ON THE COMPLETE INTEGRAL CLOSURE OF THE REES ALGEBRA

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1. Introduction and background.

In this work we analyze the complete integral closure of graded domains constructed using and manipulating some specific ideal filtrations in a domain R. Among the objects that encode information on the asymptotic behavior of the ideal I, the Rees algebra of I,

$$\mathcal{R}(I) = \oplus_{n\geq 0} I^n t^n \subseteq R[t],$$

is the chief example of blowup algebra, where by this name one refers to those algebraic structures that are loosely related, when R is Noetherian, to the concept of blowing up a variety along a subvariety (see [26]). A wealth of work has been produced on the integral closure of $\mathcal{R}(I)$ in its field of quotients in a Noetherian setting. Much less is known on the complete integral closure of $\mathcal{R}(I)$ when R is not necessarily Noetherian; the difficulties naturally arising because of the erratic behavior of the complete integral closure.

Let R be a an integral domain and K the quotient field of R. A nonzero element x of K is said to be *almost integral* on R if there is a nonzero element d of R such that $dx^i \in R$, for $i \geq 0$. This is equivalent to saying that all the powers of x belong to a finite R-module, [6, Chap. V, §1, n.4], or that they generate a fractionary ideal of R, [7, Lemma 3.1]. The complete integral closure of R in K, denoted $\widetilde{R}$, is the set of the elements of K almost integral over R and one says that R is completely integrally closed if $R = \widetilde{R}$. Every element integral over R is almost integral over R and the converse holds in the Noetherian case. Thus, if R' denotes the integral closure of R in K, one has $R \subseteq R' \subseteq \widetilde{R}$ and $R' = \widetilde{R}$ in the Noetherian case. It is well known that $\widetilde{R}$ is always integrally closed, see [25, page 172].

The complete integral closure however is not a proper closure operation. Many examples have been given to show that the complete integral closure of a domain might not be completely integrally closed itself, see [9] and [10]. In [23], Sheldon provides an example of a two-dimensional Prüfer domain whose complete integral closure is not a completely integrally closed domain. In [14], Hill constructs an example where iterating the process of taking the complete integral closure does not lead to a completely integrally closed domain in a finite number of steps.

Among valuation rings, the only completely integrally closed rings are the rank-one valuation rings, see [6, Chap. VI, §4, Proposition 9] and [19]. Any intersection of completely integrally closed domains with the same quotient field is completely integrally closed. Hence, Krull domains, which are locally finite intersection of discrete valuation rings of rank one (DVR) in their quotient field, are a class of completely integrally closed domains that are not necessarily Noetherian, see [6, Chap. VII, §3, Definition 3]. Intersections of rank-one valuation rings, however, do not exhaust the class of completely integrally closed domains as proved in [23]. It is further evidence of anomalous behavior the fact that a localization of a completely integrally closed domain is not necessarily completely integrally closed; an example is given by the ring of entire functions over the complex field [11].

In this work we examine the structure of the complete integral closure of $\mathcal{R}(I)$ and we give sufficient and necessary conditions on the ring R and the ideal I for $\mathcal{R}(I)$ being completely integrally closed or Krull. As in the Noetherian case, it turns out that the complete integral closure of $\mathcal{R}(I)$ is deeply connected with the structure of other blowup algebras like the extended Rees algebra and the associated graded ring. Namely

$$\mathcal{R}(I)^e = \mathcal{R}(I)[t^{-1}] = \oplus_{n\in\mathbb{Z}} I^n t^n \subseteq R[t, t^{-1}],$$
$$\mathcal{G}(I) = \oplus_{n\geq 0}(I^n/I^{n+1}) \cong \mathcal{R}(I)/I\mathcal{R}(I) \cong \mathcal{R}(I)^e/t^{-1}\mathcal{R}(I)^e,$$

where $I^n = R$ for all $n \leq 0$. Barshay proves in [5, Theorem 5] that if I is an ideal such that $\mathcal{G}(I)$ is reduced, then $\mathcal{R}(I)$ is integrally closed in $R[t]$. In particular, if R is integrally closed and $\mathcal{G}(I)$ is reduced, then $\mathcal{R}(I)$ is integrally closed. In this way, when R is Noetherian, Barshay's result gives a very simple sufficient condition to guarantee the complete integral closure of $\mathcal{R}(I)$. When R is not Noetherian the situation is more complicated and it is not difficult to give an example of a domain and an ideal such that the associated graded ring is reduced, $\mathcal{R}(I)$ is integrally closed but not completely integrally closed (see Example 3.8).

After having studied in Section 2 the structure of the complete integral closure of a graded domain, in Section 3 we prove that if $\mathcal{G}(I)$ is a domain, then $\mathcal{R}(I)$ is completely integrally closed (Krull) if and only if R is completely integrally closed (Krull) and $\bigcap_{n\geq 0} I^n = 0$; see Theorem 3.6 and Corollary 3.16. Further, if $I\mathcal{R}(I)$ can be written as an irredundant finite intersection of strongly primary ideals (e.g. $\mathcal{G}(I)$ reduced with finitely many minimal primes) and $P_1, \dots, P_n$ are the associated primes of $I\mathcal{R}(I)$, then $\mathcal{R}(I)$ is completely integrally closed (Krull) if and only if R is completely integrally closed (Krull) and $\mathcal{R}(I)_{P_i}$ is completely integrally closed (Krull) for each $i = 1, \dots, n$, if and only if R is completely integrally closed (Krull) and $\mathcal{R}(I)_{P_i}$ is a DVR for each $i = 1, \dots, n$; see Theorem 3.13. Besides, the condition that $I\mathcal{R}(I)$ is as an irredundant finite intersection of strongly primary ideals is also necessary for $\mathcal{R}(I)$ being a Krull domain; see Corollary 3.15.

In Section 4 we concentrate on the Rees algebra of a prime ideal $\mathfrak{p}$ of R. We prove that if $\mathcal{G}(\mathfrak{p}R_{\mathfrak{p}})$ is a domain, then the complete integral closure of $\mathcal{R}(\mathfrak{p})$ is contained in the symbolic algebra of $\mathfrak{p}$; see Proposition 4.1. When $R_{\mathfrak{p}}$ is a DVR and $\mathcal{G}(\mathfrak{p})$ is reduced with only finitely many minimal primes, we obtain that $\mathcal{G}(\mathfrak{p})$ is a domain if and only if $\mathcal{R}(\mathfrak{p})$ is completely integrally closed and the prime ideal $\mathfrak{q} = Q \cap R$ has height at most one, for any height-one prime ideal Q of $\mathcal{R}(\mathfrak{p})$; see Theorem 4.6.

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2. The complete integral closure of a graded domain

In this section we examine the general structure of the complete integral closure of graded domains and in particular of the Rees algebra of an ideal.

Let $A = \oplus_{\gamma\in\Gamma} A_\gamma$ be a domain graded by a torsionless additive monoid Γ and let $Q(A)$ be its quotient field. We denote by $H = \bigcup_{\gamma\in\Gamma} A_\gamma$ the set of homogeneous elements of A. Then H is a multiplicative set of A and A_H is graded by the group $< \Gamma >$. That is $A_H = \oplus_{\alpha\in<\Gamma>} (A_H)_\alpha$, where

$$(A_H)_\alpha = \{a/b \mid a \in A_\beta, b \in A_\gamma \text{ and } \alpha = \beta - \gamma\}.$$

A_H is called the *homogeneous quotient field* of A, even though in general it is not a field. It is clear that A_H is contained in $Q(A)$ and that $(A_H)_0$ is a field containing the quotient field of A_0. We say that an element of $Q(A)$ is *homogeneous of degree* α if it is contained in $(A_H)_\alpha$.

It has been proved in [3, Proposition 2.1] that, regardless to A, A_H is always completely integrally closed.

Next result guarantees that the complete integral closure of a graded structure is still graded. The same assertion can be found in [27, Proposition 4.17], we give the proof for completeness.

Lemma 2.1: *If $A = \oplus_{\gamma\in\Gamma} A_\gamma$, then $\widetilde{A} = \oplus_{\alpha\in<\Gamma>} \widetilde{A} \cap (A_H)_\alpha$.*

Proof: Since A_H is completely integrally closed, then $\widetilde{A} \subseteq A_H$. Whence any element $x \in \widetilde{A}$ is a sum of finitely many homogeneous elements x_{α_j} of $Q(A)$. We have to show that $x_{\alpha_j} \in \widetilde{A}$ for each j. Let $d = d_{\gamma_1} + \cdots + d_{\gamma_t} \in A$ be such that $dx^n \in A$ for each $n \geq 0$. It is then clear that $d_{\gamma_1}(x_{\alpha_1})^n \in A$ and so $x_{\alpha_1} \in \widetilde{A}$. Hence $x - x_{\alpha_1} \in \widetilde{A}$ and, by induction, $x_{\alpha_j} \in \widetilde{A}$ for each j. $\square$

The analogous statement for the integral closure is in [28, Vol II, Theorem 11]. Next result, a direct consequence of Lemma 2.1, appears also in [3, Proposition 5.2].

Lemma 2.2: *If $A = \oplus_{\gamma\in\Gamma} A_\gamma$ is a graded domain, the following are equivalent:*

(1) *A is completely integrally closed in $Q(A)$;*
(2) *A is completely integrally closed in A_H;*
(3) *A_0 is completely integrally closed in $(A_H)_0$ and each homogeneous element of $Q(A)$ of nonzero degree which is almost integral over A is already in A.*

Observe that, if x is an homogeneous element of $Q(A)$ that is almost integral over A, then there is a nonzero homogeneous element h of A such that $hx^n \in A$ for $n \geq 0$.

Now assume that $\Gamma = \mathbb{N}$, the natural integers. In this case, let y be any nonzero element of $Q(A)$ of degree 1. If $x \in (A_H)_z$, $z \in \mathbb{Z}$, then $x/y^z \in (A_H)_0$. Hence $(A_H)_z = (A_H)_0 y^z$ and $A_H = (A_H)_0[y, y^{-1}]$. Therefore $Q(A) = (A_H)_0(y)$. Moreover y is trascendental over the field $(A_H)_0$ [28, Vol. II, page 157]. In particular, if A is either $R[t]$, $R[t,t^{-1}]$, $\mathcal{R}(I)$ or $\mathcal{R}(I)^e$, the homogeneous quotient field of A is $K[t,t^{-1}]$ and the quotient field of A is $K(t)$, where K is the quotient field of R.

Next results make use of the following observation: A domain R is completely integrally closed if and only if the polynomial ring $R[X_1,\ldots,X_n]$ is completely intergrally closed, see [6, Chap. V, §4, Proposition 14] and [9, Corollary 3].

Proposition 2.3: *Let R be a domain with quotient field K, then the following are equivalent:*

(1) *R is completely integrally closed in K;*
(2) *$R[t]$ is completely integrally closed in $K(t)$;*
(3) *$R[t,t^{-1}]$ is completely integrally closed in $K(t)$.*

Proof: The equivalence of (1) and (2) is proved in [6, Chap. V, §4, Proposition 14].

We now prove that (1) implies (3). By Lemma 2.2, we just need to show that each homogeneous element of $K(t)$ which is almost integral over $R[t,t^{-1}]$ is already in $R[t,t^{-1}]$. Let $(a/b)t^c$ be an element of $K(t)$ almost integral over $R[t,t^{-1}]$. Thus there is a nonzero element $dt^e \in R[t,t^{-1}]$ such that $d(a/b)^n t^{e+nc} \in R[t,t^{-1}]$ for all $n \geq 0$. Therefore $d(a/b)^n \in R$ for all $n \geq 0$. This concludes the argument since now we have that a/b is almost integral over R and therefore it is in R by hypothesis.

To prove that (3) implies (1) just observe that $R = R[t,t^{-1}] \cap K$. $\square$

Proposition 2.4: $\widetilde{\mathcal{R}(I)} \subseteq \widetilde{R}[t]$.

Proof: Let K be the quotient field of R. Since $K[t]$ is completely integrally closed, then $\widetilde{\mathcal{R}(I)} \subseteq K[t]$. Let $(a/b)t^m \in K[t]$ be almost integral over $\mathcal{R}(I)$ and let $dt^k \in \mathcal{R}(I)$, $d \neq 0$, be such that $dt^k((a/b)t^m)^n = d(a/b)^n t^{k+mn} \in \mathcal{R}(I)$ for each $n \geq 0$. Then $d(a/b)^n \in R$, that is $(a/b) \in \widetilde{R}$. Hence $\widetilde{\mathcal{R}(I)} \subseteq \widetilde{R}[t]$. $\square$

Recall that an integral domain R is called a *Krull domain* if there exists a family $\{\nu_\iota\}_{\iota\in I}$ of discrete valuations on the quotient field of R, $Q(R)$, such that the intersection of the rings of the ν_ι is R and, for all nonzero $x \in Q(R)$, the set of indices $\iota \in I$ such that $\nu_\iota(x) \neq 0$ is finite.

It is well known that if F is a field and $\{R_i\}_{1\leq i\leq n}$ is a finite family of subrings of F which are Krull domains, then $\bigcap_{1\leq i\leq n} R_i$ is still a Krull domain; see [6].

Proposition 2.5: *If $\mathcal{R}(I)$ is completely integrally closed (respectively a Krull domain) then R is completely integrally closed (respectively a Krull domain).*

Proof: It is enough to observe that $R = \mathcal{R}(I) \cap K$. $\square$

Proposition 2.6: *The following conditions are equivalent:*

(1) *$\mathcal{R}(I)$ is completely integrally closed in $K(t)$;*
(2) *$\mathcal{R}(I)^e$ is completely integrally closed in $K(t)$;*
(3) *R is completely integrally closed in K and $\mathcal{R}(I)$ is completely integrally closed in $R[t]$.*

Proof: Proposition 2.4 and Proposition 2.5 guarantee that (1) implies (3).

To prove that (3) implies (2) observe that, by Proposition 2.2, we just need to consider the homogeneous elements. Let $(a/b)t^c \in K(t)$ be an element almost integral on $\mathcal{R}(I)^e$. There is $dt^e \in \mathcal{R}(I)^e$, $d \in R - \{0\}$, such that $d(a/b)^n t^{e+cn} \in \mathcal{R}(I)^e$ for $n \geq 0$. In particular a/b is almost integral over R and therefore it belongs to R. Let $a/b = \alpha \in R$. Thus $(a/b)t^c = \alpha t^c \in R[t, t^{-1}]$. If $c \leq 0$, then $\alpha t^c \in \mathcal{R}(I)^e$. If $c > 0$, then $\alpha t^c \in R[t]$ and if we show that αt^c is almost integral over $\mathcal{R}(I)$ we are done by hypothesis. Let n' be the least nonnegative integer for which $f = e + cn' > 0$. Let $\epsilon = idt^f$ with i any nonzero element in $I^{cn'}$. Thus ϵ is an element of $\mathcal{R}(I)$. Since $d\alpha^n \in I^{e+nc}$, we get $id\alpha^n \in I^{cn'+e+cn} = I^{f+cn}$ for all $n \geq 0$. Thus $\epsilon(\alpha t^c)^n = idt^f\alpha^n t^{nc}$ is an element of $\mathcal{R}(I)$ for all n as wanted.

Eventually, (2) implies (1) since $\mathcal{R}(I) = \mathcal{R}(I)^e \cap K[t]$ is the intersection within $K(t)$ of two completely integrally closed domains. $\square$

3. The complete integral closure of the Rees algebra

In this section we give necessary and sufficient conditions for the Rees algebra of an ideal to be completely integrally closed or Krull. Because of Proposition 2.6 we start by studying when the Rees algebra is completely integrally closed in $R[t]$. As in the Noetherian case, the complete integral closure of $\mathcal{R}(I)$ depends on the properties of the associated graded ring $\mathcal{G}(I)$.

The remark below gathers some well known properties of $\mathcal{G}(I)$.

Remark 3.1: For any ideal I the *order* function $\nu_I : R \longrightarrow \mathbb{Z}_{\geq 0} \cup \{\infty\}$ is defined as $\nu_I(a) = \sup\{n \in \mathbb{Z} \,|\, a \in I^n\}$. Here $\nu_I(a) = \infty$ if and only if a belongs to I^n for all $n \geq 0$. It is well known that $\nu_I(a+b) \geq \min\{\nu_I(a), \nu_I(b)\}$, with equality holding if $\nu_I(a) \neq \nu_I(b)$, and $\nu_I(ab) \geq \nu_I(a) + \nu_I(b)$. Furthermore, $\nu_I(ab) = \nu_I(a) + \nu_I(b)$ for all the elements a and b if and only if $\mathcal{G}(I)$ is a domain. In this case, the ideal I and the ideal $P_0 = \bigcap_{n \geq 0} I^n$ are prime ideals. Thus R/P_0 is a domain and the order function ν_I induces a discrete valuation on R/P_0 centered in I/P_0. It follows that, if $\mathcal{G}(I)$ is a domain and $\bigcap_{n \geq 0} I^n = (0)$, then ν_I is a discrete valuation on R centered in I. The condition $\nu_I(a^d) = d\nu_I(a)$ for all $a \in A$, $d \geq 0$ is equivalent to the weaker condition that $\mathcal{G}(I)$ is reduced.

Given $a \in I^n - I^{n+1}$, denote by a^* the image of a in $\mathcal{G}(I)$, in other words, the class of a in I^n/I^{n+1}. This is called the *leading form* of a. If a belongs to all the powers of I, set $a^* = 0$.

Remark 3.2: The complete integral closure of $\mathcal{R}(I)$ in $R[t]$ is linked to the nature of the zero divisors of $\mathcal{G}(I)$. If there are an homogeneous zero divisor $a^* \in \mathcal{G}(I)$ and an homogeneous element $i^* \in (0 :_{\mathcal{G}(I)} a^*)$, such that $(ia^j)^* a^* = 0$ for all $j \geq 1$, then $\mathcal{R}(I)$ is not completely integrally closed in $R[t]$. In fact, assuming $a \in I^m - I^{m+1}$ and $i \in I^l - I^{l+1}$, we have $(it^l)(at^{m+1})^j = ia^j t^{mj+l+j}$. One can proceed by induction on j to prove that ia^j is in I^{mj+l+j} for all $j \geq 1$. Let $j = 1$. By hypothesis $I^* a^* = 0$, thus $ia \in I^{m+l+1}$ as wanted. Let $j > 1$ and assume $ia^j \in I^{mj+l+j}$. This means that $ia^j \in I^r - I^{r+1}$ with $r \geq mj + l + j$. By hypothesis, $(ia^j + I^{r+1})(a + I^{m+1}) = 0$, thus $ia^{j+1} \in I^{r+m+1} \subseteq I^{mj+l+j+m+1} = I^{m(j+1)+l+(j+1)}$ as wanted. In conclusion at^{m+1} is an element of $R[t] - \mathcal{R}(I)$ which is completely integral on $\mathcal{R}(I)$.

Proposition 3.3: *If $\mathcal{R}(I)$ is completely integrally closed in $R[t]$, then one has $\bigcap_{n \geq 0} I^n = 0$.*

Proof: If it were not so, there would be a nonzero element $d \in I^n$ for all $n \geq 0$. This would imply that $dt^i \in I^i t^i \in \mathcal{R}(I)$ for all $i \geq 0$ while $t \notin \mathcal{R}(I)$. Consequently $\mathcal{R}(I)$ would not be completely integrally closed in $R[t]$. $\square$

Remark 3.4: If R is a domain, $I \subseteq R$ an ideal and $\mathfrak{p}$ a prime ideal containing I such that $R_{\mathfrak{p}}$ is a Noetherian ring, then the intersection of all the powers of I is zero. In fact if x belongs to I^n for all $n \geq 0$, the image of x vanishes after localizing at $\mathfrak{p}$. It follows that x itself must be zero. Conditions that force the intersection of all the powers of an ideal to be equal to zero in the general case have been studied by D. D. Anderson, [1].

Proposition 3.5: *If $\mathcal{G}(I)$ is a domain and I is an ideal of R with $\bigcap_{n\geq 0} I^n = 0$, then $\mathcal{R}(I)$ is completely integrally closed in $R[t]$.*

Proof: If it were not so, there would be a nonzero element $at^m \in R[t] - \mathcal{R}(I)$ and an element $it^l \neq 0$ in $\mathcal{R}(I)$ such that

$$(it^l)(at^m)^j \in \mathcal{R}(I)$$

for all $j \geq 0$. Being a different from zero, there would be an integer n such that $a \in I^n - I^{n+1}$ with $n < m$ since $at^m \notin \mathcal{R}(I)$. Being $it^l \in \mathcal{R}(I)$ different from zero too, $i \in I^{l+h} - I^{l+h+1}$ for some $h \geq 0$. Moreover $\nu_I(a^{h+1}) = (h+1)\nu_I(a) = (h+1)n$ since $\mathcal{G}(I)$ is a domain. One thus would have $ia^{h+1} \in I^{l+m(h+1)}$ with $m \geq n+1$. Having $I^{l+m(h+1)} \subseteq I^{l+(n+1)(h+1)} = I^{l+n(h+1)+h+1}$, the product

$$(i + I^{l+h+1})(a^{h+1} + I^{(h+1)n+1}) = ia^{h+1} + I^{l+h+n(h+1)+1}$$

would be equal to zero in $\mathcal{G}(I)$ contradicting the hypothesis that $\mathcal{G}(I)$ is a domain. $\square$

Theorem 3.6: *If $\mathcal{G}(I)$ is a domain, then $\mathcal{R}(I)$ is completely integrally closed if and only if R is completely integrally closed and $\bigcap_{n\geq 0} I^n = 0$.*

Proof: Use Proposition 2.6 and Proposition 3.3. $\square$

Remark 3.7: A *fractional divisorial ideal* of a domain R is an ideal of type $(R{:}_K J)$ for some fractional ideal J of R. Hence a fractional ideal I of R is divisorial if and only if $I = (R{:}_K (R{:}_K I))$. If R is completely integrally closed and $\mathfrak{p}$ is a divisorial prime, then $R_{\mathfrak{p}}$ is a DVR [8, Theorem 2.1], thus $\bigcap_{n\geq 0} \mathfrak{p}^n = 0$. It follows that, if $\mathfrak{p}$ is a divisorial prime of a completely integrally closed domain R and $\mathcal{G}(\mathfrak{p})$

is a domain, then $\mathcal{R}(\mathfrak{p})$ is completely integrally closed. Observe that $R_\mathfrak{p}$ a DVR forces $\mathcal{R}(\mathfrak{p}R_\mathfrak{p})$ to be completely integrally closed as well.

The condition that $\mathcal{G}(I)$ be a domain does not force the intersection of all the powers of I to be equal to zero, as it is shown in the following example.

Example 3.8: Let R be a valutation domain of dimension one that is not a DVR and let $\mathfrak{m}$ be its maximal ideal. R is a completely integrally closed domain with $\mathfrak{m} = \mathfrak{m}^2$. Thus $\bigcap_{n\geq 0} \mathfrak{m}^n = \mathfrak{m} \neq 0$. Since $\mathcal{R}(\mathfrak{m}) = R + t\mathfrak{m}[t] = R + \mathfrak{m}[t]$ and $\mathfrak{m}\mathcal{R}(\mathfrak{m}) = \mathfrak{m}[t]$, the associated graded ring of $\mathfrak{m}$ is

$$\mathcal{G}(\mathfrak{m}) \cong \mathcal{R}(\mathfrak{m})/\mathfrak{m}\mathcal{R}(\mathfrak{m}) = \mathcal{R}(\mathfrak{m})/\mathfrak{m}[t] = R/\mathfrak{m}.$$

Note that $\mathcal{R}(\mathfrak{m})$ is integrally closed but not completely integrally closed. In fact its complete integral closure is $R[t]$.

It is simple to provide an example of an ideal I in a completely integrally closed domain R for which $\mathcal{R}(I)$ is completely integrally closed and $\mathcal{G}(I)$ is not even reduced. Consider, for example, a principal ideal $I = aR$ in a completely integrally closed domain R. Since $\mathcal{R}(I) = R[at]$ is a polynomial ring over R, it is completely integrally closed, however $\mathcal{G}(I)$ is not a domain if I is not prime and it is not reduced if $a = b^2$.

In what follows we study the case when the associated graded ring is not a domain. To do so we are led to impose conditions on the structure of the extended ideal $I\mathcal{R}(I)$. In particular we study the case in which $I\mathcal{R}(I)$ has a primary decomposition.

Definition 3.9: The prime ideals of a domain R maximal among the ideals of zero-divisors modulo an ideal I are called the *maximal primes* of I. $\mathrm{Max}(I)$ denotes the set of maximal primes of I.

If $a \in R - \{0\}$, as in the proof of [17, Theorem 53], one has

$$R = R_a \bigcap_{P\in \mathrm{Max}\,(aR)} R_P.$$

Proposition 3.10: *If R is completely integrally closed (respectively a Krull domain) and $\mathcal{R}(I)_P$ is completely integrally closed (respectively a Krull domain) for each maximal prime P of $I\mathcal{R}(I)$, then $\mathcal{R}(I)$ is completely integrally closed (respectively a Krull domain).*

Proof: By Proposition 2.6 it is enough to show that $\mathcal{R}(I)^e$ is completely integrally closed. Since $\mathcal{R}(I)/I\mathcal{R}(I) \cong \mathcal{R}(I)^e/t^{-1}\mathcal{R}(I)^e$, P is a maximal prime of $I\mathcal{R}(I)$ if and only if $Q = (P, t^{-1})$ is a maximal prime of $t^{-1}\mathcal{R}(I)^e$. Observe also that $\mathcal{R}(I)_P \cong \mathcal{R}(I)^e_Q$. Since

$$\mathcal{R}(I)^e = \mathcal{R}(I)^e_{t^{-1}} \bigcap_{Q \in \mathrm{Max}\ (t^{-1}\mathcal{R}(I)^e)} \{\mathcal{R}(I)^e_Q\},$$

by Proposition 2.3, if R is completely integrally closed (respectively Krull), then $\mathcal{R}(I)^e_{t^{-1}} = R[t, t^{-1}]$ is completely integrally closed. Thus, if $\mathcal{R}(I)_P$ is completely integrally closed for all P, then $\mathcal{R}(I)^e$ is completely integrally closed being the intersection of completely integrally closed domains with the same quotient field. The same proof shows that if R is a Krull domain and $\mathcal{R}(I)_P$ is Krull for each maximal prime P of $I\mathcal{R}(I)$, then $\mathcal{R}(I)^e$ is Krull and consequently $\mathcal{R}(I) = \mathcal{R}(I)^e \cap R[t]$ is Krull as well. $\square$

Definition 3.11: An ideal of a ring R is said to be *strongly primary* if it is primary and contains a power of its radical.

Next lemma is a direct consequence of well known results. We give the proof for completeness.

Lemma 3.12: *Let R be a domain and I an ideal such that $I = L_1 \cap \cdots \cap L_n$ is an irredundant finite intersection of strongly primary ideals. Then $\sqrt{L_i} = P_i = (I :_R x_i)$ for some $x_i \in R - P_i$ and the the maximal primes of I are the ideals which are maximal among the P_i.*

Proof: By [4, Theorem 4.5], each P_i is the radical of $(I :_R x)$ for any x not in L_i. Going modulo I one may assume $I = 0$. Thus Ann (x) is contained in P_i for any x not in L_i. Since L_i is strongly primary it contains a power of its radical, thus there is an integer m for which P_i^m is contained in L_i. Let $I_i = \bigcap_{j \neq i} L_j$. Then we have

$I_iP_i^m = 0$. Let n be the least non-negative integer such that $I_iP_i^n = 0$ and let x be a nonzero element of $I_iP_i^{n-1}$. Since $P_ix = 0$, for such an x we have $P_i \subseteq \mathrm{Ann}\,(x)$, and hence $P_i = \mathrm{Ann}\,(x)$. Finally, since the set of zero-divisors modulo I is the union of $P_1, \cdots, P_n$ [4, Proposition 4.7], each ideal of zero-divisors is contained in some P_i. $\square$

Observe that, with the notation of the previous Lemma, if $I = aR$ is principal, then the ideals P_i are divisorial. In fact $P_i = (aR{:}_R\, x_i) = (R{:}_K\,(1, x_i a^{-1}))$.

Theorem 3.13: *Assume that $I\mathcal{R}(I)$ can be written as an irredundant finite intersection of strongly primary ideals (e.g. $\mathcal{G}(I)$ reduced with finitely many minimal primes) and let $P_1, \ldots, P_n$ be the associated primes of $I\mathcal{R}(I)$. Then the following are equivalent:*

(1) *$\mathcal{R}(I)$ is completely integrally closed (respectively Krull);*
(2) *R is completely integrally closed (respectively Krull) and $\mathcal{R}(I)_{P_i}$ is completely integrally closed (respectively Krull) for each $i = 1, \ldots, n$;*
(3) *R is completely integrally closed (respectively Krull) and $\mathcal{R}(I)_{P_i}$ is a DVR for each $i = 1, \ldots, n$.*

Proof: Assume that $\mathcal{R}(I)$ is completely integrally closed (respectively Krull). Then R is completely integrally closed (respectively Krull) by Proposition 2.5. Since $\mathcal{R}(I)/I\mathcal{R}(I) \cong \mathcal{R}(I)^e/t^{-1}\mathcal{R}(I)^e$, the ideals $Q_i = (P_i, t^{-1})$ are the primes associated with the principal ideal $t^{-1}\mathcal{R}(I)^e$. Hence they are divisorial by Lemma 3.12. Since $\mathcal{R}(I)^e$ is completely integrally closed by Proposition 2.6, then $\mathcal{R}(I)^e_{Q_i} \cong \mathcal{R}(I)_{P_i}$ is a DVR for each $i = 1, \ldots, n$ by [8, Theorem 2.1]. This proves that (1) implies (3).

That (3) implies (2) is clear.

Eventually, that (2) implies (1) follows from Proposition 3.10. $\square$

A simple consequence of the previous result and Theorem 3.6 is that, if $\mathcal{G}(I)$ is a domain and R is completely integrally closed, then $\bigcap_{n\geq 0} I^n = 0$ if and only if $\mathcal{R}(I)_P$ is a DVR where P is the prime ideal $I\mathcal{R}(I)$.

Corollary 3.14: *Let R be a completely integrally closed domain (respectively Krull), I an ideal of R such that $\mathcal{G}(I)$ is reduced with only finitely many minimal* *primes, $P_1, \dots, P_n$. Let $\mathfrak{p}_i$ be the lift to R of the contraction of P_i to R/I. If $\mathcal{G}(I_{\mathfrak{p}_i})$ is reduced and $R_{\mathfrak{p}_i}$ is Noetherian for each $i = 1, \dots, n$, then $\mathcal{R}(I)$ is completely integrally closed (respectively Krull).*

Proof: By [5], the hypothesis $R_{\mathfrak{p}_i}$ integrally closed and $\mathcal{G}(I_{\mathfrak{p}_i})$ reduced for each $i = 1, \dots, n$, implies that $\mathcal{R}(I_{\mathfrak{p}_i})$ is integrally closed for each $i = 1, \dots, n$. Since $\mathcal{R}(I)_{P_i}$ is isomorphic to a localization of $\mathcal{R}(I_{\mathfrak{p}_i})$, which is Noetherian, then $\mathcal{R}(I)_{P_i}$ is normal, hence completely integrally closed. The proof now follows from Theorem 3.13. □

Note that, if $\mathcal{R}(I)$ is Krull, the contractions to R of the associated primes of $I\mathcal{R}(I)$ are exactly the associated primes of I^n for all $n >> 0$, see [24, Lemma 2.3].

Corollary 3.15: *Let R be a Krull domain, then the following are equivalent:*

(1) *$\mathcal{R}(I)^e$ is Krull;*
(2) *$\mathcal{R}(I)$ is Krull;*
(3) *$\mathcal{R}(I)$ is completely integrally closed and $I\mathcal{R}(I)$ is an irredundant finite intersection of strongly primary ideals;*
(4) *$I\mathcal{R}(I) = P_1^{(n_1)} \cap \cdots \cap P_s^{(n_s)}$ and $\mathcal{R}(I)_{P_i}$ is a DVR for each i.*

Proof: The fact that (4) implies (1) and (1) implies (2) follows by the proof of Proposition 3.10.

Since the ideal $\mathcal{R}(I)_+ = \oplus_{n\geq 1} I^n t^n$ is a height-one prime ideal of $\mathcal{R}(I)$, then, if $\mathcal{R}(I)$ is Krull, $\mathcal{R}(I)_+$ is divisorial. It follows that also the ideal $I\mathcal{R}(I) = \oplus_{n\geq 1} I^{n+1} t^n = t^{-1}\mathcal{R}(I)_+$ is divisorial. Hence $I\mathcal{R}(I)$ is of type $P_1^{(n_1)} \cap \cdots \cap P_s^{(n_s)}$ for some height-one prime ideals $P_1, \dots, P_s$ of $\mathcal{R}(I)$ and $\mathcal{R}(I)_{P_i}$ is a DVR for each $i = 1, \dots, s$. This proves that (2) implies (4).

Theorem 3.13 proves that (4) implies (3).

At last (3) implies (2) since by Theorem 3.13, if $\mathcal{R}(I)$ is completely integrally closed and $I\mathcal{R}(I)$ is an irredundant finite intersection of strongly primary ideals, then $\mathcal{R}(I)_{P_i}$ is a DVR for each prime P_i associated to I. Again by Theorem 3.13, if R is Krull, then $\mathcal{R}(I)$ is Krull. □

In the Noetherian case the equivalence between (2) and (4) has been proved in [13, Proposition 2.13].

Corollary 3.16: *If $\mathcal{G}(I)$ is a domain, then $\mathcal{R}(I)$ is Krull if and only if R is Krull and $\bigcap_{n\geq 0} I^n = 0$.*

Proof: It follows by Proposition 2.5, Theorem 3.6 and Corollary 3.15. $\square$

Therefore, if R is the integral closure of a Noetherian domain and $\mathcal{G}(I)$ is a domain, then $\mathcal{R}(I)$ is a Krull domain. In fact, in this case, for every ideal I, the intersection of all the powers of I is zero, see [1, Theorem 5].

Remark 3.17: A *Mori* domain is a domain with the ascending chain condition on divisorial ideals. It is well known that a Krull domain is a completely integrally closed Mori domain. Thus we are led to ask if an analogue of Corollary 3.15 holds for Mori domains. The difficulty lies in the fact that if R is Mori, the polynomial ring $R[t]$ need not be Mori [22]. However, assuming that $R[t]$ is a Mori domain, by using the same argument given in the Krull case, we can conclude that $\mathcal{R}(I)$ is Mori if and only if $I\mathcal{R}(I)$ has finitely many maximal primes P_i and $\mathcal{R}(I)_{P_i}$ is Mori for each i. In fact, like in the Krull case, a height-one prime ideal of a Mori domain is divisorial. Hence, if $\mathcal{R}(I)$ is Mori, $I\mathcal{R}(I)$ is divisorial and it has finitely many maximal primes [15, Corollary 2.2]. Further, localizations of Mori domains are Mori and finite intersections of Mori domains are Mori, see [20].

We do not know whether the fact that $\mathcal{R}(I)$ is Mori forces $R[t]$ to be Mori.

4. The Rees algebra of a prime ideal

In this section we analyze the complete integral closure of the Rees algebra of a prime ideal $\mathfrak{p}$ of R in relation with the structure of the associated graded ring and the symbolic algebra of $\mathfrak{p}$. The *symbolic algebra of* $\mathfrak{p}$ is the Rees algebra associated to the filtration given by the symbolic powers of $\mathfrak{p}$, namely

$$\mathcal{S}(\mathfrak{p}) = \oplus_{n\geq 0}\mathfrak{p}^{(n)}.$$

Recall that the *n-th symbolic power of a prime ideal* $\mathfrak{p}$ in a ring R is defined to be the ideal $\mathfrak{p}^n R_{\mathfrak{p}} \cap R$. If $\mathfrak{p}^n$ has a primary decomposition, $\mathfrak{p}^{(n)}$ is its $\mathfrak{p}$-primary component.

Proposition 4.1: *Let R be a completely integrally closed domain, $\mathfrak{p}$ a prime ideal of R such that $\bigcap_{n\geq 0}\mathfrak{p}^{(n)} = (0)$ (e.g. $R_{\mathfrak{p}}$ Noetherian) and $\mathcal{G}(\mathfrak{p}R_{\mathfrak{p}})$ is a domain. Then the complete integral closure of $\mathcal{R}(\mathfrak{p})$ is contained in $\mathcal{S}(\mathfrak{p})$.*

Proof: By Proposition 2.4, $\widetilde{\mathcal{R}(\mathfrak{p})}$ is contained in $R[t]$. Let at^n be a nonzero homogeneous element of $R[t]$ almost integral over $\mathcal{R}(\mathfrak{p})$. We show that $a \in \mathfrak{p}^{(n)}$, thus showing that at^n belongs to $\mathcal{S}(\mathfrak{p})$.

Since at^n is almost integral over $\mathcal{R}(\mathfrak{p})$, there is a non zero element $bt^m \in \mathcal{R}(\mathfrak{p})$ such that $ba^i \in \mathfrak{p}^{m+ni}$ for all $i \geq 0$. If $n = 0$, then $ba^i \in \mathfrak{p}^m \subseteq R$ for all $i \geq 0$. Since R is completely integrally closed it follows that a belongs to R. Assume now that $n \geq 1$. Let ν be the order function of $\mathfrak{p}R_{\mathfrak{p}}$. Since $a \neq 0$ and $\bigcap_{n\geq 0}\mathfrak{p}^n R_{\mathfrak{p}} = 0$, we may assume that $\nu(a/1) = l$ is finite. Since $\mathcal{G}(\mathfrak{p}R_{\mathfrak{p}})$ is a domain, we have that $\nu((b/1)(a^i/1)) = \nu(b/1) + i\nu(a/1) = \nu(b/1) + il$. On the other hand, $\nu((b/1)(a^i/1)) \geq m+in$, because $ba^i \in \mathfrak{p}^{m+ni}$. Thus $\nu(b/1) \geq m+i(n-l)$. It follows that $b/1 \in \mathfrak{p}^{m+(n-l)i}R_{\mathfrak{p}}$ for all i. If $(n-l) > 0$, then $b/1 \in \bigcap_{i\geq 0}\mathfrak{p}^{n+(n-l)i}R_{\mathfrak{p}} = 0$, contradicting the fact that $b \neq 0$. It follows that $l \geq n$ and then $a \in \mathfrak{p}^{(l)} \subseteq \mathfrak{p}^{(n)}$. $\square$

It is well known that, under the same hypotheses, the integral closure of the Rees algebra of $\mathfrak{p}$, $\mathcal{R}_a(\mathfrak{p})$, is contained in the symbolic algebra. In fact, if $\mathcal{G}(\mathfrak{p}R_{\mathfrak{p}})$ is a domain, the powers $\mathfrak{p}^n R_{\mathfrak{p}}$ are integrally closed since they are valuation ideals of the $\mathfrak{p}$-adic valuation of $R_{\mathfrak{p}}$.

Remark 4.2: Observe that if R and $\mathcal{R}(\mathfrak{p}R_{\mathfrak{p}})$ are both completely integrally closed, the symbolic algebra of $\mathfrak{p}$ over R is completely integrally closed since $\mathcal{S}(\mathfrak{p}) = \mathcal{R}(\mathfrak{p}R_{\mathfrak{p}}) \cap R[t]$, and $R[t]$ is completely integrally closed. (Recall that when $R_{\mathfrak{p}}$ is a DVR then $\mathcal{R}(\mathfrak{p}R_{\mathfrak{p}})$ is completely integrally closed). Similarly, if R is Krull, then $\mathcal{S}(\mathfrak{p})$ is Krull if and only if $\mathcal{R}(\mathfrak{p}R_{\mathfrak{p}})$ is completely integrally closed; see [18] or [24, Lemma 2.5] in the Noetherian case.

In conclusion, if R is completely integrally closed and $R_{\mathfrak{p}}$ is a DVR, we have

$$\mathcal{R}(\mathfrak{p}) \subseteq \mathcal{R}_a(\mathfrak{p}) \subseteq \widetilde{\mathcal{R}(\mathfrak{p})} \subseteq \mathcal{S}(\mathfrak{p}) = \widetilde{\mathcal{S}(\mathfrak{p})}.$$

It is therefore natural to ask:

Question 4.3: *Let R be completely integrally closed and $R_{\mathfrak{p}}$ a DVR. Is $\widetilde{\mathcal{R}(\mathfrak{p})}$ completely integrally closed? Under which conditions is $\widetilde{\mathcal{R}(\mathfrak{p})}$ equal to $\mathcal{S}(\mathfrak{p})$?*

Recalling the results in [9], one may observe that $\widetilde{\mathcal{R}(\mathfrak{p})} = \mathcal{S}(\mathfrak{p})$ if the conductor of $\mathcal{S}(\mathfrak{p})$ in $\mathcal{R}(\mathfrak{p})$ is nonzero, i.e. if $\mathcal{S}(\mathfrak{p})$ is contained in a finite $\mathcal{R}(\mathfrak{p})$-submodule of the quotient field, see [9, Lemma 5]. Again by [9, Corollary 6], $\widetilde{\mathcal{R}(\mathfrak{p})}$ is completely integrally closed if the conductor of $\widetilde{\mathcal{R}(\mathfrak{p})}$ in $\mathcal{R}(\mathfrak{p})$ is nonzero.

Obvious, but restrictive, cases in which $\widetilde{\mathcal{R}(\mathfrak{p})} = \mathcal{S}(\mathfrak{p})$ are those when either $\mathcal{R}(\mathfrak{p})$ or $\mathcal{R}_a(\mathfrak{p})$ coincide with $\mathcal{S}(\mathfrak{p})$.

Remark 4.4: If $S = R - \mathfrak{p}$, let S^* denote the multiplicatively closed set in $\mathcal{G}(\mathfrak{p})$ whose elements are the leading forms of elements in S. It is clear that $\mathcal{R}(\mathfrak{p}R_{\mathfrak{p}}) \cong \mathcal{R}(\mathfrak{p}) \otimes_R R_{\mathfrak{p}} \cong \mathcal{R}(\mathfrak{p})_S$ and that $\mathcal{G}(\mathfrak{p}R_{\mathfrak{p}}) \cong \mathcal{G}(\mathfrak{p}) \otimes_R R_{\mathfrak{p}} \cong \mathcal{G}(\mathfrak{p})_{S^*}$. The kernel of the natural map from $\mathcal{G}(\mathfrak{p})$ to $\mathcal{G}(\mathfrak{p}R_{\mathfrak{p}})$ is $\oplus_{n\geq 0}((\mathfrak{p}^n \cap \mathfrak{p}^{(n+1)})/\mathfrak{p}^{n+1})$. Notice that the elements of this kernel are zero-divisors in $\mathcal{G}(\mathfrak{p})$. Let

$$P = \mathfrak{p}\mathcal{R}(\mathfrak{p}R_{\mathfrak{p}}) \cap \mathcal{R}(\mathfrak{p}) = \oplus_{n\geq 0}(\mathfrak{p}^n \cap \mathfrak{p}^{(n+1)})t^n.$$

Each prime ideal Q of $\mathcal{R}(\mathfrak{p})$ containing $\mathfrak{p}\mathcal{R}(\mathfrak{p})$ such that $Q \cap R = \mathfrak{p}$, contains P. In fact $Q \cap S = \emptyset$ and going modulo $\mathfrak{p}\mathcal{R}(\mathfrak{p})$ one obtains a prime Q^* of $\mathcal{G}(\mathfrak{p})$ that inverting S^* remains prime. Since Q^* is equal to the contraction of its extension via S^*, it is obvious that the kernel of the canonical homomorphism in contained in Q^*. Therefore $P \subseteq Q$.

Furthermore, if $\mathcal{G}(\mathfrak{p}R_{\mathfrak{p}})$ is a domain, then P is the only minimal prime ideal containing $\mathfrak{p}\mathcal{R}(\mathfrak{p})$ such that $P \cap R = \mathfrak{p}$. In fact, being the contraction of a prime, it is prime and any other prime of that type must contain it.

Consequently, if $\mathcal{G}(\mathfrak{p}R_{\mathfrak{p}})$ is a domain, $\mathcal{R}(\mathfrak{p}) = \mathcal{S}(\mathfrak{p})$ if and only if $P = \mathfrak{p}\mathcal{R}(\mathfrak{p})$ if and only if $\mathfrak{p}\mathcal{R}(\mathfrak{p})$ is a prime ideal of $\mathcal{R}(\mathfrak{p})$ if and only if $\mathcal{G}(\mathfrak{p})$ is a domain; see [21, Corollary 1.2].

In general, even assuming $R_{\mathfrak{p}}$ a DVR, $\widetilde{\mathcal{R}(\mathfrak{p})}$ can be different from the symbolic algebra as shown in the following example.

Example 4.5: Let K be a field and R the 2-dimensional normal domain $R = K[|x,y,z|]/(x^2 - yz)$. Consider the height-one prime ideal $\mathfrak{p} = (x,y)R$. Clearly

$R_{\mathfrak{p}}$ is a DVR. Actually, $\mathfrak{p}$ is an almost complete intersection and it is generated by a d-sequence, see [16]. The symmetric and the Rees algebra of $\mathfrak{p}$ are the same and $\mathcal{R}(\mathfrak{p})$ is isomorphic to $R[U,V]/(yU-xV, xU-zV)$ while $\mathcal{G}(\mathfrak{p})$ is isomorphic to $K[z,U,V]/(zV)$ which is not a domain. Therefore $\mathcal{R}(\mathfrak{p})$ does not coincide with $\mathcal{S}(\mathfrak{p})$ since this happens if and only if $\mathcal{G}(\mathfrak{p})$ is a domain; see Remark 4.4. However $\mathcal{R}(\mathfrak{p})$ is integrally closed (hence completely integrally closed since R is Noetherian) because $\mathcal{G}(\mathfrak{p})$ is reduced, see [5].

The last observation of Remark 4.4 can be improved when $R_{\mathfrak{p}}$ is a DVR.

Theorem 4.6: *Let R be completely integrally closed and $\mathfrak{p}$ a prime ideal of R such that $R_{\mathfrak{p}}$ is a DVR and $\mathcal{G}(\mathfrak{p})$ is reduced with finitely many minimal primes. Then the following are equivalent:*

(1) *$\mathcal{G}(\mathfrak{p})$ is a domain;*

(2) *$\mathcal{R}(\mathfrak{p})$ is completely integrally closed and, for any height-one prime ideal Q of $\mathcal{R}(\mathfrak{p})$, the prime ideal $\mathfrak{q} = Q \cap R$ has height at most one;*

(3) *$\mathcal{R}(\mathfrak{p})$ is completely integrally closed and, for any associated prime of $\mathfrak{p}\mathcal{R}(\mathfrak{p})$, the prime ideal $\mathfrak{q} = Q \cap R$ has height at most one.*

Proof: Since $R_{\mathfrak{p}}$ is a DVR, $\mathcal{G}(\mathfrak{p}R_{\mathfrak{p}})$ is a domain. Thus, by Remark 4.4, $P = \mathfrak{p}\mathcal{R}(\mathfrak{p}R_{\mathfrak{p}}) \cap \mathcal{R}(\mathfrak{p})$ is the only minimal prime ideal containing $\mathfrak{p}\mathcal{R}(\mathfrak{p})$ such that $P \cap R = \mathfrak{p}$. Since, by hypothesis, $\mathfrak{p}\mathcal{R}(\mathfrak{p})$ has a primary decomposition given by finitely many primes which are minimal over it and P is formed by zero-divisors, we deduce by Lemma 3.12 that P appears in this decomposition. Therefore we may write $\mathfrak{p}\mathcal{R}(\mathfrak{p}) = P \cap Q_1 \cap \ldots \cap Q_s$ where P is the only one with $P \cap R = \mathfrak{p}$. Hence, $\mathfrak{q}_i = Q_i \cap R$ is prime in R of height at least 2, for each $i = 1, \ldots, s$. This shows that (1) and (3) are equivalent. In fact, if $\mathcal{G}(\mathfrak{p})$ is a domain we have $\mathfrak{p}\mathcal{R}(\mathfrak{p}) = P$ and $\mathcal{R}(\mathfrak{p})$ is completely integrally closed by Theorem 3.6. Viceversa, if (3) holds, clearly the Q_i's do not appear in the primary decomposition of $\mathfrak{p}\mathcal{R}(\mathfrak{p})$. Therefore $\mathfrak{p}\mathcal{R}(\mathfrak{p}) = P$ and $\mathcal{G}(\mathfrak{p})$ is a domain.

It is clear that (2) implies (3). To see that (3) implies (2), recall that, by Theorem 3.13, if $\mathcal{R}(\mathfrak{p})$ is completely integrally closed, then all the primes ideals associated to $\mathfrak{p}\mathcal{R}(\mathfrak{p})$ have height one. Besides, if Q is a prime ideal of $\mathcal{R}(\mathfrak{p})$ not containing $\mathfrak{p}\mathcal{R}(\mathfrak{p})$ and $\mathfrak{q} = Q \cap R \neq 0$, then $\mathfrak{q}$ is a prime ideal of R not containing

$\mathfrak{p}$ and $\mathcal{R}(\mathfrak{p})_{R-\mathfrak{q}} \cong \mathcal{R}(\mathfrak{p}R_{\mathfrak{q}}) \cong R_{\mathfrak{q}}[x]$. Thus Q is the only prime contracting to $\mathfrak{q}$ and the ideals Q and $\mathfrak{q}$ have the same height. $\square$

Definition 4.7: One says that an extension $A \subseteq B$ of Krull rings with the property that $P \cap A$ has height at most one, for any height-one prime ideal Q of B, satisfies condition PDE (*Pas d'éclatement*) or NBU (*No blowing up*), see [7].

Remark 4.8: Recall that the divisorial prime ideals of a Krull domain are exactly the height-one primes (see Remark 3.7 for the definition of divisorial ideal). Denote by $\mathcal{C}(A)$ the *divisor class group* of the Krull domain A, that is the quotient group $\mathcal{D}(A)/\mathcal{P}(A)$, where $\mathcal{D}(A)$ is the free abelian group generated by the height-one primes of A and $\mathcal{P}(A)$ is the group of principal ideals. Condition PDE implies that the map $\psi : \mathcal{D}(A) \longrightarrow \mathcal{D}(B)$ defined by $I \mapsto (B\colon(B\colon IB))$ is a homomorphism; see [2, Proposition 5.1].

If $\mathcal{R}(I)$ is Krull, $\mathcal{C}(\mathcal{R}(I))$ is generated by the classes of the height-one primes containing some nonzero constant and the map $\bar{\psi} : \mathcal{C}(R) \longrightarrow \mathcal{C}(\mathcal{R}(I))$ induced by ψ is always injective with a left inverse induced by the contraction, because $\psi(I) \cap R = (\mathcal{R}(I)\colon(\mathcal{R}(I)\colon I\mathcal{R}(I))) \cap R = I$, for any divisorial ideal I of R; see [2, Theorem 6.4].

Theorem 4.9: *Let R be a Krull domain and $\mathfrak{p}$ a divisorial prime of R such that $\mathcal{G}(\mathfrak{p})$ is reduced with only finitely many minimal primes. Then the following are equivalent:*

(1) *$\mathcal{G}(\mathfrak{p})$ is a domain;*

(2) *$\mathcal{R}(\mathfrak{p})$ is Krull and the extension $R \subseteq \mathcal{R}(\mathfrak{p})$ satisfies condition PDE;*

(3) *$\mathcal{R}(\mathfrak{p})$ is Krull and the map $\bar{\psi} : \mathcal{C}(R) \longrightarrow \mathcal{C}(\mathcal{R}(\mathfrak{p}))$ is an isomorphism.*

Proof: (2) implies (1) by Theorem 4.6. Besides, if (1) holds, then $\mathcal{R}(\mathfrak{p})$ is Krull by Corollary 3.16 and the inclusion $R \subseteq \mathcal{R}(\mathfrak{p})$ satisfies condition PDE by Theorem 4.6 once again.

To prove that (2) implies (3), it is enough to show that $\bar{\psi}$ is surjective. Notice that, via (1), $\mathcal{G}(\mathfrak{p})$ is a domain and then $\mathfrak{p}\mathcal{R}(\mathfrak{p}) = P$ is prime. Since $\mathfrak{p}\mathcal{R}(\mathfrak{p})$ is divisorial, $\psi(\mathfrak{p}) = (\mathcal{R}(\mathfrak{p})\colon(\mathcal{R}(\mathfrak{p})\colon\mathfrak{p}\mathcal{R}(\mathfrak{p}))) = \mathfrak{p}\mathcal{R}(\mathfrak{p})$. On the other hand, any height-one prime Q of $\mathcal{R}(\mathfrak{p})$ different from P and containing some nonzero constant contracts to a height-one prime $\mathfrak{q}$ of R and is the only prime of $\mathcal{R}(\mathfrak{p})$ contracting to $\mathfrak{q}$. Since $\mathcal{D}(\mathcal{R}(\mathfrak{p}))$ is generated by the height-one primes, $\psi(\mathfrak{q}) = (\mathcal{R}(\mathfrak{p})\colon (\mathcal{R}(\mathfrak{p})\colon \mathfrak{q}\mathcal{R}(\mathfrak{p}))) = Q$. It follows that ψ is surjective.

Finally, (3) implies (1). In fact, if $\bar{\psi}$ is bijective, then ψ is surjective and there must be a divisorial ideal J of R such that $\psi(J) = P = \mathfrak{p}\mathcal{R}(\mathfrak{p}R_{\mathfrak{p}}) \cap \mathcal{R}(\mathfrak{p})$. Since $J = \psi(J) \cap R = P \cap R = \mathfrak{p}$, then $P = \psi(\mathfrak{p}) = \mathfrak{p}\mathcal{R}(\mathfrak{p})$ and $\mathfrak{p}\mathcal{R}(\mathfrak{p})$ is prime. □

Among the works that, in the Noetherian case, have investigated the relationship among the class groups of R, $\mathcal{R}(I)$ and $\mathcal{S}(I)$, we point out [12], [13] and [24].

The previous results motivates the following question.

Question 4.10: *Let R be a completely integrally closed domain and let $\mathfrak{p}$ be a prime ideal of R such that $R_{\mathfrak{p}}$ is a DVR and $\mathcal{G}(\mathfrak{p})$ is reduced with only finitely many minimal primes. Is $\mathcal{R}(\mathfrak{p})$ completely integrally closed?*

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