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Polymeric phase of simplicial quantum gravity

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Abstract

We deduce the appearance of a polymeric phase in 4-dimensional simplicial quantum gravity by varying the values of the coupling constants and discuss the geometric structure of the phase in terms of ergodic moves. A similar result is true in 3-dimensions. © 1998 Elsevier Science B.V.

1. Introduction

Path-integral approach to quantum gravity leads in a natural way to geometric and probabilistic problems that have a great interest both in physics and mathematics [1,2]. A basic issue of the theory is to make sense of a formal probability measure on Riemannian structures associated with the partition function

$$\mathscr{Z} = \sum_{\text{Top}} \int_{\frac{\text{Riem}(M)}{\text{Diff}(M)}} \mathscr{D}[g(M)]\, e^{-S[g,M]} \tag{1}$$

where the action S is defined as a functional over Riemannian manifolds (M,g)

$$S[M,g] = \Lambda \int_{(M,g)} dx^4 \sqrt{g} - \frac{1}{16\pi G} \int_{(M,g)} dx^4 \sqrt{g}\, R \tag{2}$$

with Λ the cosmological constant and G the gravitational constant. $\mathscr{Z}$ is a badly ill-defined quantity since both the sum over topologies and the integration on the space of Riemannian structures $\frac{\text{Riem}(M)}{\text{Diff}(M)}$ cannot be given any sensible mathematical status.

The idea of simplicial quantum gravity is the classical and often useful idea of bypassing such issues by discretizing the theory and recovering the continuum one with a suitable limiting procedure. Following Regge [8] we use as discrete Riemannian manifolds piecewise flat manifolds obtained by gluing together a finite number of simplices. Briefly: inequivalent (in the sense of Tutte) dynamical triangulations will simulate the inequivalent Riemannian structures; the vector

$$f = (N_0, N_1, N_2, N_3, N_4) \tag{3}$$

with N_i the number of i dimensional simplices is called the f vector of the triangulation and the curvature is concentrated on the $d-2$ dimensional simplices (bones): the curvature on the bone b is

$$K(b) = \frac{48\sqrt{15}}{a^2} \left[\frac{2\pi - q(b)\cos^{-1}\frac{1}{4}}{q(b)} \right] \tag{4}$$

where $q(b)$ is the number of d dimensional simplices incident on b and a is the length of the sides

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of the gluing simplices. The discrete counterpart of the action (2) is

$$S = k_4 N_4 - k_2 N_2 \tag{5}$$

where k_4 depends linearly on the inverse of the gravitational constant and on the cosmological constant, whereas k_2 is proportional to the inverse of the gravitational constant. Restricting to triangulations with spherical topology, the partition function of the discrete theory is:

$$\mathscr{Z}(k_2,k_4) = \sum_{T \in S^4} e^{k_2 N_2(T) - k_4 N_4(T)}. \tag{6}$$

This can be written more explicitly in the form

$$\mathscr{Z}(k_2,k_4) = \sum_{N_4} \sum_{N_2} W(N_2,N_4) e^{k_2 N_2 - k_4 N_4} = \sum_{N_4} Z(N_4,k_2) e^{-k_4 N_4} \tag{7}$$

where $W(N_2,N_4)$ is the number of inequivalent spherical triangulations with N_4 simplices and N_2 bones, and $Z(N_4,k_2)$ is the canonical partition function (at fixed volume). It is important to stress that N_4 and N_2 completely determine the f vector due to the Dhen-Sommerville relations:

$$\sum_{i=0}^{4} (-1)^i N_i(T) = \chi(T) \tag{8}$$

$$\sum_{i=2k-1}^{4} (-1)^i \frac{(i+1)!}{(i-2k+2)!(2k-1)!} N_i(T) = 0 \tag{9}$$

with $k = 1,2$. Eq. (8) is just the Euler-Poincaré equation while (9) are consequence of the fact that the link of every $2k$-simplex is an odd dimensional sphere, and hence has Euler number zero. Our interest will be concentrated on the statistical system (7).

2. Geometrical constraints and ergodic moves

There exists a set of ergodic moves (elementary surgery operations) in 3 and 4 dimensions called generically (k,l) moves [5]: k and l are integers numbers such that $k + l = d + 2$. The moves consist in cutting out a subcomplex made up of k simplices substituting it with a complex of l simplices with the same boundary. In particular, if s^d is the d-dimensional simplex, the k complex is the star of a $d-k+1$ simplex in ∂s^{d+1} and the l complex is the complement (see [5] for a detailed description). In this way all spherical triangulations can be constructed starting from the basic ∂s^{d+1} with a finite number of moves (actually in $4d$ this is true only for smooth triangulations). Following this construction we can give a characterization of the generic f vector by analyzing how (k,l) moves modify it:

$$(1,5) \to \Delta_{1,5} f = (1,5,10,10,4) \tag{10}$$

$$(2,4) \to \Delta_{2,4} f = (0,1,4,5,2) \tag{11}$$

$$(3,3) \to \Delta_{3,3} f = (0,0,0,0,0) \tag{12}$$

and obviously $\Delta_{5,1} f = -\Delta_{1,5} f$ and $\Delta_{4,2} f = -\Delta_{2,4} f$. If $n_{k,l}$ is the number of moves of the type (k,l) the corresponding f vector will be

$$f = (6 + x_1, 15 + 5x_1 + x_2, 20 + 10x_1 + 4x_2, 15 + 10x_1 + 5x_2, 6 + 4x_1 + 2x_2) \tag{13}$$

with $x_1 = n_{1,5} - n_{5,1}$ and $x_2 = n_{2,4} - n_{4,2}$.

This characterization of the f vector is equivalent to (3) with the Dhen-Sommerville relations. This is not enough to completely determine the possible f vectors since it is not always possible to perform a (k,l) move. It is always possible apply a move of the type (1,5) but to apply the reverse move we must start from a triangulation that has a vertex with a star made of 5 simplices. It is often possible to apply a move of type (2,4) but in order to apply the reverse move we must start from a triangulation that has an edge with a star made of 4 simplices. A result in this direction is the following, essentially due to Walkup [9]:

Theorem *For any combinatorial triangulation of a 4 sphere the inequality*

$$N_1 \geq 5N_0 - 15 \tag{14}$$

holds with equality if and only if it is a stacked sphere.

A d-dimensional stacked sphere is a triangulation obtained from ∂s^{d+1} applying only $(1,d+1)$ moves.

To this lower bound theorems we can add a more trivial upper bound

$$N_1 \leq \frac{N_0(N_0-1)}{2} \qquad (15)$$

that only says that evidently the edges must be less than all the possible couple of vertices, together with the obvious condition

$$N_0 \geq d+2 \qquad (16)$$

Translating this inequalities in terms of moves we obtain

$$x_2 \geq 0, \quad x_1 \geq 0, \quad x_1^2 + x_1 - 2x_2 \geq 0 \qquad (17)$$

It is interesting also to write explicitly these constraints in terms of the parameters that are usually used, N_2 and N_4, since we will be interested in the statistic behavior at fixed volume:

$$N_2 \leq \tfrac{5}{2}N_4 + 5, \quad N_2 \geq 2N_4 + 8 \qquad (18)$$

$$9N_2^2 - 18N_2 - 12N_2N_4 + 24N_4 + 4N_4^2 \geq 0 \qquad (19)$$

It will be very useful [2] in the study of the asymptotic behavior of our model to introduce the parameter $\xi = N_2/N_4$. Also the geometrical constraints become more easy, in the large volume limit, using this parameter; Eqs. (18) tell to us that

$$\xi \leq \frac{5}{2} + \frac{5}{N_4}, \quad \xi \geq 2 + \frac{8}{N_4} \qquad (20)$$

that in the limit $N_4 \to \infty$ become

$$2 \leq \xi \leq \tfrac{5}{2} \qquad (21)$$

From Eq. (19) we obtain

$$9\xi^2 - 18\frac{\xi}{N_4} - 12\xi + \frac{24}{N_4} + 4 \geq 0 \qquad (22)$$

and using the bounds (21) we obtain

$$9\xi^2 - 12\xi + 4 = (3\xi - 2)^2 \geq 0 \qquad (23)$$

that is always true.

These asymptotic conditions are consequences of Walkups theorem whose demonstration is in fact quite not trivial. However, we can give a simple argument in terms of moves providing an intuitive picture: as we have already stressed it is far more easy to perform a (k,l) move with $k<l$ than the reverse one and such a move increases the volume of the manifold while the reverse move decreases it; so we can conclude that, when the number of simplices is large, almost all (in fact Walkups theorem say all (17)) triangulations are obtained with a number of (k,l) moves greater than (l,k). So we can obtain the conditions (21) as limiting values in the boundaries of the allowed region:

$$\xi_{\min} = \lim_{x_2 \to \infty} \xi = \lim_{x_2 \to \infty} \frac{20 + 10x_1 + 4x_2}{6 + 4x_1 + 2x_2} = 2 \qquad (24)$$

$$\xi_{\max} = \lim_{x_1 \to \infty} \xi = \lim_{x_1 \to \infty} \frac{20 + 10x_1 + 4x_2}{6 + 4x_1 + 2x_2} = \frac{5}{2} \qquad (25)$$

A funny dynamical interpretation of the constraints can be given in terms of equilibrium points of the ''moves operators'': starting from a triangulation with $\xi = N_2/N_4$ we have a jump

$$\Delta_{1,5}\xi = \frac{N_2 + 10}{N_4 + 4} - \frac{N_2}{N_4} \qquad (26)$$

The equilibrium condition is

$$\Delta_{1,5}\xi = 0 \leftrightarrow \xi = \frac{5}{2} \qquad (27)$$

and this equilibrium point is stable in the sense that

$$\Delta_{1,5}\xi > 0 \leftrightarrow \xi < \tfrac{5}{2}, \quad \Delta_{1,5}\xi < 0 \leftrightarrow \xi > \tfrac{5}{2} \qquad (28)$$

Likewise for move (2,4)

$$\Delta_{2,4}\xi = 0 \leftrightarrow \xi = 2 \qquad (29)$$

and also this equilibrium point is stable in the sense (28). This simple analysis explain why only the region (21) is spanned when constructing spherical triangulations with a large number of simplexes: because points ξ that are outside this interval are attracted towards it.

3. Asymptotic behavior of canonical measure

The behavior of the system conditioned to fixed volume is described by canonical partition function

$$Z(k_2, N_4) = \sum_{T_{N_4}} e^{k_2 N_2} = \sum_{N_2} W(N_2, N_4) e^{k_2 N_2} \qquad (30)$$

We can study it by using the parameter ξ [2]:

$$Z(k_2, N_4) = \sum_k W(N_4, \xi_k N_4) e^{k_2 N_4 \xi_k} \qquad (31)$$

Since the number of triangulations with N_4 simplexes is asymptotically exponentially bounded (see [2] for a demonstration), the asymptotic behavior of $W(N_4,\xi_k N_4)$ can be formalized in the form

$$W(N_4,\xi_k N_4) \sim f(N_4,\xi_k)\, e^{N_4 s(\xi_k)} \tag{32}$$

with $f(N_4,\xi)$ that has typically a polynomial or subexponential asymptotic behavior in N_4. The measure induced in this way in the space of triangulations is defined by the probabilities

$$\mu^C_{k_2,N_4}(\xi_k) = \frac{f(N_4,\xi_k)\, e^{N_4(s(\xi_k)-k_2\xi_k)}}{Z(k_2,N_4)} \tag{33}$$

The asymptotic behavior of probability measures defined in this way is a classical problem of probability theory and under general conditions the result is

$$\mu^C_{k_2,N_4} \Rightarrow_{N_4\to\infty} \sum_i \mu_i \delta(\xi-\xi_i^*) \tag{34}$$

The points ξ_i^* are defined by the condition

$$s(\xi_i^*) - k_2\xi_i^* = \sup_{\xi_{\min}\le\xi\le\xi_{\max}}[s(\xi)-k_2\xi] \tag{35}$$

and the convergence (34) is very fast; namely considered a set A such that $\xi_i^* \notin A$ we have

$$\mu^C_{k_2,N_4}(A) \sim \frac{e^{N_4(\sup_{\xi\in A}[s(\xi)-k_2\xi])}}{e^{N_4(s(\xi^*)-k_2\xi^*)}} \sim e^{-KN_4} \tag{36}$$

with $K>0$; that says that the probability of deviant events A goes to zero exponentially fast: this fact is usually referred as the deviations are large.

This general argument can be formalized in this particular case in terms of Laplaces method: the form of the partition function

$$Z(k_2,N_4) = \sum_k f(N_4,\xi_k)\, e^{N_4(s(\xi_k)+k_2\xi_k)} \tag{37}$$

has the structure of a Riemann sum

$$Z(k_2,N_4) \sim \sum_k N_4 f(N_4,\xi_k)\, e^{N_4(s(\xi_k)+k_2\xi_k)}\Delta(\xi_k) \tag{38}$$

namely $\xi_{k+1}-\xi_k \sim 1/N_4$ and we have a sum of $\frac{\xi_{\max}-\xi_{\min}}{\Delta\xi} \sim N_4$ terms. So for large N_4 (38) is well approximated by the continuum version (see [2] for details and an explicit form of $s(\xi)$ and $\tilde{f}(N_4,\xi)$)

$$Z(k_2,N_4) \sim \int_{\xi_{\min}}^{\xi_{\max}} \tilde{f}(N_4,\xi)\, e^{N_4(s(\xi)+k_2\xi)}d\xi \tag{39}$$

Laplaces theorem says that when N_4 is large almost all the contribution to the value of the integral (39) comes from the region near the point(s) ξ^*; and this is a result of type (34).

4. Polymeric phase

We are now interested in a theoretic interpretation of numerical results that give a strong evidence of the appearance of a polymeric phase for k_2 large enough [3]. We will show that this phenomena is a direct consequence of the concentration of the measure illustrated in the previous chapter and we will analyze the geometrical characteristic of this phase.

When k_2 is large, triangulations with large ξ are favorite; we translate this simple idea into a mathematical language: the points of maximum in the exponent of the expression (39) are determined by the condition

$$s'(\xi) + k_2 = 0 \tag{40}$$

from this it is immediately to deduce that if $k_2 > -\inf_{\xi_{\min}\le\xi\le\xi_{\max}} s'(\xi)$ the expression (40) is always greater than zero and the canonical measure concentrates on $\xi_{\max}$:

$$\mu^C_{k_2,N_4}(\xi) \Rightarrow_{N_4\to\infty} \delta(\xi-\xi_{\max}) \tag{41}$$

These triangulations are well described by the lower bound result of Walkup and we can conclude that in this region of the parameter k_2 the dominant configurations in the statistical sum (30) are essentially stacked spheres. A more precise statement could be that for the dominant configurations the most important phenomena is the stacking ($(1,d+1)$ moves); a general characterization can in fact be given: starting from $y_1 = \frac{N_4-2y_2-6}{4}$ we obtain

$$\xi = \frac{5}{2} - \frac{y_2}{N_4} + \frac{5}{N_4} \tag{42}$$

We can easily conclude that triangulations with large N_4 characterized by a $\xi = \xi_{\max} = \frac{5}{2}$ are obtained with the condition

$$\lim_{N_4\to\infty} \frac{y_2}{N_4} = 0 \tag{43}$$

This condition is satisfied not only by stacked spheres, which are defined by the relation $y_2 = 0$, but also by triangulations with $y_2 = C < \infty$ that can be constructed, for example, by stacking starting from a generic triangulation (defined by $y_2 = C$) and not from the basic triangulation ∂s^5; and more in general the condition is satisfied also by triangulations constructed with a number of moves y_2 that grows with a power in N_4 smaller than one.

The practical consequence of this assertions is that we can study the statistical property of simplicial quantum gravity for that region of k_2 by studying a more simple model obtained by restricting the space of configurations: the smaller statistical system that we will consider is the system constituted by only stacked spheres. We have just showed that this in fact is a further simplification but what happens is that this subsystem contains all the principal features of interest. In general the following trivial relation is true:

$$Z(k_2, N_4) > \sum_{(\mathrm{S.S.})_{N_4}} e^{k_2 N_2} \tag{44}$$

but for k_2 large enough this relation becomes

$$Z(k_2, N_4) \gtrsim \sum_{(\mathrm{S.S.})_{N_4}} e^{k_2 N_2} \tag{45}$$

where the abbreviation (S.S.) means obviously (Stacked Spheres) and the symbol $\gtrsim$ means that the exact relation is $\geq$ but the asymptotic behavior is the same $\sim$.

The study of the partition function for the stacked spheres system (45) is an easier problem: the condition for stacked spheres $y_2 = 0$ tells to us that $N_4 = 6 + 4y_1$ and $N_2 = 20 + 10y_1$ and we obtain the relation $N_2 = \frac{5}{2}N_4 + 5$. This latter tells to us that the number of bones is determined by the number of simplices. Consequently, the explicit expression of the partition function is:

$$Z_{\mathrm{S.S.}}(k_2, N_4) = W_{\mathrm{S.S.}}(N_4)\, e^{k_2(\frac{5}{2}N_4 + 5)} \tag{46}$$

The problem that remain to face is the calculation of the number of inequivalent stacked spheres with N_4 simplices: we will do it in an approximate way stressing above all the fact that the structure of the different configurations is typical of branched polymers with a fundamental element (monomer) that builds up a tree configuration, and we will also calculate the entropy exponent γ and show that it is $\frac{1}{2}$ as is typical of branched polymers.

The tree structure that is behind stacked spheres can be easily reconstructed from the following geometrical interpretation of (1,5) moves: starting from s^5 we construct the basic spherical 4-d triangulation as ∂s^5; a (1,5) move is obtained by substituting a 4-d simplex of this triangulation with 5 simplices as discussed in Section 2; but we can proceed also in a different way by constructing a new triangulation of the 5-d ball gluing a second s^5 trough a 4-d face to the beginning simplex. The spherical triangulation ∂B where B is the new triangulating ball obtained in this way is equivalent to the triangulation obtained with the (1,5) move. This construction is general: we obtain every stacked sphere as the boundary ∂B_n of triangulations of the 5-d ball obtained by gluing a s^5 to B_{n-1} through a 4-d face of ∂B_{n-1} and this correspondence is easily seen to be one to one [9]. It is also easily to see that ∂B_n is a stacked sphere with $2 + 4n$ simplices. This construction is illuminating in characterizing a polymeric phase in simplicial quantum gravity. The monomer with which the polymer is builded is provided by the s^5 simplices and the polymer structure is obtained by analyzing the only tree-like triangulations of 5-d ball whose boundary are stacked spheres. A good insight in the structure of such polymers and also a tool for calculating $W_{\mathrm{S.S.}}(N_4)$ is obtained by analyzing the 1-d skeleton of the dual of $B_n s$. To every s^5 it is associated a point and in each such points there are 6 lines incident that correspond to the 6 4-d faces; when two s^5 are glued along a face the corresponding points are joined by a line. The graphs obtained in this way are all the possible trees the incidence numbers of which are only 6 and 1: the vertices with incidence 6 represent the 5-d simplices glued together and the vertices with incidence 1 represents the free 4-d faces of ∂B_n. This construction is not enough to reconstruct the ball B_n, and the corresponding stacked sphere. This reconstruction would be possible only from the knowledge of the full dual structure, nonetheless the above partial construction will be enough to calculate the right asymptotic behavior.

The problem of counting such trees is equivalent to a problem of counting isomers in chemistry: a solution is given in the classical paper of Otter [7].

The asymptotic expression of the number of not isomorphic trees with n vertex and ramification number not greater than m is:

$$T_n^{<m} \sim c(m)\frac{\alpha(m)^n}{n^{\frac{5}{2}}} \tag{47}$$

This is easily seen to be also the solution of our counting problem: namely the following relation holds:

$$T_n^{<m} = T^{(1,m)}_{((m-2)n+2,n)} \tag{48}$$

The notation of the left side was already explained and the symbol on the right side means the number of trees with $(m-2)n+2$ vertices with number of incidence 1 and n vertices with number of incidence m. The equality is verified by constructing explicitly a one to one correspondence: starting from a tree in $T^{(1,m)}_{((m-2)n+2,n)}$ we delete all vertices with number of incidence 1 and the corresponding lines and we obtain an element in $T_n^{<m}$; the reverse correspondence is obtained by joining new vertices with ramification number 1 to the old vertices until all the old vertices reach the exact ramification number m.

In order to count the number of stacked spheres with N_4 simplexes we have to estimate the number of inequivalent trees with $\frac{N_4-2}{4}$ vertices with incidence 6 and N_4 vertices with incidence 1; we get

$$W_{\mathrm{S.S.}}(N_4) \gtrsim T^{(1,6)}_{\left(N_4,\frac{N_4-2}{4}\right)} = T^{<6}_{\frac{N_4-2}{4}} = c\frac{\alpha^{\frac{N_4-2}{4}}}{\left(\frac{N_4-2}{4}\right)^{\frac{5}{2}}} \sim \cos t\frac{\left(\alpha^{\frac{1}{4}}\right)^{N_4}}{N_4^{\frac{5}{2}}} \tag{49}$$

With this rough but effective asymptotic estimate we get informations about the canonical partition function by using relations (45), (46)

$$Z(k_2,N_4) \gtrsim C(k_2)\frac{1}{N_4^{\frac{5}{2}}}e^{N_4\left(\frac{1}{4}\log\alpha+\frac{5}{2}k_2\right)} \tag{50}$$

Thus, we have obtained an expression of the form

$$f_{k_2}(N_4)\,e^{N_4 k_4^c(k_2)} \tag{51}$$

with a subleading asymptotics $f_{k_2}(N_4)$ of polynomial type. The subleading asymptotics is particularly important because from it we can deduce the entropy exponent γ: the general form is $f_{k_2}(N_4) \sim N_4^{\gamma-3}$ that in our case gives $\gamma - 3 = -\frac{5}{2}$ and we obtain $\gamma = \frac{1}{2}$ as is typical of branched polymers and as comes out from numerical simulations [3]. From the knowledge of this exponent we can for example obtain the critical behavior of susceptibility [3]

$$\sum_r G(r,k_2,k_4) = \frac{d^2}{dk_4^2}\mathscr{Z}(k_2,k_4) \sim (k_4-k_4^c)^{-\gamma} = (k_4-k_4^c)^{-\frac{1}{2}} \tag{52}$$

where $G(r,k_2,k_4)$ is the correlation function [3].

From expression (50) we can also get an estimate of the critical line that we expect to be quit good when the parameter k_2 is large enough:

$$k_4^c(k_2) \gtrsim \tfrac{1}{4}\log\alpha + \tfrac{5}{2}k_2 \tag{53}$$

This turn out to be in fact compatible both with numerical and analytical [2] results.

5. Conclusions

There exists a Walkups theorem also for the 3-dimensional case [6,9], and all the previous analysis can be repeated.

The asymptotic behavior of canonical measure suggested in Section 3 stresses the peculiar character of simplicial quantum gravity as a critical system: the measure concentrates on different regions of the space of configurations for different values of k_2. This characteristics allows, for example, to compute the mean value of geometrical objects restricting on a smaller region of the configuration space.

$$E_{\mu^C_{k_2,N_4}}(f) \Rightarrow_{N_4\to\infty} E_{\delta(\xi-\xi^*(k_2))}(f) \tag{54}$$

This is exactly the procedure followed to describe the structure of the polymeric phase and more informations could be obtained with a detailed study of statistical mechanics of stacked spheres (correlations functions, for example).

A further step must be the comprehension of the geometric structure of the crumpled phase. This is connected with the discovery of upper bound theorems that substitute the trivial bound (15); recent results (see [4], for example) suggest that the region of the phase space that corresponds to the crumpled phase is characterized by the appearance of singular structures.

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