that the singular triangulations we are explicitly considering are characterized by the dominance of just one singular vertex. Actually, as the parameters α and β vary, triangulations of $\mathbb{S}^4$ of the form $B^4 \cup_{S^3} C(\partial B^4)$ are not the only ones possible whose average incidence $b(4,2)$ takes on the value (80), *at least as* $N_4(S^4) \to \infty$. As a matter of fact, in the infinite volume limit (but not at finite volume), triangulations with more than one singular vertex and with singular edges are still characterized by the average incidence (80). Their dominance in the class of triangulations considered, as $N_4(S^4) \to \infty$, is driven by a rather simple entropic mechanism which we discuss in detail in this section.

Implicitly, the occurrence of more than one singular vertices may still be described by the construction $B^4 \cup_{S^3} C(\partial B^4)$, since one may simply consider the new singular vertices and edges to be located in the ball B^4. However, the interplay between dominance of one or more (edge-connected) singular vertices is most easily seen from a simple variant of the construction leading to (80). The generic singular triangulation of $\mathbb{S}^4$ is still realized by gluing two 4-balls along an isometric $\mathbb{S}^3$ boundary which is again assumed to be a stacked 3-sphere, i.e. as $B^4_{\mathrm{es}} \cup_{S^3} B^4$. However, one of the two balls, say the one denoted by B^4_{es} (es being an acronym for *edge-singular*), is no longer taken of the form of a cone $C(\partial B^4)$ over the $\mathbb{S}^3$ boundary, but more generally is provided by a triangulation with f-vector

$$
\begin{aligned}
N_0(B^4_{\mathrm{es}}) &= \frac{1}{3}\sum_{j=1}^{k} N_3(B^3(j)) + k,\\
N_1(B^4_{\mathrm{es}}) &= \frac{5}{3}\sum_{j=1}^{k} N_3(B^3(j)) + \frac{1}{2}\sum_{l=1}^{k-1} N_2(S^2(l)) + 3(k-1),\\
N_2(B^4_{\mathrm{es}}) &= \frac{10}{3}\sum_{j=1}^{k} N_3(B^3(j)) + 2\sum_{l=1}^{k-1} N_2(S^2(l)) + 2(k-1),\\
N_3(B^4_{\mathrm{es}}) &= 3\sum_{j=1}^{k} N_3(B^3(j)) + \frac{5}{2}\sum_{l=1}^{k-1} N_2(S^2(l)),\\
N_4(B^4_{\mathrm{es}}) &= \sum_{j=1}^{k} N_3(B^3(j)) + \sum_{l=1}^{k-1} N_2(S^2(l)).
\end{aligned}
\tag{92}
$$

To grasp the geometrical origin of this f-vector imagine k distinct 3-spherical disks, $B^3(j)$, joined through $(k-1)$ $\mathbb{S}^2$-boundaries, $\mathbb{S}^2(l)$; a sort of 3-dimensional *peanut-shell* with k-bulges and $k-1$ necks. This gives rise to a 3-spherical peanut-shell $\mathbb{S}^3$, and we get a 4-dimensional ball B^4_{es} out of this $\mathbb{S}^3$ by considering $(k-1)$ edges $\{\sigma^1(l)\}_{l=1,\dots,k-1}$ not belonging to $\mathbb{S}^3$ and connecting k vertices $\{\sigma^0(j)\}_{j=1,\dots,k} \notin \mathbb{S}^3$. The 4-ball B^4_{es} is defined by requiring that the generic 2-spheres $\mathbb{S}^2(l)$ are the links (in B^4_{es}) of the corresponding edges $\sigma^1(l)$, $l = 1, \dots, k-1$. Moreover, the complex obtained by B^4_{es} by removing the $(k-1)$ stars (in B^4_{es}) of the edges $\sigma^1(l)$, is assumed to be the disjoint union of k cones $C(B^3(j))$ over the 3-spherical disks $B^3(j)$, with apices in the k vertices $\sigma^0(j)$. This construction can be roughly described as a 3-dimensional peanut shell containing one rather than k distinct 4-dimensional nuts. It is easily verified that the f-vector $N_i(B^4_{es})$ (92) describes this generalized *peanut triangulation* and that it represents a triangulated 4-ball with $N_0(B^4_{es})$ vertices, k of which, $\{\sigma^0(j)\}_{j=1,\dots,k}$ are interior vertices (i.e. $\sigma^0(j) \notin \partial B^4_{es}$), with $N_3(B^3(j)) + N_2(S^2(j))$ 4-simplices σ^4 incident on the jth of them. The jth of the $k-1$ interior links $\sigma^1(l)$, connects the vertex $\sigma^0(j)$ with $\sigma^0(j+1)$, and $N_2(S^2(j))$ 4-dimensional simplices σ^4 are incident on it. Thus, if some, say $1 \leqslant s \leqslant k$, of the $\{N_3(B^3(j)\}$ and the corresponding $s-1$ of the $\{N_2(S^2(l))\}$ grow with the simplicial volume of the $\mathbb{S}^4 \supset B^4_{es}$, (not necessarily with the same rate), the triangulation of B^4_{es} just constructed contains s *singular* vertices connected by $s-1$ *singular* edges. Note that if we take the boundary of this triangulated B^4_{es} we obtain a stacked 3-sphere $\mathbb{S}^3$ with f-vector

$$
\begin{aligned}
N_0(S^3) &= \frac{1}{3}\sum_{j=1}^{k} N_3(B^3(j)), \\
N_1(S^3) &= \frac{4}{3}\sum_{j=1}^{k} N_3(B^3(j)), \\
N_2(S^3) &= 2\sum_{j=1}^{k} N_3(B^3(j)), \\
N_3(S^3) &= \sum_{j=1}^{k} N_3(B^3(j)).
\end{aligned}
\tag{93}
$$

This $\mathbb{S}^3$ boundary of the 4-dimensional ball B^4_{es} may be profitably thought of as resulting from the connected sum, along isometric $\mathbb{S}^2$-boundaries of k distinct stacked 3-spheres $\mathbb{S}^3_i$, $i = 1, \dots, k$, to be considered as the links (in an $\mathbb{S}^4$) of a corresponding singular vertex. In this way the singular ball B^4_{es} (and the corresponding $\mathbb{S}^4$) can be considered as the kinematical set up for discussing the *interaction* of k distinct singular vertices (of the type considered in the previous section). This picture allows us also to prove an elementary but important result showing that, in the class of triangulations considered, the order of singularity of a singular edge $\sigma^1(l)$ in B^4_{es} is subdominating with respect to the order of singularity of the corresponding vertices $\sigma^0(l-1)$ and $\sigma^0(l)$. In other words, in the large volume limit, the number of 4-simplices incident on $\sigma^1(l)$ grows slower than the number of 4-simplices incident on the vertices $\sigma^0(l-1)$ and $\sigma^0(l)$. As usual, the number of incident 4-simplices can be considered as the (possibly) singular volume associated to the corresponding edge or vertex. Thus, if we denote such simplicial volumes by $\mathrm{Vol}(\sigma^1(l)) = \#\{\sigma^4 \cap \sigma^1(l)\}$, $\mathrm{Vol}(\sigma^0(l-1)) = \#\{\sigma^4 \cap \sigma^0(l-1)\}$, and $\mathrm{Vol}(\sigma^0(l)) = \#\{\sigma^4 \cap \sigma^0(l)\}$, we have the following

Lemma 2. In the class of triangulations considered for B^4_{es},

$$\lim_{N_4(B^4_{\mathrm{es}})\to\infty} \frac{\mathrm{Vol}(\sigma^1(l))}{\mathrm{Vol}(\sigma^0(l))} = 0. \tag{94}$$

(Obviously the same holds with $\mathrm{Vol}(\sigma^0(l))$ replaced by $\mathrm{Vol}(\sigma^0(l-1))$). In order to prove this result we may consider, without loss of generality, a B^4_{es} whose $\mathbb{S}^3$-boundary consists of two stacked 3-spheres $\mathbb{S}_1$ and $\mathbb{S}_2$ joined through an isometric $\mathbb{S}^2$. The more general case can be proved similarly without much effort. Again without loss of generality we may assume that the two isometric copies of $\mathbb{S}^2$ along which the two $\mathbb{S}^3_1$ and $\mathbb{S}^3_2$ are glued are the links of a vertex in the corresponding $\mathbb{S}^3_i$ (as remarked in the previous paragraph, this can be always arranged; also it can be easily shown that there are stacked 3-spheres with a marked vertex whose 2-spherical link grows linearly with the simplicial volume of the 3-sphere, see Section 3.1 for an example in 4 dimensions. The above lemma states that, in the case of stacked 3-spheres, this large volume behavior for an $\mathbb{S}^2$ cannot hold if such a 2-sphere is a joining neck). It immediately follows that the f-vector of $\partial B^4_{\mathrm{es}} = \mathbb{S}^3_1 \cup_{S^2} \mathbb{S}^3_2$ can be written in terms of the f-vector of $\mathbb{S}^3_i$, $i = 1, 2$, and of $\mathbb{S}^2$ as

$$\begin{aligned}
N_0(S^3) &= \frac{1}{3}[N_3(S^3_1) + N_3(S^3_2)] - N_2(S^2) + N_0(S^2) - 6, \\
N_1(S^3) &= \frac{4}{3}[N_3(S^3_1) + N_3(S^3_2)] - 4N_2(S^2) + N_1(S^2) - 4, \\
N_2(S^3) &= 2[N_3(S^3_1) + N_3(S^3_2)] - 4N_2(S^2), \\
N_3(S^3) &= N_3(S^3_1) + N_3(S^3_2) - 2N_2(S^2).
\end{aligned} \tag{95}$$

Since $N_1(S^2) = (3/2)N_2(S^2)$ we immediately get

$$\frac{N_3(S^3)}{N_1(S^3)} = \frac{3}{4} \cdot \frac{N_3(S^3_1) + N_3(S^3_2) - 2N_2(S^2)}{N_3(S^3_1) + N_3(S^3_2) - \frac{15}{8}N_2(S^2) - \frac{12}{4}}, \tag{96}$$

which implies that $N_3(S^3)/N_1(S^3) > \frac{3}{4}$ as long as $N_2(S^2)/N_3(S^3_i) = O(1)$ in the large volume limit (i.e. as $N_3(S^3) + N_2(S^2) \to \infty$). Thus $\partial B^4_{\mathrm{es}} = \mathbb{S}^3_1 \cup_{S^2} \mathbb{S}^3_2$ can be a stacked 3-sphere if and only if

$$\lim_{N_3(S^3)\to\infty} \frac{N_2(S^2)}{N_3(S^3_i)} = 0. \tag{97}$$

According to the remarks made above $N_2(S^2) = \mathrm{Vol}(\sigma^1(i))$ and $N_3(S^3_i) + N_2(S^2) = \mathrm{Vol}(\sigma^0(i))$, and we can write (97) as

$$\lim_{N_4(B^4_{es})\to\infty} \frac{\mathrm{Vol}(\sigma^1(i))}{\mathrm{Vol}(\sigma^0(i)) - \mathrm{Vol}(\sigma^1(i))} = 0, \tag{98}$$

from which the lemma follows. This latter result only implies that the singular volume of the edge cannot grow *linearly* with the total volume of the ball B^4_{es} (and of the resulting $\mathbb{S}^4$, see below). As we have seen in the previous section, a linear growth is instead typical for the singular volume associated to the vertices. It should be stressed that a subdominant rate of growth (say with some fractional power of the total volume of B^4_{es}), is well in agreement with (97). As a matter of fact, subdominant powers for the volume growth associated with a singular edge are the ones typically experienced in numerical simulations [5]. Note that triangulations with $\mathrm{Vol}(\sigma^1(i))$ as large as kinematically possible, thus growing with $[N_4(B^4_{es})]^\delta$ for some $0<\delta<1$, entropically dominate over triangulations of B^4_{es} with $\mathrm{Vol}(\sigma^1(i)) = O(1)$. This remark follows as a direct consequence of the fact that triangulating B^4_{es} under the hypothesis $N_2(S^2) = O(1)$, while sufficient to assure the validity of (97), it is not a necessary condition. It generates a subclass of constrained configurations in the class of triangulations of B^4_{es} considered. Conversely, triangulations with $N_2(S^2) \propto [N_4(B^4_{es})]^\delta$ are, according to (97), unconstrained, and as such much more numerous at least in the large volume limit. As in the previous section, we obtain a 4-sphere $\mathbb{S}^4_{es}$, (es again for *edge-singular*), by gluing a generic triangulated ball B^4 with stacked $\mathbb{S}^3$ boundary to the singular B^4_{es} defined by (92), viz., $\mathbb{S}^4_{es} \simeq B^4 \cup_{S^3} B^4_{es}$. It is easily checked that the f-vector of such an $\mathbb{S}^4_{es}$ is given by

$$\begin{aligned}
N_0(S^4_{es}) &= N_0(S^3) + N_0(\hat{B}^4) + k,\\
N_1(S^4_{es}) &= N_1(S^3) + N_0(S^3) + N_1(\hat{B}^4) + \frac{1}{2}\sum_{l=1}^{k-1} N_2(S^2(l)) + 3(k-1),\\
N_2(S^4_{es}) &= N_2(S^3) + N_1(S^3) + N_2(\hat{B}^4) + 2\sum_{l=1}^{k-1} N_2(S^2(l)) + 2(k-1),\\
N_3(S^4_{es}) &= N_3(S^3) + N_2(S^3) + N_3(\hat{B}^4) + \frac{5}{2}\sum_{l=1}^{k-1} N_2(S^2(l)),\\
N_4(S^4_{es}) &= N_3(S^3) + N_4(B^4) + \sum_{l=1}^{k-1} N_2(S^2(l)),
\end{aligned} \tag{99}$$

where $N_i(S^3)$ denotes the f-vector of the joining stacked 3-sphere $\partial(B^4_{es}) \simeq \mathbb{S}^3$, and $N_i(\hat{B}^4)$ is the f-vector of the interior of B^4. According to (97), $N_2(S^2)/N_3(S^3)$ is asymptotically $o(1)$, thus, in the large volume limit, the average incidence $b(4,2)$ of such a triangulated $\mathbb{S}^4_{es}$ is still provided by the expression (80) introduced in the previous section, viz.,

$$\lim_{N_4(S^4)\to\infty} b(4,2)|_{S^4_{es}} = 10\cdot\frac{12+2\alpha}{30+3\beta}, \tag{100}$$

where the two parameters β and α are again defined by (78) and (79), respectively. Before we proceed any further, we should emphasize that (100) strictly speaking only holds in the limit $N_4(S^4) \to \infty$, and that *at finite* (but large) volume $N_4(S^4_{es})$, we have $b(4,2)|_{S^4_{es}} > b(4,2)$ with

$$b(4,2)|_{S^4_{es}} = 10 \cdot \frac{12+2\alpha}{30+3\beta} + \eta \frac{\sum_{l=1}^{k-1} N_2(S^2(l))}{N_3(S^3)}, \tag{101}$$

for a suitable α- and β-dependent constant $\eta > 0$ which can be easily worked out. For instance, for the relevant case $k = 2$ (i.e. two singular vertex connected by a subsingular edge), we get to leading order in $N_2(S^2)/N_3(S^3)$

$$b(4,2)|_{S^4_{es}} = 10 \cdot \frac{12+2\alpha}{30+3\beta} + 10 \cdot \frac{6+3\beta-4\alpha}{100+\beta^2+20\beta} \left[\frac{N_2(S^2)}{N_3(S^3)}\right]. \tag{102}$$

Since according to Lemma 2 the ratio $\sum_{l=1}^{k-1} N_2(S^2(l))/N_3(S^3)$ can go to zero, in the large volume limit, as slowly as $N_3(S^3)^{\delta-1}$ for some $0 < \delta < 1$, we get

Lemma 3. At finite volume $N_4(S^4)$, the singular-vertex triangulations $\mathbb{S}^4_{sv} \simeq B^4 \cup_{S^3} C(\partial B^4)$, considered in Section 3.3, are closer to the kinematical boundary $b(4,2) = 4$ than the edge-singular triangulations $\mathbb{S}^4_{es} \simeq B^4 \cup_{S^3} B^4_{es}$.

We stress that this result does not imply that the singular-vertex triangulations $\mathbb{S}^4_{sv} \simeq B^4 \cup_{S^3} C(\partial B^4)$ entropically dominate in the large volume limit. For, according to (100), the edge-singular triangulations become more and more important as the volume increases, and eventually in the infinite volume limit the triangulated spheres $\mathbb{S}^4_{es}$ enter in full entropic competition with the triangulated $\mathbb{S}^4_{sv}$ considered in the previous section. Actually this entropic competition comes into play quite rapidly as the volume increases. For instance, from (102), one gets that, for the dominating configurations at $h = 0$,

$$b(4,2)|_{S^4_{es}} \simeq \frac{110}{27} + \frac{100}{324} \cdot \left[\frac{N_2(S^2)}{N_3(S^3)}\right]. \tag{103}$$

Numerical simulations at $N_4(S^4) = 32\,000$ (see, e.g., Refs. [5,7]) show evidence that $N_2(S^2)/N_3(S^3) < 1/10$, thus the average incidence $b(4,2)|_{S^4_{es}}$ of $\mathbb{S}^4_{es}$ differs (at $h = 0$) from the average incidence $b(4,2)|_{S^4_{vs}}$ of $\mathbb{S}^4_{vs}$ by less than 3/100. Therefore it is important to understand how, as $N_4(S^4)$ increases, the k distinct singular vertices (and the corresponding $k-1$ subsingular connecting edges) interact among them, and which configuration actually dominates in the large volume limit. As we have seen in Section 3.3, the various singular triangulations of the 4-sphere considered there are parameterized by the ratio between the total simplicial volume of the given $\mathbb{S}^4_{sv}$ and the simplicial volume of its singular part (see (85) and (88)). If we consider a similar ratio also for $\mathbb{S}^4_{es}$, i.e.

$$\frac{\mathrm{Vol}(S^4_{es})}{\mathrm{Vol}(\mathrm{sing})} = \frac{N_4(S^4_{es})}{N_4(B^4_{es})}, \tag{104}$$

then, as is easily verified, this ratio is still provided, in the large volume limit, by (85). It follows that the entropic comparison between the single singular vertex triangulations $\mathbb{S}^4_{sv}$ and the multiple singular vertices triangulations $\mathbb{S}^4_{es}$ should be carried out at a fixed value of the ratio $\mathrm{Vol}(S^4_{es})/\mathrm{Vol}(\mathrm{sing}) = \mathrm{const.} = \mathrm{Vol}(S^4_{sv})/\mathrm{Vol}_{\mathrm{sing}}(\sigma^0)$.

In our case

$$N_4(B^4_{es}) = \sum_{j=1}^{k} N_3(B^3(j)) + \sum_{l=1}^{k-1} N_2(S^2(l))$$
$$= N_3(S^3) + \sum_{l=1}^{k-1} N_2(S^2(l)). \tag{105}$$

According to the remarks following Lemma 2, unconstrained triangulations of $\mathbb{S}^4_{es}$ generally have $\sum_{l=1}^{k-1} N_2(S^2(l)) = O(N_3(S^3)^\delta)$ for some $0 < \delta < 1$. Thus

$$\lim_{N_4(S^4)\to\infty} N_4(B^4_{es})/N_3(S^3) = 1, \tag{106}$$

and working at constant ratio (104) (in the infinite volume limit $N_4(S^4_{es}) \to \infty$) implies that we have to consider triangulations of B^4_{es} with

$$N_3(S^3) = A_1 \cdot N_4(S^4) \tag{107}$$

and

$$A_2 \leqslant \sum_{l=1}^{k-1} N_2(S^2(l)) \leqslant A_3 \cdot N_3(S^3)^\delta, \tag{108}$$

for some positive constants A_1, A_2, and A_3.

Guided by these considerations we can easily get a set of entropic rules for determining which configuration dominates in the set of singular triangulations of $\mathbb{S}^4$. We start by an obvious adaptation of an argument in [5], according to which the number of distinct triangulations associated with a singular vertex (the *local entropy of the vertex*) is provided by the number of distinct triangulations of the link of the given vertex. The link, $\mathrm{link}(\sigma^0(j))$, around the jth singular vertex $\sigma^0(j) \in B^4_{es}$, is a 3-sphere $\mathbb{S}^3(j)$, and any two such links, $\mathbb{S}^3(j)$ and $\mathbb{S}^3(j+1)$, associated with two singular vertex connected by a singular edge $\sigma^1(j)$, have a non-empty intersection $\mathbb{S}^2(j)$ (the link of the connecting edge $\sigma^1(j)$). Thus, the inclusion-exclusion principle implies that the number, $\mathrm{Card}[B^4_{es}(S^3(1),\dots,S^3(k);S^2(1),\dots,S^2(k-1))]$, of distinct triangulations of B^4_{es} with given singular vertices $\{S^3(j)\}_{j=1,\dots,k}$ and given singular edges $\{S^2(l)\}_{l=1,\dots,k-1}$ is provided by

$$\mathrm{Card}[B^4_{es}(S^3(1),\dots,S^3(k);S^2(1),\dots,S^2(k-1)))] = \frac{\prod_{j=1}^{k}\mathrm{Card}[S^3(j)]}{\prod_{l=1}^{k-1}\mathrm{Card}[S^2(l)]}, \tag{109}$$

where Card$[S^3(j)]$ and Card$[S^2(l)]$ respectively denote the number of distinct triangulations of the 3-spherical links of the jth singular vertex and of the 2-spherical singular link of the lth singular edge. Since each $S^3(j)$ is a stacked 3-sphere (hence with an average incidence $b(3,1) = \frac{9}{2}$) the microcanonical partition function (9) immediately provides the leading order asymptotics both for Card$[S^3(j)]$ and Card$[S^2(l)]$, viz.,

$$\mathrm{Card}[S^3(j)]_{N_3(S^3(j))\gg 1} \simeq \left[\frac{(b(3,1)-\hat{q}+1)^{b(3,1)-\hat{q}+1}}{(b(3,1)-\hat{q})^{b(3,1)-\hat{q}}}\right]^{N_1(S^3(j))}, \tag{110}$$

where $b(3,1) = \frac{9}{2}$, $\hat{q} = 3$. Since $N_1(S^3(j)) = \frac{4}{3}N_3(S^3(j))$, we get

$$\mathrm{Card}[S^3(j)]_{N_3(S^3(j))\gg 1} \simeq \left[\frac{(\frac{5}{2})^{5/2}}{(\frac{3}{2})^{3/2}}\right]^{\frac{4}{3}N_3(S^3(j))}. \tag{111}$$

Similarly, by setting $b(2,1) = 6$, $\hat{q} = 3$, and $N_0(S^2(l)) = 2 + \frac{1}{2}N_2(S^2(l))$, (9) provides

$$\begin{aligned}\mathrm{Card}[S^2(l)]_{N_2(S^2(l))\gg 1} &\simeq \left[\frac{(b(2,1)-\hat{q}+1)^{b(2,1)-\hat{q}+1}}{(b(2,1)-\hat{q})^{b(2,1)-\hat{q}}}\right]^{N_0(S^2(l))} \\ &= \left[\frac{4^4}{3^3}\right]^{\frac{1}{2}N_2(S^2(l))}.\end{aligned} \tag{112}$$

Thus, by setting $C(2) \doteq [4^4/3^3]^{1/2}$ and $C(3) \doteq [(5/2)^{5/2}/(3/2)^{3/2}]^{4/3}$, we eventually get

$$\begin{aligned}&\mathrm{Card}[B^4_{\mathrm{es}}(S^3(1),\ldots,S^3(k);S^2(1),\ldots,S^2(k-1))] \\ &\simeq \exp\left[\left(\sum_{j=1}^{k} N_3(S^3(j))\right)\ln C(3) - \left(\sum_{l=1}^{k-1} N_2(S^2(l))\right)\ln C(2)\right].\end{aligned} \tag{113}$$

Since

$$\sum_{j=1}^{k} N_3(S^3(j)) = N_3(S^3) + 2\sum_{l=1}^{k-1} N_2(S^2(l)), \tag{114}$$

where $S^3 = \partial B^4_{\mathrm{es}}$ is the stacked boundary of B^4_{es}, we can rewrite (113) as

$$\mathrm{Card}[B^4_{\mathrm{es}}(S^3(1),\ldots;S^2(1),\ldots,)] \simeq C(3)^{N_3(S^3)}\left[\frac{C(3)^2}{C(2)}\right]^{\sum_{l=1}^{k-1} N_2(S^2(l))} \tag{115}$$

(by exploiting (105) this expression can be also rewritten in terms of $N_4(B^4_{\mathrm{es}})$). Since $C(3)/C(2) > 1$, we have that triangulations of B^4_{es} with large $\sum_{l=1}^{k-1} N_2(S^2(l))$ are dominant in the infinite volume limit. This implies that the simplicial volume of the $k-1$ edges connecting the k vertices is as large as possible. Note that (115) does not depend on the particular $S^3(j)$ or $S^2(l)$ but only on the fixed quantities $N_3(S^3)$ and $\sum_{l=1}^{k-1} N_2(S^2(l))$ determining the ratio between $N_4(S^4)$ and the volume of the singular part B^4_{es} of $\mathbb{S}^4$ (see (104) and (105)).

Thus, among all possible triangulations with k distinct singular vertices connected by $k-1$ distinct edges, those entropically favored, as k varies, are the less constrained ones, namely triangulations with just one singular edge connecting two singular vertices: the triangulations of B^4_{es} with $k=2$. For such triangulations the S^3 links of the singular vertices and the S^2 link of the connecting edge are as large as kinematically possible. Note that for the triangulated $B^4 = C(\partial B^4)$ considered in Section 3.3 we have

$$\mathrm{Card}[C(\partial B^4)] \simeq C(3)^{N_3(S^3)}, \tag{116}$$

and in the large volume limit

$$\mathrm{Card}[B^4_{es}(S^3(1),\dots;S^2(1),\dots,)] \simeq C(3)^{N_3(S^3)} \cdot \left[\frac{C(3)^2}{C(2)}\right]^{\sum_{l=1}^{k-1} N_2(S^2(l))}$$
$$> \mathrm{Card}[C(\partial B^4)] \simeq C(3)^{N_3(S^3)}. \tag{117}$$

Since, as $N_4(S^4)$ increases, the triangulations $\mathbb{S}^4_{es}$ enter more and more in entropic competition with the single singular vertex triangulations $\mathbb{S}^4_{sv}$, (117) directly implies the following basic result:

Lemma 4. For a given ratio

$$\frac{\mathrm{Vol}(S^4_{es})}{\mathrm{Vol}(\mathrm{sing})} = \frac{N_4(S^4_{es})}{N_4(B^4_{es})} = \frac{22+6h}{9}, \tag{118}$$

with $h = 0,1,2,\dots$, the singular triangulations of $\mathbb{S}^4$ which are closer to the kinematical boundary $b(4,2)=4$, and which entropically dominate in the large volume limit $N_4(S^4) \to \infty$, are realized by triangulations $\mathbb{S}^4_{es}$ with one subsingular edge connecting two singular vertices, and are characterized by the average incidence

$$b_h(4,2) = 10 \cdot \frac{22+6h}{54+15h}. \tag{119}$$

The last part of this lemma, concerning the h-parameterization of the singular triangulations, is an immediate consequence of the expressions (100) and (101) for the average incidence of $\mathbb{S}^4_{es}$ and of the results of Section 3.3. Results which characterize the sets of value of α and β giving the closest approach of $b(4,2) = 10 \cdot (12+2\alpha)/(30+3\beta)$ to the kinematical boundary $b(4,2)=4$ as the ratio $\mathrm{Vol}(S^4_{es})/\mathrm{Vol}(\mathrm{sing})$ varies. The geometrical analysis just discussed and Lemma 4 appear in good qualitative agreement with the picture which emerges from recent Monte Carlo simulations [7] concerning the study of singular structures in 4D simplicial gravity. According to such a numerical analysis there are, *at finite volume*, two pseudo-critical couplings (and hence corresponding pseudo-critical incidences $b(4,2)$) separately associated with the creation of singular edges and singular vertices. This behavior seem to correspond to the different entropic relevance of the single singular vertex triangulations $\mathbb{S}^4_{sv}$ and of the singular edge triangulations $\mathbb{S}^4_{es}$ discussed above. In the simulations the two pseudo-critical couplings lock into a single critical point in the large volume limit. This merging appears to be related to the full entropic competition between $\mathbb{S}^4_{sv}$ and $\mathbb{S}^4_{es}$ which dominates our geometrical picture in the infinite volume limit. Explicitly, the average incidence $b(4,2)|_{S^4_{es}}$ (see 101) is slightly larger (at finite volume) than $b(4,2)|_{S^4_{sv}}$. Thus, if we apply formula (14) relating the average incidence $b(4,2)$ to a value of the coupling k_2, we find that the set of $k_2(S^4_{es})$'s corresponding to $b(4,2)|_{S^4_{es}}$ (as h varies) is slightly smaller than the corresponding set of $k_2(S^4_{vs})$'s associated with $b(4,2)|_{S^4_{vs}}$. Anticipating the analysis of Section 4, this remark implies that there are indeed two pseudo-critical points respectively associated with edge-singular $\mathbb{S}^4_{es}$ and vertex-singular $\mathbb{S}^4_{vs}$ triangulations, say $k_2^{crit}(S^4_{es};N_4)$ and $k_2^{crit}(S^4_{vs};N_4)$, with

$$k_2^{crit}(S^4_{es};N_4) \leqslant k_2^{crit}(S^4_{vs};N_4), \tag{120}$$

and coalescing in just one critical point as N_4 gets larger and larger. Obviously, what one actually sees at a given finite volume mostly depends on the rate $N_2(S^2)/N_3(S^3)$ (see 101) which controls how fast the two average incidences $b(4,2)|_{S^4_{es}}$ and $b(4,2)|_{S^4_{vs}}$ approach each other. On this rate we are not yet able to say anything substantial. As recalled (see (103)) computer simulations indicates that at relatively large volumes (typically $N_4 = 32\,000$) the term $N_2(S^2)/N_3(S^3)$ is already so small that $b(4,2)|_{S^4_{es}} \simeq b(4,2)|_{S^4_{vs}}$ up to a few percent, and edge-singular triangulations are to all effects as close to the kinematical boundary $b(4,2) = 4$ as the $\mathbb{S}^4_{vs}$ are. Thus they do entropically dominate.

3.5. *The characterization of the critical incidence*

Since in the infinite volume limit both singular configurations $\mathbb{S}^4_{sv}$ and $\mathbb{S}^4_{es}$ are characterized by the same average incidence (80), we can use indifferently both for characterizing the critical incidence $b_0(4)$ signaling the closest approach of generic singular triangulations to the kinematical boundary $b(4,2) = 4$. The single singular vertex configurations $\mathbb{S}^4_{sv}$ are somehow easier to handle than $\mathbb{S}^4_{es}$, thus for definiteness we describe the characterization of the critical incidence (and the corresponding critical gravitational coupling) by referring explicitly to $\mathbb{S}^4_{sv} \simeq B^4 \cup_{S^3} C(\partial B^4)$. In any case, one should keep in mind that the extension of the analysis to $\mathbb{S}^4_{es}$ can be carried out without difficulty along the same lines.

How can we characterize the critical incidence $b_0(4)$? A glance at Table 1 clearly shows that, as $\mathrm{Vol}(S^4)/\mathrm{Vol}_{\mathrm{sing}}(\sigma^0)$ increases, the values of $b(4,2)|_h$ are very close to each other. This remark implies that triangulations with $b(4,2)|_{h=0} = 110/27$, even if entropically dominating in $\mathbb{S}^4_{sv} \simeq B^4 \cup_{S^3} C(\partial B^4)$, cannot be taken as the mark of the real critical incidence. As a matter of fact, for values of h close to the leading configuration at $h = 0$, there can be statistical competition between such singular triangulations, at least as $N_4 \to \infty$. The critical incidence b_0 is actually obtained by averaging the distinct $b(4,2)|_h$'s over the set of corresponding singular triangulations.

To characterize such average we exploit the fact that the singular triangulations we are considering have their singular part constructed as a cone over a stacked 3-sphere $\mathbb{S}^3$. If we join, through the identification of a marked $\sigma^3 \in \mathbb{S}^3$, two stacked 3-spheres, $\mathbb{S}^3(1)$ and $\mathbb{S}^3(2)$ we get another stacked 3-sphere $\mathbb{S}^3(3) = \mathbb{S}^3(1)\#\mathbb{S}^3(2)$, and all (voluminous) stacked spheres can be obtained in this way. Thus, if we construct the cone over this connected sum of stacked 3-spheres we can sweep all possible voluminous (i.e. large N_4) singular triangulations of the type we are considering. Explicitly, let us denote the singular triangulations of $\mathbb{S}^4$, obtained from the stacked 3-spheres $\mathbb{S}^3(1)$ and $\mathbb{S}^3(2)$, by $\mathbb{S}^4(1) \doteq B^4(1) \cup_{S^3(1)} C(\mathbb{S}^3(1))$ and $\mathbb{S}^4(2) \doteq B^4(2) \cup_{S^3(2)} C(\mathbb{S}^3(2))$, respectively. If $\mathbb{S}^3(3) = \mathbb{S}^3(1)\#_f\mathbb{S}^3(2)$, where f is an homeomorphism between two marked $\sigma^3(1) \in \mathbb{S}^3(1)$ and $\sigma^3(2) \in \mathbb{S}^3(2)$, then

$$\mathbb{S}^4(3) = \mathbb{S}^4(1)\#_{f*}\mathbb{S}^4(2) = (B^4(1) \cup_f B^4(2)) \cup_{S^3(3)} C(\mathbb{S}^3(3)), \tag{121}$$

where $f*$ is the extension of f to the cone over the marked σ^3, and every singular triangulations of $\mathbb{S}^4$ over a stacked 3-sphere can be obtained in this way.

The analytical counterpart of (121) follows directly from the last of relations (82) characterizing the f-vector of the ball B^4 as the parameters α and β (thus h) vary. From it we get

$$\begin{aligned} N_4[B^4(1) \cup_f B^4(2)] &= \left[\frac{13+6h}{9}\right] N_1(S^3(1)) + \left[\frac{13+6h}{9}\right] N_1(S^3(2)) \\ &= N_4[B^4(1)] + N_4[B^4(2)], \end{aligned} \tag{122}$$

where we have discarded constant terms which are $o(1)$ in the large N_4 limit. To exploit this information let

$$\mathcal{T}_h[\mathrm{Vol}(B^4) = N] \doteq \mathrm{Card}\left\{\mathbb{S}^4 \colon \frac{\mathrm{Vol}_{\mathrm{norm}}(S^4)}{\mathrm{Vol}_{\mathrm{sing}}(\sigma^0)} = \frac{6h+22}{9};\ \mathrm{Vol}(B^4) = N\right\} \tag{123}$$

be the cardinality of the set of distinct singular triangulations of the ball B^4, constructed over a stacked $\mathbb{S}^3$, with given ratio $\mathrm{Vol}(S^4)/\mathrm{Vol}_{\mathrm{sing}}(\sigma^0)$ and $N_4(B^4) = N$. According to the behavior of this set of triangulations under the connected sum we have

$$\mathcal{T}_h\left[\mathrm{Vol}(B^4) = N(1) + N(2)\right] = \mathcal{T}_h\left[\mathrm{Vol}(B^4) = N(1)\right] \cdot \mathcal{T}_h\left[\mathrm{Vol}(B^4) = N(2)\right]. \tag{124}$$

It is easily verified that this relation implies that the leading asymptotics of $\mathcal{T}_h(\mathrm{Vol}(B^4))$ is provided by

$$\mathcal{T}_h(\mathrm{Vol}(B^4)) = c(B^4; h)^{N_4(B^4)}, \tag{125}$$

where $\ln c(B^4; h)$ is the specific entropy for the generic $\sigma^4 \in \mathbb{S}^4_{\mathrm{vs}}$.

Since there is a unique cone $C(\mathbb{S}^3)$ over the stacked sphere boundary $\partial B^4 \simeq \mathbb{S}^3$, (125) provides also the leading exponential asymptotics to the number of distinct triangulations of $\mathbb{S}^4_{\mathrm{vs}}$ with given N_4 and given h, viz.,

$$\mathrm{Card}\{\mathbb{S}^4_{vs}\} \propto c(B^4;h)^{N_4(B^4)}. \tag{126}$$

Actually, when $h \gg 1$ and $N_3(S^3) = O(1)$, for each triangulation of $\mathbb{S}^3$, there can be a worth of $\mathrm{Aut}(S^3)$ inequivalents cones, $\mathrm{Aut}(S^3)$ denoting the automorphisms group of the given triangulation, for simplicity we disregard here these correction factors. Note also that the above construction applies to the edge-singular spheres $\mathbb{S}^4_{es}$ with minor modifications.

According to (64),

$$N_4(B^4) = N_4(S^4) - N_3(S^3) = N_4(S^4)\frac{13+6h}{22+6h}, \tag{127}$$

thus we get that to leading order

$$\mathrm{Card}\{\mathbb{S}^4_{vs}\} = c(B^4;h)^{N_4(S^4)-N_3(S^3)} \doteq s(h)^{N_4(S^4)}, \tag{128}$$

where we have introduced the specific entropy, $\ln s(h)$, of a $\sigma^4 \in \mathbb{S}^4{}_{sv}$ according to

$$\ln s(h) \doteq \lim_{N_4(S^4)\to\infty} \frac{\ln \mathrm{Card}\{\mathbb{S}^4_{vs}\}}{N_4(S^4)} = \frac{13+6h}{22+6h}\ln c(B^4;h). \tag{129}$$

In order to characterize $\ln s(h)$, note that triangulations of the form $\mathbb{S}^4_{vs}$ describe, for $h = 0$, the generic singular triangulations of $\mathbb{S}^4$ realizing the closest approach to the kinematical boundary $b(4,2) = 4$. Conversely, and as already stressed, the triangulations $\mathbb{S}^4_{vs}$ reduce, as $h \to \infty$, to the generic (branched polymer) triangulations of $\mathbb{S}^4$ (with a rooted σ^4). These remarks imply that corresponding to $h = 0$ and $h = h_{max}$ we must have

$$\begin{aligned} \ln s(h=0) &= \ln c(S^4; h=0), \\ \ln s(h=h_{max}) &= \ln c(S^4; h=h_{max}), \end{aligned} \tag{130}$$

where h_{max} is characterized by the value of the ratio (88) evaluated for the smallest possible $N_3(S^3) = 5$, i.e. $h_{max} = \frac{3}{10}N_4 - \frac{11}{3}$, and where $\ln c(S^4;h)$ is the specific entropy associated with the microcanonical partition function (9), i.e.

$$c(S^4;h) \simeq \left[\frac{(b(4,2)-2)^{b(4,2)-2}}{(b(4,2)-3)^{b(4,2)-3}}\right]^{10/b(4,2)}, \tag{131}$$

with $b(4,2) = 10\cdot(22+6h)/(54+15h)$ (the actual specific entropy contains a constant factor which is of no relevance for the present considerations, see (9)).

Since $c(S^4;h)$ is a slowly varying function of h, the specific entropy $\ln s(h)$ can be characterized as the convex combination of $\ln s(h=0)$ and $\ln s(h=h_{max})$ over the interval $0 \leqslant h \leqslant h_{max}$, viz.,

$$\ln s(h) = \frac{h}{h_{max}}\ln s(h=h_{max}) + \left(1 - \frac{h}{h_{max}}\right)\ln s(h=0). \tag{132}$$

In other words, we are considering $\ln s(h)$ as the convex combination of the extreme pure phases ($h = 0$: crumpling, and $h \to \infty$: branched polymer). A straightforward computation provides

$$s(h) = c(S^4; h = 0) \cdot \left[\frac{c(S^4; h = 0)}{c(S^4; h = h_{\max})} \right]^{-\frac{10}{3N_4}h} . \tag{133}$$

Since in the large $N_4(S^4)$ limit, $\ln[c(S^4; h = 0)/c(S^4; h = h_{\max})] \simeq 0.06$ we eventually get for the leading asymptotics

$$\mathrm{Card}\{\mathbb{S}^4_{vs}\} = c(S^4; h = 0)^{N_4} e^{-\frac{h}{5}}. \tag{134}$$

It is worth stressing that a completely analogous result holds for $\mathrm{Card}\{\mathbb{S}^4_{es}\}$, since, as $N_4 \to \infty$, the set of edge-singular triangulations (with one edge connecting two singular vertices), $\mathbb{S}^4_{es}|_{k=2}$, is as close to the kinematical boundary $b(4,2) = 4$ as the triangulations $\mathbb{S}_{vs}$. The two class $\mathbb{S}^4_{es}$ and $\mathbb{S}^4_{vs}$ only differ in the subleading asymptotics. According to (134), the average value of $b(4,2)|_h$ over the set of singular triangulations considered is given, in the large N_4 limit, by

$$\langle b(4,2)_{\text{sing}} \rangle |_{h_{\max}} = \frac{\sum_{h=0}^{h_{\max}} b(4,2)|_h \exp[-\frac{h}{5}]}{\sum_{h=0}^{h_{\max}} \exp[-\frac{h}{5}]}. \tag{135}$$

By approximating the numerator with an integral, we get

$$\langle b(4,2)_{\text{sing}} \rangle |_{h_{\max}} = 4 + \frac{4}{15} \cdot \frac{e^{\frac{18}{25}} [E_1(\frac{18}{25}) - E_1(\frac{h_{\max}}{5} + \frac{18}{25})]}{5(1 - e^{-\frac{h_{\max}}{5}})}, \tag{136}$$

where $E_1(x)$ is the exponential integral function. In the large volume limit $h_{\max} \to \infty$, and the above expression reduces to

$$\langle b(4,2)_{\text{sing}} \rangle = 4 + \frac{4}{75} e^{\frac{18}{25}} E_1 \left(\frac{18}{25} \right) \simeq 4.0394361235. \tag{137}$$

As stressed, a similar analysis carried out for the class of singular triangulations $\mathbb{S}^4_{es}$ would provide the same $\langle b(4,2)_{\text{sing}} \rangle$. It follows that, as $N_4(S^4) \to \infty$, (137) is the value of the incidence $b(4,2)$ statistically dominating in both sets $\mathbb{S}^4_{sv}$ and $\mathbb{S}^4_{es}$. As argued in the previous sections, these triangulations are the ones characterizing the smallest possible $b(4,2)$ marking the onset of the dominance of singular geometries. Thus, we can identify $\langle b(4,2)_{\text{sing}} \rangle$ with the *critical* incidence b_0 (see Section 2.1) characterizing the transition between the weak and the strong coupling phase of the theory, i.e.

$$b_0(4) \doteq \langle b(4,2)_{\text{sing}} \rangle \simeq 4.0394361235. \tag{138}$$

Together with the critical incidence $\langle b(4,2) \rangle$ it is worthwhile to compute the infinite volume average, over the set of singular triangulations $\mathbb{S}^4_{sv}$ or $\mathbb{S}^4_{es}$, of the local volume of the singular part of the triangulation, Vol(sing). Note that for the class of triangulations $\mathbb{S}^4_{sv}$, $\mathrm{Vol(sing)} = \mathrm{Vol}(\sigma^0)$, whereas for the triangulations of $\mathbb{S}^4_{es}$ dominating in the infinite volume limit, we have

$$\mathrm{Vol(sing)} \simeq 2\mathrm{Vol}(\sigma^0), \tag{139}$$

since according to the remarks of Section 3.4 and Lemma 4, in such a limit, triangulations with just two singular vertices (connected by a subsingular edge) dominate.

For both class of triangulations $\mathrm{Vol}(S^4)/\mathrm{Vol}(\mathrm{sing}) = (22+6h)/9$, and the required average is provided by

$$\left\langle \frac{\mathrm{Vol}(\mathrm{sing})}{\mathrm{Vol}(S^4)} \right\rangle \Bigg|_{h_{\max}} = \frac{\sum_{h=0}^{h_{\max}} \exp[-\frac{h}{5}]\,\frac{9}{6h+22}}{\sum_{h=0}^{h_{\max}} \exp[-\frac{h}{5}]}. \tag{140}$$

(Strictly speaking, this ensemble average explicitly refers to the single singular vertex triangulations $\mathbb{S}^4_{\mathrm{sv}}$, however, as stressed before, this ensemble average differs from the $\mathbb{S}^4_{\mathrm{es}}$ ensemble average by corrections which vanish as $N_4(S^4) \to \infty$.)

By approximating as usual the summations with an integral we get

$$\left\langle \frac{\mathrm{Vol}(\mathrm{sing})}{\mathrm{Vol}(S^4)} \right\rangle \Bigg|_{h_{\max}} = \frac{3e^{11/15}}{10(1-e^{-h_{\max}/5})} \left[E_1\left(\frac{11}{15}\right) - E_1\left(\frac{h_{\max}}{5} + \frac{11}{15}\right) \right]. \tag{141}$$

According to (139), we get for the average local volume of the (two) most singular vertices, the explicit expression

$$\langle \mathrm{Vol}(\sigma^0) \rangle |_{h_{\max}} = \frac{3e^{11/15}}{20(1-e^{-h_{\max}/5})} \left[E_1\left(\frac{11}{15}\right) - E_1\left(\frac{h_{\max}}{5} + \frac{11}{15}\right) \right] \cdot N_4, \tag{142}$$

which, in the infinite volume limit, reduces to

$$\langle \mathrm{Vol}(\sigma^0) \rangle = \frac{3e^{11/15}}{20} E_1\left(\frac{11}{15}\right) \cdot N_4. \tag{143}$$

Note that the value of the critical average incidence $\langle b(4,2) \rangle \simeq 4.03943\ldots$ shows that the leading configurations contributing to the singular geometry of $\mathbb{S}^4_{\mathrm{es}}$ are, loosely speaking, those for which $h \leqslant 6$ (see Table 2). Thus, a rough indicator of what is the average singular volume for $b(4,2)$ sufficiently smaller than $\langle b(4,2) \rangle \simeq 4.03943\ldots$ (viz., when in the polymeric phase) can be obtained by considering the average

$$\left\langle \frac{\mathrm{Vol}(\mathrm{sing})}{\mathrm{Vol}(S^4)} \right\rangle \Bigg|_{\mathrm{poly}} = \frac{\sum_{h\geqslant 6}^{h_{\max}} \exp[-\frac{h}{5}]\,\frac{9}{6h+22}}{\sum_{h\geqslant 6}^{h_{\max}} \exp[-\frac{h}{5}]}. \tag{144}$$

Explicitly we get

$$\langle \mathrm{Vol}(\sigma^0) \rangle_{\mathrm{poly}} = \frac{3e^{29/15}}{20} E_1\left(\frac{29}{15}\right) \cdot N_4, \tag{145}$$

which can be interpreted as the contribution to $\langle \mathrm{Vol}(\sigma^0) \rangle$ coming from the non-singular geometries in $\mathbb{S}^4_{\mathrm{es}}$.

4. The critical coupling k_2^{crit}

The kinematical picture which emerges from the above analysis is immediately connected to the thermodynamical behavior of 4D dynamical triangulations by recalling the results of Section 2.2 according to which, as k_2 varies the distribution of triangulated manifolds is strongly peaked around triangulations with an average incidence given by $3[A(k_2)/(A(k_2)-1)]$ (see (14)). Thus by solving for k_2 the equation

$$\langle b(4,2)_{\rm sing}\rangle = 3\left(\frac{A(k_2)}{A(k_2)-1}\right), \tag{146}$$

we get an estimate of the value of k_2 corresponding to which singular triangulations start dominating the canonical partition function (4) in the *infinite volume limit*. Recall that singular triangulations are those characterizing the subexponential subleading asymptotics (see Theorem 5.2.1, pp. 106–118 of Ref. [1])

$$W[N_2, b(4,2)] \simeq e^{(\alpha_4 b(4,2))N_2} \times \left[\frac{(b-\hat{q}+1)^{b-\hat{q}+1}}{(b-\hat{q})^{b-\hat{q}}}\right]^{N_2} e^{[-m(b(4,2))N_4^{1/n_H}]} N_2{}^{-11/2}, \tag{147}$$

with $m(b(4,2) > 0$ (see (9) for the general expression; the above expression can be obtained from (9) by setting $n=4$, $\alpha_2 = 0$, $\tau(b) = 0$, and $D = 0$ since we are considering $\mathbb{S}^4$, we have also dropped a few unessential constant terms). Thus we can identify the k_2 solution of Eq. (146) with the *critical value*, $k_2^{\rm crit}$, of the inverse gravitational coupling marking the transition between the strong and weak coupling in 4D simplicial quantum gravity. Introducing in (146) the values $\langle b(4,2)_{\rm sing}\rangle \simeq 4.0394361235$ obtained above for the kinematical bound controlling the occurrence of generic singular triangulations, we get for the critical coupling the explicit value

$$k_2^{\rm crit} \simeq 1.3093. \tag{148}$$

4.1. A model for pseudo-criticality at finite $N_4(S^4)$

It is very interesting to compare the value for $k_2^{\rm crit}$, already in very good agreement with what is found by means of Monte Carlo simulations, with the other k_2^h's obtained by solving Eq. (146) with the left member $\langle b_{\rm sing}(4,2)\rangle$ replaced by the values $b_h(4,2)$ provided by (87). In this way we get Table 2.

According to the remarks in the previous paragraph, k_2^h, $h = 1,2,\ldots$, can be interpreted as the values of the inverse gravitational coupling corresponding to which the subleading singular configurations comes into play. In other words, corresponding to such values of k_2 there are *distinct peaks* in the distribution of singular triangulations of $\mathbb{S}^4_{\rm es}$. The *leading peak* is at $k_2 = k_2^{\rm crit} \simeq 1.24465$, this corresponds to the dominance of singular triangulations for which $\mathrm{Vol}(S^4)/\mathrm{Vol}_{\rm sing}(\sigma^0)|_h = 22/9$; the *first subleading peak* occurs at $k_2 = k_2^{h=1} \simeq 1.2744$, corresponding to the subdominance of singular triangulations for which $\mathrm{Vol}(S^4)/\mathrm{Vol}_{\rm sing}(\sigma^0)|_h = 28/9$; the *second subleading peak* occurs at $k_2 = k_2^{h=2} \simeq 1.2938$ and is associated with the subdominance of singular triangulations for which $\mathrm{Vol}(S^4)/\mathrm{Vol}_{\rm sing}(\sigma^0)|_h = 34/9$, and so on. In the large N_4 limit there is enough *phase space* for having all such peaks contributing to the partition function of the theory, and the presence of the subdominating peaks lowers the critical incidence from its *bare* value

Table 2
Some of the values of k_2^h obtained by solving Eq. (146) for $b_h(4,2)$ as h varies. Such values appear strikingly near to the values of the pseudo-critical points found in Monte Carlo simulations as the size of the triangulations considered is increased

h	$b(2,4)$	$\mathrm{Vol}(S^4)/\mathrm{Vol}_{\mathrm{sing}}(\sigma^0)\vert_h$	k_2^h
0	$\frac{110}{27} \simeq 4.07407$	2.444	$\simeq 1.24465$
1	$\frac{280}{69} \simeq 4.0579$	3.111	$\simeq 1.2744$
2	$\frac{340}{84} \simeq 4.04761$	3.777	$\simeq 1.2938$
3	$\frac{400}{99} \simeq 4.0404$	4.444	$\simeq 1.30746$
4	$\frac{460}{114} \simeq 4.03508$	5.111	$\simeq 1.31762$
5	$\frac{520}{129} \simeq 4.03100$	5.777	$\simeq 1.32545$

$b(4,2)|_{h=0}$ to $\langle b(4,2)_{\mathrm{sing}}\rangle$, and shifts the critical k_2^{crit} from the *bare* value 1.24465 to its effective value $k_2^{\mathrm{crit}} \simeq 1.3093$. Using a field-theoretic image, one may say that in the large volume limit the fluctuations associated with the various subdominating peaks in the distribution of singular triangulations dress the bare critical incidence.

Conversely, at a finite value of N_4 one would expect that the resulting average $\langle b(4,2)_{\mathrm{sing}}\rangle(N_4)$, computed from (135) with $h \leqslant \bar{h}(N_4) \leqslant h_{\max}$, for some $\bar{h}(N_4)$, is larger than the limiting value $\langle b(4,2)_{\mathrm{sing}}\rangle$. Corresponding to this $\langle b(4,2)_{\mathrm{sing}}\rangle(N_4)$ one gets an N_4-dependent pseudo-critical point $k_2^{\mathrm{crit}}(N_4)$ smaller than the actual k_2^{crit}. Roughly speaking, at finite volume, there is no phase space available for having all subdominating peaks competing with each other according to their relative entropic relevance. Moreover, at finite volume we should distinguish which kind of singular geometry we are dealing with. According to Lemma 3 and (101), the average incidence is larger for the edge-singular triangulations $\mathbb{S}^4_{es}$ than for the single singular vertex triangulations $\mathbb{S}^4$. Thus, corresponding to $\mathbb{S}^4$ or $\mathbb{S}^4_{es}$ we should get a slightly different sequence of pseudo-critical points, (according to (146), $k_2^{\mathrm{crit}}(N_4)|(S^4_{es}) \leqslant k_2^{\mathrm{crit}}(N_4)|(S^4)$), a difference which fades away as the volume increases.

In order to make contact with numerical simulations is worthwhile to develop an analytical model taking care of these *finite size* effects. Again for simplicity, let us limit our analysis to the vertex singular triangulations $\mathbb{S}^4_{vs}$, with the understanding that what we say can be easily extended to the edge-singular triangulations $\mathbb{S}^4_{es}$ with minor modifications. The starting point of our analysis is the entropic formula (134) expressing, as h varies, the entropy of the triangulations $\mathbb{S}^4_{vs}$ as convex combinations of its extreme two *pure phases* associated with crumpling ($h = 0$) and polymerization ($h = h_{\max} \to \infty$). Rather than use directly (134) we should refer to the conditional entropy

$$\frac{\mathrm{Card}\mathbb{S}^4_{vs}}{\mathrm{Card}\mathbb{S}^4}, \tag{149}$$

which provides the contribution of the triangulations $\mathbb{S}^4_{vs}$ to the set of all possible triangulations of $\mathbb{S}^4$, at fixed volume.

From (134) and (9) we get, to leading order in the large $N_4(S^4)$ limit,

$$\frac{\mathrm{Card}\mathbb{S}^4_{\mathrm{vs}}}{\mathrm{Card}\mathbb{S}^4} \simeq \Omega^{N_4(S^4)} e^{-\frac{h}{5}}, \tag{150}$$

where Ω is the h-dependent constant

$$\Omega \doteq \frac{c(S^4; h=0)}{c(S^4; h)} \simeq 33.97082 \cdot \left[\frac{(b(4,2)-2)^{b(4,2)-2}}{(b(4,2)-3)^{b(4,2)-3}}\right]^{-10/b(4,2)}, \tag{151}$$

with $b(4,2) = 10 \cdot (22+6h)/(54+15h)$.

The expression (150) for the conditional entropy, holds at finite, sufficiently large $N_4(S^4)$, and, since $\mathrm{Card}\mathbb{S}^4_{\mathrm{vs}}/\mathrm{Card}\mathbb{S}^4 \leqslant 1$, it implies that, at *finite volume*, triangulations $\mathbb{S}^4$ with $h \gg N_4(S^4) \ln \Omega$ are entropically suppressed. This remark implies that the configurations $\mathbb{S}^4_{\mathrm{vs}}$, which actually contribute in characterizing the critical incidence, have an entropic cut at some value of h, say $\bar{h}(N_4) = O(N_4(S^4) \ln \Omega)$. The specific entropy $\ln c(S^4; h)$ of $\{\mathbb{S}^4\}$ changes very slowly with h, thus at finite $N_4(S^4) \doteq N$, we may tentatively write

$$\left(\frac{\mathrm{Card}\mathbb{S}^4_{\mathrm{vs}}}{\mathrm{Card}\mathbb{S}^4}\right)_N = \Omega_0^{N_4} e^{-\frac{h}{5}}, \tag{152}$$

for $0 \leqslant h \leqslant \bar{h}(N)$ whereas

$$\left(\frac{\mathrm{Card}\mathbb{S}^4_{\mathrm{vs}}}{\mathrm{Card}\mathbb{S}^4}\right)_N = \Omega^{N_4}_{h=h_{\max}}, \tag{153}$$

for $\bar{h}(N)\rangle h \leqslant h_{\max}$, and where $\Omega_0 = 1+\epsilon$, $\epsilon\rangle 0$, is a suitable constant not differing much from 1 (according to (151) $\Omega|(h=1) \simeq 1.01234$, and $\Omega|(h=10^5) \simeq 1.0615$). In other words, we are assuming that for $0 \leqslant h \leqslant \bar{h}(N)$ the system may exist as a mixture of its two extreme pure phases, whereas for $h > \bar{h}(N)$ it collapses into its branched polymer phase. It is worthwhile stressing that more realistically one may consider, in place of (152), a convex combination of the extreme phase $h=0$ and the (non-extreme) phase corresponding to $h = \bar{h}(N)$. By exploiting (133), this prescription can be worked out without difficulty. However, it gives rise to a rather complex scaling behavior of the resulting entropy. Moreover, the fact that $c(S^4; h)$ is a slow varying function of h, makes, as we shall see, the simpler (152) quite accurate and much easier to handle.

A qualitative characterization of $\bar{h}(N)$ as N varies can be easily obtained by the obvious scaling properties of (152). If we consider triangulations $\mathbb{S}^4_{\mathrm{vs}}$ with two distinct volumes, say $N_4(S^4) = N(1)$ and $N_4(S^4) = N(2)$, then

$$\left(\frac{\mathrm{Card}\mathbb{S}^4_{\mathrm{vs}}}{\mathrm{Card}\mathbb{S}^4}\right)_{N_4=N(1)} = \left(\frac{\mathrm{Card}\mathbb{S}^4_{\mathrm{vs}}}{\mathrm{Card}\mathbb{S}^4}\right)_{N_4=N(2)}, \tag{154}$$

provided that $\bar{h}(N)$ scales with N according to

$$\bar{h}(N(2)) = \bar{h}(N(1)) + 5[N(2) - N(1)] \ln \Omega_0. \tag{155}$$

This scaling relation implies that $\bar{h}(N)$ has a linear dependence on $N_4(S^4)$ according to

$$\bar{h}(N_4) = 5N_4 \ln \Omega_0 + \xi, \tag{156}$$

where ξ is a suitable constant. This rather simple argument does not yet provide the actual value of the constants Ω_0 and ξ. However, confrontation with numerical data at $N_4(S^4) = 32\,000$ indicates as reliable candidates the values

$$5 \ln \Omega_0 = \frac{1}{16\,000}, \quad \xi = -1. \tag{157}$$

Note that the above condition for Ω_0 implies $\Omega_0 \simeq 1.0000125$, a value which is perfectly consistent with the above characterization of Ω (see (151). It also indicates that the triangulations $\mathbb{S}^4_{\rm vs}$ (actually the entropically dominating $\mathbb{S}^4_{\rm es}$), do saturate the possible set of triangulations of $\mathbb{S}^4$ in the strong coupling phase.

The N_4-dependent pseudo-critical incidence $\langle b(4,2)_{\rm sing}\rangle(N_4)$ and the associated pseudo-critical point $k_2^{\rm crit}(N_4)$ can be easily obtained from (136) by replacing $h_{\rm max}$ with $\bar{h}(N)$, viz.,

$$\langle b(4,2)_{\rm sing}\rangle(N_4) = 4 + \frac{4}{15} \cdot \frac{e^{\frac{18}{25}}[E_1(\frac{18}{25}) - E_1(\frac{\bar{h}(N)}{5} + \frac{18}{25})]}{5(1 - e^{-\frac{\bar{h}(N)}{5}})}, \tag{158}$$

and by solving for k_2 Eq. (146) with $\langle b(4,2)_{\rm sing}\rangle(N_4)$ in place of $\langle b(4,2)_{\rm sing}\rangle$.

By exploiting these results we get an overall analytic picture of the large volume behavior of 4-dimensional simplicial quantum gravity which is in a surprising agreement with the Monte Carlo simulations of the real system [7].

5. Comparison with numerical work

At this stage it is indeed useful to discuss the status of our geometrical results in the light of the most recent numerical work. This comparison is particularly important since, as recalled in the introductory remarks, the current perspective on 4-dimensional simplicial quantum gravity has undergone a rather drastic change. As a matter of fact, recent Monte Carlo simulations seem to accumulate more and more evidence for a first-order nature of the transition separating the strong and the weak coupling regime of the theory. Taken at face value this result suggests that dynamical triangulations is not likely to be a viable model of quantum gravity unless one adds additional terms to the action. It is perhaps fair to say that the geometrical analysis of the previous paragraphs bears relevance to such an issue. The characterization of the critical coupling $k_2^{\rm crit}$ and the existence of *entropically subdominating peaks* in the distribution of singular triangulations strongly indicates that this geometrical picture may be responsible for the phenomenology we see in numerical work. Let us start by noticing that in numerical work is difficult to resolve the various contribution to the distribution of singular triangulations coming from the various peaks geometrically found by our analysis. The resolving power depends, among other parameters, on the size of the triangulations, and

Table 3
The value of the analytical pseudo-critical points $k_2^{\rm crit}(N_4)$ versus their Monte Carlo counterparts. These values are computed under the hypothesis that the linear dependence of $\bar{h}(N_4)$ from $N_4(S^4)$ is given by $h(N_4) = N_4/16\,000 - 1$

N_4	h	Analytical $k_2^{\rm crit}(N_4)$	Monte Carlo $k_2^{\rm crit}(N_4)$
32 000	1	1.25795	1.258
48 000	2	1.26752	1.267
64 000	3	1.27466	1.273

as a rough indicator, the larger the size the bigger is the set of subdominating singular triangulations which come into play. Obviously the first subdominant terms are the most relevant ones, and as suggested in the previous section, an interesting value to look at for comparison with Monte Carlo data is the value of the inverse gravitational coupling corresponding to the pseudo-critical average incidence $\langle b(2,4)_{\rm sing}\rangle(N_4)$. As recalled there, by solving for k_2 the equation

$$\langle b(4,2)_{\rm sing}\rangle(N_4) = 3\left(\frac{A(k_2)}{A(k_2)-1}\right), \tag{159}$$

we obtain the value of $k_2^{\rm crit}(N_4)$ corresponding to which we expect to see a clear signature of the dominance of singular geometries in the set of triangulated 4-spheres of volume N_4. This is actually a pseudo-critical point, the location of which depends on N_4. Numerically one finds that as the *volume* N_4 of the triangulation increases, the corresponding pseudo-critical point $k_2^{\rm crit}(N_4)$ increases too, (see, e.g., Ref. [6]). Simulations and extrapolation to triangulations with size $N_4 = 48\,000$ and $N_4 = 64\,000$ locate the corresponding $k_2^{\rm crit}(N_4)$ at 1.267 and 1.273, respectively. According to (156) the actual dependence of the number of dominating peaks, $\bar{h}(N_4)$, as a function of the volume N_4 of the triangulation, is linear according to

$$\bar{h}(N_4) = \frac{N_4}{16\,000} - 1, \tag{160}$$

for $N_4(S^4) \geqslant 32\,000$, where the actual value, ($5\ln\Omega_0 = 1/16\,000$ and $\xi = -1$), of the constants comes from comparison with the numerical data provided at $N_4(S^4) = 32\,000$ by Ref. [6]. With this expression of $\bar{h}(N_4)$ we obtain, from (159) and (158), Table 3. The agreement between the analytical pseudo-critical points and the Monte appears surprisingly good, and suggests that the identification of our $k_2^{\rm crit}(N_4)$ with the pseudo critical $k_2^{\rm crit}(N_4)$ found in Monte Carlo simulations is not a mere coincidence. An important implication of this identification, if correct, is that the growth with N_4 of $k_2^{\rm crit}(N_4)$ is due to the increasing contribution of the subdominating singular triangulations. This result provides a nice explanation to the fact that Monte Carlo data seem to indicate that the major part of the finite size effects come from the crumpled phase [18].

By extrapolating the actual measurements, the Monte Carlo simulations locate the critical point around $k_2^* \simeq 1.327$ or around $k_2^* \simeq 1.293$ (depending on whether the data fit used is modeled after a second-order or a first-order transition, respectively) [6].

Again, our analytical result $k_2^{\rm crit} \simeq 1.3093$ appears in quite good agreement with the numerical data (curiously enough our $k_2^{\rm crit}$ is, with a good approximation, the average of the above two numerical data), and moreover its analytical characterization provides a natural entropic explanation to the structure and location of the associated finite-size pseudo-critical points.

Another distinct feature of recent numerical works concerns the bimodality in the distribution of singular vertices seen during Monte Carlo simulations exactly around $k_2^{\rm crit}(N_4 = 32\,000) \simeq 1.258$ [6]. In this respect, particularly interesting are Refs. [7,13], where long run histories (at $N_4(S^4) = 32\,000$) provide a reliable measurement of the average maximum vertex order near the critical point. In these simulations the system wanders between two states characterized by two quite distinct values of the average maximum vertex order. In one case, this maximum is close to 3000, while for the other the figure it is close to 1000. A correlation analysis shows that this metastability corresponds to tunneling back and forth from a branched polymer state (average vertex order $\simeq$ 1000) containing no singular vertex and a crumpled state (average vertex order $\simeq$ 3000) with one or two singular vertices.

According to our analysis, this behavior is the one exactly coded into the entropy formulae (152) and (153) which exactly describe a finite size tunnelling between a crumpled state (described by (152)) and a branched polymer state (described by (153). A good indication of the average vertex order, as we approach the transition point for increasing k_2, is provided by (143) At $N_4(S^4) = 32\,000$ this analytic formula yields

$$\langle \mathrm{Vol}(\sigma^0)\rangle_{N_4=32\,000} = \frac{3e^{11/15}}{20} E_1\left(\frac{11}{15}\right) \cdot (N_4 = 32\,000) \simeq 3400. \qquad (161)$$

Conversely, if we approach the transition point by lowering k_2, then a reliable indication is provided by (145). Explicitly we get

$$\langle \mathrm{Vol}(\sigma^0)\rangle_{\rm poly} = \frac{3e^{29/15}}{20} E_1\left(\frac{29}{15}\right) \cdot (N_4 = 32\,000) \simeq 1770. \qquad (162)$$

Such results appear quite in reasonable agreement with the values of $\langle \mathrm{Vol}(\sigma^0)\rangle_{N_4=32\,000}$ obtained during the simulations and mentioned before. Such data suggests that the bimodality seen in the numerical simulation has its origin in the presence of subdominating singular triangulations. In particular, due to finite size effects the set of subdominating singular triangulations $\mathbb{S}^4_{\rm es}$ for $h = 0, 1, \ldots, 6$ seems to provide a metastable cluster of configurations that entropically dominate the crumpled state. Taken at face value, this set of results seem to indicate, at least to the indulgent reader, a variety of viewpoints on the actual status of a theoretical interpretation of the numerical simulations:

(I) The bimodality as well as the implied first-order interpretation of the transition between weak and strong coupling is a finite size effect related to (i) The saturation of the triangulations of $\{\mathbb{S}^4\}$ with $\mathbb{S}^4_{\rm es}$ in the strong coupling phase; (ii) The slow dependence of the specific entropy, $\ln c(S^4; h)$, of $\{\mathbb{S}^4\}$ from the parameter h controlling the volume of the singular part of the triangulation. This slow variation may be responsible

Table 4
A comparison between the analytical values and the available Monte Carlo data for the first two cumulants of the distribution of the number of vertices of the triangulation

k_2	$c_1(N_4;k_2)$	c_1(Monte Carlo)	$c_2(N_4;k_2)$	c_2(Monte Carlo)
1.240	0.1935053	0.18970(12)	0.109062	0.141(7)
1.246	0.1945674	0.19150(11)	0.1194586	0.144(8)
1.252	0.1956271	0.19399(32)	0.1465348	0.254(35)
1.258	0.1966846	0.19712(20)	0.3996907	0.316(8)
1.264	0.1977398	0.20052(21)	0.1844987	0.118(20)
1.270	0.1987927	0.20085(27)	0.1274851	0.118(20)

of the fact that the tunnelling does not disappear as the volume of the triangulations increases.

Obviously, this latter remark can be easily turned inside out to favor a less optimistic point of view:

(II) The slow h-variation in $\ln c(S^4;h)$ may well be such as to maintain the bimodality for larger and larger volumes: we have a genuine first-order transition. It is rather clear that our analysis, being based on a sort of mean field approximation, cannot distinguish clearly between such two scenarios: we need sharper entropic estimates. Even if shamefully low in providing answers to the headlines that numerical simulations score, we wish to conclude with a final example pointing to a constructive way of using our analytical entropy estimates. This final point concerns the k_2 dependence of the two normalized cumulants of the distribution of the number of vertices of the triangulation, $c_1(N_4;k_2)$ and $c_2(N_4;k_2)$ whose analytic expression is explicit provided by (28) and (29). Strictly speaking, these expressions are accurate only near the actual critical average incidence b_0, however we can use them quite safely in a rather larger range of variation of k_2 (due to the slow variation of $b(2,4)$ as a function of k_2). Accurate Monte Carlo measurements of such cumulants have been reported in [6], and by referring to these data for $N_4 = 32\,000$, the comparison between MC data and our analytic results for $c_1(N_4;k_2)$ and $c_2(N_4;k_2)$ are shown in Table 4. The agreement between the analytical cumulant $c_1(k_2;N_4)$ and its Monte Carlo counterpart is particularly good; (note that for a better comparison with the numerical data we have actually used in (29) an average between $b(4,2)|_{h=0}$ and $b(4,2)|_{h=1}$ so as to shift from $k_2^{\rm crit} \simeq 1.3093$ to a pseudo-critical $k_2^{\rm crit}(N_4) \simeq 1.258$). Slightly less impressive is the agreement between the second cumulants, but this is to be expected since near the pseudo-critical point $k_2^{\rm crit}(N_4)$, the second cumulant $c_2(N_4;k_2)$ fluctuates quite wildly. We wish to stress that such an agreement rests both on the rigorous asymptotics (21), (23) and on the scaling *hypotheses*

$$m(k_2) = \frac{1}{\nu}\left|\frac{1}{b(k_2)} - \frac{1}{b_0}\right|^{\nu} \tag{163}$$

and

$$\lim_{\substack{N_4 \to \infty \\ k_2 \to k_2^{\rm crit}}} \left| \frac{1}{b(k_2)} - \frac{1}{b_0} \right|^{\nu - 1} \cdot N_4^{\frac{1}{n_H} - 1} = \text{const.}, \tag{164}$$

The best agreement, used in Table 4, is obtained by choosing $\nu \approx 0.94$. Eq. (163) is nothing but a natural consequence of the vanishing of the parameter $m(b)$ for $b(2,4) \to b_0$, whereas the second condition (164) rests on a less firm ground and must be considered as a working hypothesis to be better substantiated. Some of the results discussed above show that the numerical evidence pointing toward a first-order nature of the transition can be explained in a natural geometrical framework. The bimodality, which has been underlined as a strong indication that the transition is of a first order, is well explained by the presence of entropically subdominating peaks in the distribution of singular triangulations. Similarly to what has been argued by Catterall et al. [5], the system tunnels among such distinct subdominant configurations with some of these configurations being metastable for N_4 finite (especially those with $h \simeq 0, 1, \ldots$ which dominate the crumpled phase, and those for which $h \gg 1$ characterizing the branched polymer phase). Of course the analytical arguments provided by us are all based on a kind of mean-field approximation, since we consider only a restricted class of triangulations. Mean-field analysis is in general not very reliable when it comes to predicting the *order* of a phase transition. However, in this case we have seen that combined with an additional scaling assumption, we get reasonable agreement with Monte Carlo data for both $k_2^c(N_4)$, $c_1(N_4)$ and $c_2(N_4)$. This might indicate a validity beyond that usually provided by a mean-field approximation.

A good test of the reliability of the geometric truncation used in the present work is to apply it to the more complicated system of 4D simplicial quantum gravity coupled to Abelian gauge fields. In that system one seemingly observes a new interesting phase structure [19], different from the branched polymer - crumpled phase originally reported in [20].

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