A FLOW ON NETWORK APPROACH TO SOME PROBABILISTIC PROBLEMS

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References

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- L. Bertini, A. Faggionato, D. Gabrielli – *From level 2.5 to level 2 large deviations for continuous time Markov chains* arXiv:1212.6908
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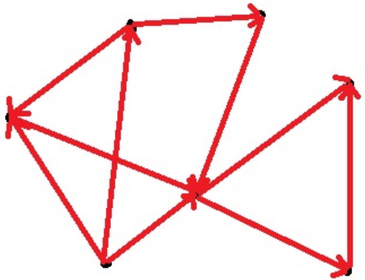
Flows on Networks

- A network is an oriented graph (finite or infinite); $x \in V$ is a vertex, $(x, y) \in E$ is an oriented edge.
- A flow $Q : E \to \mathbb{R}^+$ assigns the amount of mass $Q(x, y) \geq 0$ flowing on the edge (x, y).



Discrete divergence

The divergence of Q at $x \in V$ is

$$\begin{aligned} \operatorname{div} Q(x) &= \text{exiting flow} - \text{entering flow} \\ &= \sum_{y:(x,y)\in E} Q(x,y) - \sum_{y:(y,x)\in E} Q(y,x) \end{aligned}$$

Discrete Calculus

$$\sum_{x\in V} f(x) \operatorname{div} Q(x) = -\sum_{(x,y)\in E} Q(x,y)\big(f(y) - f(x)\big)$$

Continuity equation

μ_1 and μ_2 two probability measures on V

Q constant in time flow such that $\operatorname{div} Q = \mu_1 - \mu_2$

Consider

$$\begin{cases} \partial_t \nu + \operatorname{div} Q = 0\,, \\ \nu(0) = \mu_1 \end{cases}$$

then

$$\nu(t) = (1-t)\mu_1 + t\mu_2$$

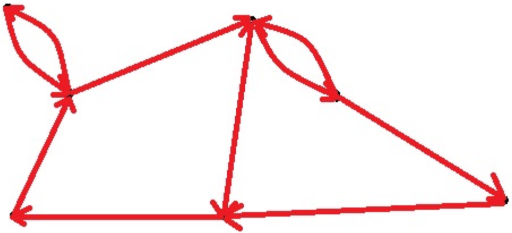
the constant flow Q transform μ_1 into μ_2 in a unitary time

Networks from Markov chains

Transition graph of a continuous time Markov chain

$$\boxed{(x,y) \in E \Longleftrightarrow r(x,y) > 0}$$

$X(t)_{t\in[0,T]}$ a trajectory



Empirical measure and flow

Empirical measure

$$\mu_T(x) = \frac{1}{T}\int_0^T \delta_{x,X(s)} ds$$

probability measure on V

Empirical flow

$$Q_T(x,y) = \frac{1}{T}\Big| \{\text{jumps } x \to y \text{ in } [0,T]\} \Big|$$

Observe that $\operatorname{div} Q_T(x) = 0 \ \forall x \neq X(0), X(T)$.

Large deviations

μ_T satisfies a LDP when $T \to +\infty$ (Donsker-Varadhan) with rate functional

$$\mathbb{P}\left(\mu_T \simeq \mu\right) \sim e^{-T I_{dv}(\mu)}$$

$$I_{dv}(\mu) = \sup_f \left\{ \sum_{(x,y)\in E} \mu(x) r(x,y) \big(1 - e^{f(y)-f(x)}\big) \right\}$$

(μ_T, Q_T) satisfy a joint LDP with explicit rate functional

$$I(\mu, Q) = \sum_{(x,y)\in E} \left[Q(x,y) \log \frac{Q(x,y)}{\mu(x) r(x,y)} + \mu(x) r(x,y) - Q(x,y) \right]$$

if $\operatorname{div} Q = 0$ and $+\infty$ otherwise

Structure of the rate functional

$$I(\mu, Q) = \sum_{(x,y)\in E} I^P_{\mu(x)r(x,y)}\big(Q(x,y)\big)$$

where if N_T is Poisson process of parameter λ the $\frac{N_T}{T}$ satisfies a LDP with rate

$$I^P_\lambda(x) = x\log\frac{x}{\lambda} + \lambda - x$$

By contraction $\Longrightarrow$ Donsker-Varadhan

$$\inf_Q I(\mu, Q) = I_{dv}(\mu)$$

Idea of the proof

Consider two Markov chains with transition rates $r(x,y)$ and $r'(x,y)$; call $r(x)=\sum_y r(x,y)$, $r'(x)=\sum_y r'(x,y)$; then

$$\frac{d\mathbb{P}}{d\mathbb{P}'}(X)=e^{-T\left[\sum_{(x,y)\in E}Q_T(x,y)\log\frac{r'(x,y)}{r(x,y)}+\sum_{x\in V}\mu_T(x)\left(r(x)-r'(x)\right)\right]}$$

is a function of the empirical measure μ_T and flow Q_T associated to the trajectory $X=X(t)_{t\in[0,T]}$

Motivations

- Scaling limit for specific models (and not)
- Many interesting observables are function of empirical measure and flow

Empirical current $J_T(x,y) = Q_T(x,y) - Q_T(y,x)$

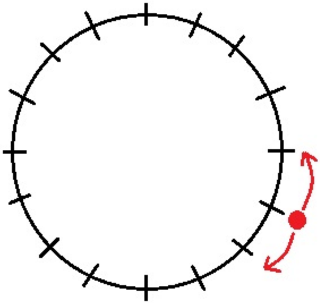
Total activity $N_T = \sum_{(x,y)\in E} Q_T(x,y)$

Gallavotti-Choen functional $W_T = \frac{1}{2}\sum_{(x,y)\in E} J_T(x,y)\log\frac{r(x,y)}{r(y,x)}$

One particle on a ring

One particle on a ring

$r(x, x+1) = r(x+1, x) = \lambda$ homogeneous symmetric, or homogeneous + one distinguished bond (tomorrow?), or inhomogeneous



One particle on a ring

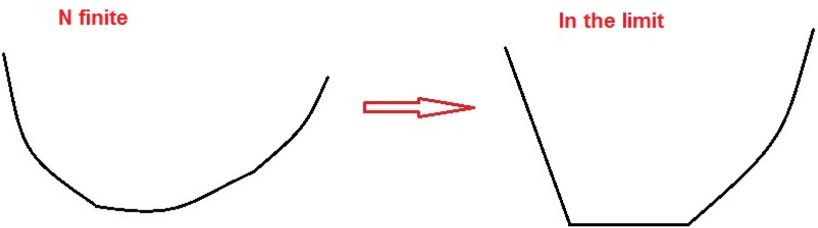
Joint LDP for (W_T, N_T) (exact formula!)

$$\Gamma(w,n) = \inf_{\mathcal{A}_{w,n}} I(\mu, Q)$$

where $\mathcal{A}_{w,n}$ is the collection of (μ, Q) s.t. $\sum_{(x,y)} Q(x,y) = n$ and $\frac{1}{2}\sum_{(x,y)} \big(Q(x,y) - Q(y,x)\big) \log \frac{r(x,y)}{r(y,x)} = w$

Minimizers give informations on the behavior of the system

Phase transitions



One particle on a ring

Simple case: LDP for empirical current J_T for the homogeneous symmetric walk

$$\Gamma_N(j) = \inf_{\mathcal{A}_j} I(\mu, Q)$$

where

$$\mathcal{A}_j = \{(\mu, Q) \,:\, Q(x, x+1) - Q(x+1, x) = j\}$$

you can solve the problem

$$\Gamma_N(j) = Nj \log\left(\frac{Nj + \sqrt{(Nj)^2 + \lambda^2}}{\lambda}\right) - \sqrt{(Nj)^2 + \lambda^2} + \lambda$$

Scaling limit

In the diffusive rescaling $\lambda = \lambda_N = \alpha N^2$

$$\lim_{N \to +\infty} \Gamma_N(j) = \frac{j^2}{2\alpha}$$

In the scaling limit you have current j at time T if $W_T \sim Tj$

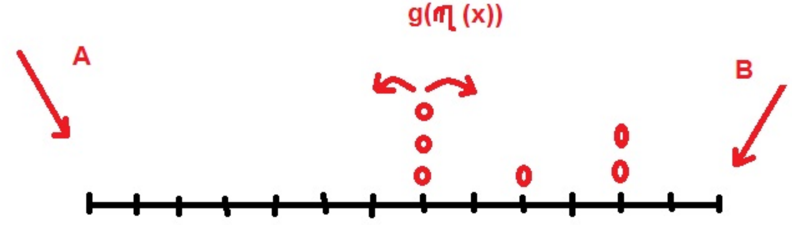
Potential applications

- current fluctuations for particle systems (TASEP yesterday); in principle also not one dimensional
- LDP for total activity of costrained lattice gases

Zero range model

Zero range model

LDP for the current through one bond of one dimensional boundary driven Z.R.



In the realm of configuration space

$$I(Q,\mu)=\sum_{\eta\in\Lambda_N}\Big[\Phi\Big(Q(\eta,\eta^{1,-}),\mu(\eta)g(\eta(1))\Big)+\Phi\Big(Q(\eta,\eta^{1,+}),\mu(\eta)A\Big)$$

$$+\sum_{x=1}^{N-1}\Phi\Big(Q(\eta,\eta^{x,x+1}),\mu(\eta)g(\eta(x))\Big)+\sum_{x=2}^{N}\Phi\Big(Q(\eta,\eta^{x,x-1}),\mu(\eta)g(\eta(x))\Big)$$

$$+\Phi\Big(Q(\eta,\eta^{N,+}),\mu(\eta)B\Big)+\Phi\Big(Q(\eta,\eta^{N,-}),\mu(\eta)g(\eta(N))\Big)\Big]$$

Projection

The flow Q on the configuration space can be projected to a flow on the simple physical space where particles move (one dimensional lattice)

$$\widetilde{Q}(x, x+1) = \sum_{\eta} Q(\eta, \eta^{x,x+1})$$

and then

$$\widetilde{j}(x, x+1) = \widetilde{Q}(x, x+1) - \widetilde{Q}(x+1, x)$$

An exact formula

LDP for the current through a bond

$$\Gamma(\widetilde{j}) = \inf_{\mathcal{A}_{\widetilde{j}}} I(\mu, Q)$$

exact but difficult

A restricted problem

If $Q(\eta,\eta')=\mu(\eta)r(\eta,\eta')$ then μ is invariant for $r \Longleftrightarrow Q$ is divergence free

We restrict to zero range perturbations of the original dynamics

$$Q(\eta,\eta')=\mu(\eta)r(\eta,\eta')e^{\gamma(\eta')-\gamma(\eta)\pm\lambda}$$

where

$$\gamma(\eta)=\sum_x \lambda(x)\eta(x)\,, \qquad \lambda \text{ parameters}$$

Then

$$\mu(\eta)=\prod_x \frac{\phi(x)^{\eta(x)}}{Z(\phi(x))g(\eta(x))!}$$

with

$$\phi(x)e^{\lambda(x+1)-\lambda(x)+\lambda}-\phi(x+1)e^{\lambda(x)-\lambda(x+1)-\lambda}=\widetilde{j}$$

A simplified functional

Inserting (μ, Q) of this form in $I(\mu, Q)$, after some algebra and

$$\sum_{\eta} \mu(\eta) g(\eta(x)) = \phi(x)$$

Sum on configuration space $\Longrightarrow$ sum on real space

$$\frac{\tilde{j}}{2}\left(\sum_{x=0}^{N} \log \frac{\tilde{j} + \sqrt{\tilde{j}^2 + 4\phi(x)\phi(x+1)}}{\sqrt{\tilde{j}^2 + 4\phi(x)\phi(x+1)} - \tilde{j}} + \log \frac{\beta}{\alpha}\right)$$

$$+A + B - \frac{1}{2}\left(\sqrt{\tilde{j}^2 + 4B\phi_N} + \sqrt{\tilde{j}^2 + 4A\phi_1}\right).$$

Minimizing the simplified problem

The parameters ϕ have no constraint, taking ∇_ϕ the stationary equations are

$$4\phi(x) = \sqrt{\tilde{j}^2 + 4\phi(x-1)\phi(x)} + \sqrt{\tilde{j}^2 + 4\phi(x)\phi(x+1)}\,, x = 1, \dots, N$$

Magically $\phi(x) = a + bx + cx^2$ with $b^2 - 4ac = \widetilde{j}$ solves !!
As $N \to +\infty$, with suitable rescaling $\Longrightarrow$ a kind of Riemannian sums converging to the corresponding continuum value of MFT

The continuum problem

MFT predicts LDP for time averaged current

$$\frac{1}{4}\inf\int_0^1 \frac{\left(\tilde{j}+G'(\rho)\nabla\rho\right)^2}{G(\rho(u))}d\,u\,. \tag{1}$$

The infimum is over all density profiles such that $G(\rho(0)) = \alpha$ and $G(\rho(1)) = \beta$. $G(x)$ is determined by the dynamics. If we call $\varphi(u) := G(\rho(u))$ we get

$$\frac{1}{4}\inf\int_0^1 \frac{\left(\tilde{j}+\varphi'(u)\right)^2}{\varphi(u)}du\,, \tag{2}$$

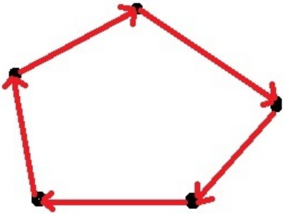
where φ is such that $\varphi(0) = \alpha$ and $\varphi(1) = \beta$. The corresponding Euler-Lagrange equation has solution $a + bx + cx^2$

Partial orders and networks

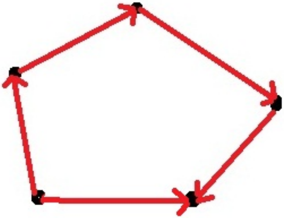
$\leq$ partial order relation on V, $x \leq y$

partial order $\Leftrightarrow$ oriented acyclic graph

An oriented acyclic graph contains no oriented cycles



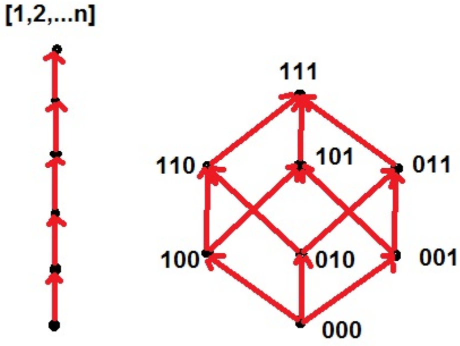
NO



YES

Hasse diagrams

∃ minimal network describing a partial order (Hasse diagram, transitive reduction)



Stochastic monotonicity

$\leq$ induces a partial order relation $\preceq$ on probability measures

$$\boxed{\mu_1 \preceq \mu_2} \text{ if and only if } \boxed{\mathbb{E}_{\mu_1}(f) \leq \mathbb{E}_{\mu_2}(f)}$$

for any function f increasing ($f(x) \leq f(y)$ when $x \leq y$)

Couplings

ρ a probability measure on $V \times V$ is a coupling between μ_1 and μ_2 if

$$\begin{cases} \sum_y \rho(x,y) = \mu_1(x) \\ \sum_x \rho(x,y) = \mu_2(y) \end{cases}$$

ρ is compatible w.r.t. $\leq$ if

$$\rho\Big(\{(x,y)\ : x \leq y\} \Big) = 1$$

Strassen Theorem

The following statements are equivalent

- $\mu_1 \preceq \mu_2$
- $\exists$ a compatible coupling ρ between μ_1 and μ_2

Intuition on the second issue: mass can be transported respecting the partial order

Strassen Theorem revisited

This intuition can be made precise

The following items are equivalent

- $\mu_1 \preceq \mu_2$
- $\exists$ a compatible coupling ρ between μ_1 and μ_2
- $\exists$ a flow Q on the Hasse diagram such that $\operatorname{div} Q = \mu_1 - \mu_2$

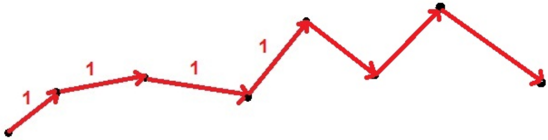
Finitely decomposable flows

Q is finitely decomposable if it can be written as $Q = \sum_n q_n I_{\gamma_n}$; γ_n self avoiding paths and $q_n \geq 0$ such that $\sum_n q_n < +\infty$

Sufficient condition: If Q has zero flux towards ∞

$$\lim_{n \to +\infty} \sum_{x \in V_v, y \notin V_n} Q(x, y) = 0$$

(V_n invading sequence) then it is finitely decomposable.



Strassen Theorem revisited (infinite case)

The following items are equivalent

- $\mu_1 \preceq \mu_2$
- $\exists$ a compatible coupling ρ between μ_1 and μ_2
- $\exists$ a finitely decomposable flow Q such that $\operatorname{div} Q = \mu_1 - \mu_2$

Constructive proof: algorithms $Q \Longrightarrow \rho$

An example

Integer lattice $\mathbb{Z}$

$\operatorname{div} Q = \mu_1 - \mu_2$ has one parameter family of solutions

$$Q(x.x+1) = \sum_{y=-\infty}^{x} \big(\mu_1(y) - \mu_2(y)\big) + c$$

Finitely decomposable $\Longleftrightarrow c = 0$

$$Q(x.x+1) = \sum_{y=-\infty}^{x} \big(\mu_1(y) - \mu_2(y)\big) = F_1(x) - F_2(x) \geq 0$$