MULTILINEAR FORMS, JACOBIAN IDEALS AND PRIMARY DECOMPOSITIONS

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The results here presented are fruit of a collaboration with Irena Swanson - New Mexico State University - Las Cruces. Some of them can be found in the joint work *Primary decompositions of Jacobian ideals: an application of* 1*-genericity*, others are new.

The goal is that of finding the primary decomposition of a wide class of ideals obtained by taking the first partial derivatives of a trilinear forms whose coefficients satisfy a weak type of genericity, *trilinear froms in general position.* More generally we are interested in understanding the structure of the ideals of minors of 1-generic matrices of linear forms.

The roots of this type of problem lie in the theory of hyperdeterminants. For example, we prove that the class of trilinear forms we analyze identifies that of the three-dimensional arrays of dimension $n \times m \times p$ with $n = m + p - 1$ (*boundary format*) for which the hyperdeterminant is non-zero.

Therefore this survey is divided in three parts: Primary decomposition, Hyperdeterminants, Trilinear forms in general position and 1-genericity.

The first two sections are of a descriptive nature, while the third one contains the specific results.

1. Primary decomposition

"... the earliest impulse toward the development of what is now commutative algebra, came from the desire of number theorists to make use of unique factorization in rings of integers in number fields other than $\mathbb{Q}$. When it became clear that unique factorization

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did not always hold, the search for the strongest available alternative began. The theory of primary decomposition is the direct result of that search."
D. Eisenbud - Commutative Algebra with a view toward Algebraic Geometry.

Among the names that built the theory of primary decomposition, one recalls: Dedekind, Lasker and E. Noether. Hers is the highly nonconstructive intuition that the existence of primary decomposition depends only on the ascending chain condition for ideals. This approach simplified and clarified all the subject opening the way to the modern comprehension of the wide class of rings that now we call Noetherian in her honor. It pushed, however, a little aside the taste (traceable in the work of Lasker) for effective, explicit results on primary decomposition. Nowadays, it is instead this 'computational flavor' that it is at the center of most of the current investigations.

Let R be a Noetherian ring (e.g. a polynomial ring over a field) and let J be an ideal in R. A *primary decomposition* of J is:

$$J = Q_1 \cap \cdots \cap Q_s,$$

where Q_i is a *primary* ideal (i.e. Q_i is a proper ideal and every zero-divisor in R/Q_i is nilpotent). For each i there is a unique smallest prime ideal P_i containing Q_i. These primes are said to be the *associated primes* of J. Eliminating the superfluous terms from the intersection and grouping together the Q_k's corresponding to the same prime P_i, one makes the primary decomposition irredundant.

In conclusion, attached to an ideal J in a Noetherian ring we have two lists: $\{P_1, \ldots, P_s\}$, the associated primes of J and $\{Q_1, \ldots, Q_s\}$, the primary components of J. Knowing the primary components one knows the associated primes too. It is possible, however, to know the associated primes without fully knowing the primary components.

These lists (perhaps the first is the most important) encrypt a great wealth of information about J. Consider, for example, the following well known three facts.

Let $I \subseteq J$ be ideals. Then $I = J$ if and only if for every P associated prime of I, $I_P = J_P$.

The zero-divisors of R/J are given by the union of all the associated primes of J. Thus if one knows the associated primes of J, one also gains insight on the homological (geometrical) properties of R/J.

Let (R, m) be an RLR (e.g. the ring of the formal power series in n variables over a field) and J an ideal of R, then the projective dimension of R/J is equal to the dimension of R if and only if m is an associated prime of J.

Recently, most of the research on primary decomposition focused on finding effective information. Without the ambition to be exhaustive, one may summarize the recent efforts as follows:

Direct algorithms to compute the primary decomposition of an ideal:

Gianni, Träger and Zacharias (SINGULAR); Eisenbud, Huneke and Vasconcelos (MACAULAY and SINGULAR); Shimoyama and Yokoyama (SINGULAR).

Asymptotic behavior of primary components and associated primes:

Katz, Katzmann, Heinzer, Huneke, McAdam, Ratliff, Sharp, Swanson, etc.

Explicit primary decomposition of particular classes of ideals:

Here the list is ample, let us just quote few classes of ideals and few names.

Monomial ideals: Bayer, Galligo and Stillman - Heinzer, Ratliff and Shah - Sturmfels, Trung and Vogel.

Binomial ideals: Eisenbud and Sturmfels.

Determinantal ideals: De Concini, Eisenbud and Procesi - Hochster - Huneke - etc.

Gaussian ideals: Corso, Vasconcelos and Villareal.

Jacobian ideals of multilinear forms: Boffi, Bruns and Guerrieri - Guerrieri and Swanson.

Permanental ideals: Laubenbacher and Swanson.

Applications to fields other than Algebra or Geometry:

Diaconis, Eisenbud and Sturmfels: It turns out that primary decompositions of binomial ideals can give useful descriptions of components of a graph arising in problems from combinatorics, statistics, and operations research.

2. Hyperdeterminants and Jacobian ideals

"The notion of determinant slowly emerged during a period of one hundred and twenty years, from the pioneering work of Leibniz to the celebrated 1812 memoir of Cauchy ... Approximately at the same time, people started to look for a multidimensional generalization of the notion of determinant, an idea that had already intrigued Vandermonde ... A few years ago, Gelfand, Kapranov and Zelevinsky arrived at ... a definition . . . motivated by the theory of general discriminants ... But amazingly enough, they realized that Cayley (1845) had already defined the very same concept, while studying certain invariants of multilinear forms. He had even suggested to give the new notion the very appropriate name of *hyperdeterminant*."

G. Boffi - *On a Certain complex related to the notion of hyperdeterminant*

Let X denote the image of the Segre embedding:

$$\mathbb{P}^{n-1} \times \mathbb{P}^{m-1} \times \mathbb{P}^{p-1} :\longrightarrow \mathbb{P}^{nmp-1}$$

which sends the point of coordinates $((\ldots : X_i : \ldots), (\ldots : Y_j : \ldots), (\ldots : Z_k : \ldots))$ to the point of coordinates $(\ldots : X_i Y_j Z_k : \ldots)$. Here $\mathbb{P}^{n-1} = \mathbb{P}\,(V_1^*)$, $\mathbb{P}^{\,m-1} = \mathbb{P}\,(V_2^*)$ and $\mathbb{P}^{\,p-1} = \mathbb{P}\,(V_3^*)$ with V_1^* isomorphic to $\mathbb{C}^n$, V_2^* to $\mathbb{C}^m$ and V_3^* to $\mathbb{C}^p$. One may also rewrite the Segre embedding as follows:

$$\mathbb{P}(V_1^*) \times \mathbb{P}(V_2^*) \times \mathbb{P}(V_3^*) :\longrightarrow \mathbb{P}(V_1^* \otimes V_2^* \otimes V_3^*),$$

where (e, f, g) is sent to $e \otimes f \otimes g$.

The projectively dual variety $X^V \subseteq (\mathbb{P}^{nmp-1})^* = \mathbb{P}(V_1 \otimes V_2 \otimes V_3)$ of X is the closure of the locus of the hyperplanes of $\mathbb{P}^{nmp-1}$ that contain the tangent space of X at some non singular point p of X. In other words, X^V is the closure of the set of the tangent hyperplanes to X.

Assuming $n \geq m \geq p \geq 2$, X^V is an hypersurface if and only if $n \leq m + p - 1$ (see [GKZ], page 46). When X^V is an hypersurface, its defining equation is a irreducible homogeneous polynomial function on the space $V_1 \otimes V_2 \otimes V_3$, the so called *$X$-discriminant.*

One calls such polynomial function *hyperdeterminant of format $n \times m \times p$.* If X^V is not an hypersurface, one sets the hyperdeterminant to be equal to 1 and refers to this case as trivial. The hyperdeterminant is denoted by Det .

Analytically, the hyperplane $\{f = 0\}$ of $\mathbb{P}^{nmp-1}$ (a point of $(\mathbb{P}^{nmp-1})^*$) belongs to X^V if and only if f vanishes at some point of X with all its first derivatives. More precisely, we can describe $\{f = 0\}$ an hyperplane in $\mathbb{P}^{nmp-1}$ as a linear form

$$f = \sum_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m \\ 1 \leq k \leq p}} a_{ijk} U_{ijk},$$

after the restriction to X, f is represented by a trilinear form

$$f(\underline{X}, \underline{Y}, \underline{Z}) = A = \sum_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m \\ 1 \leq k \leq p}} a_{ijk} X_i Y_j Z_k$$

and it is clear that $\{f = 0\}$ belongs to X^V if and only if Det $(A) = 0$ if and only if the system

$$A = A_{X_i} = A_{Y_j} = A_{Z_k} = 0$$

has a solution $(\underline{a}, \underline{b}, \underline{c})$ with $\underline{a}$, $\underline{b}$ and $\underline{c}$ non-zero vectors. (The symbols A_{X_i}, A_{Y_j} and A_{Z_k} denote the first partial derivatives of A with respect to the corresponding variables).

One says that a three-dimensional array (a trilinear form) A for which this happens is *degenerate*.

When the hyperdeterminant of format $n \times m \times p$ exists, Det is uniquely determined up to sign by the requirement that Det (A) has integral coefficients and it is irreducible over $\mathbb{Z}$.

In general, no formula for Det (A) is known, similar to the usual expression

$$\sum_{\sigma \in S_m} (-1)^{\operatorname{sign} \sigma} a_{1\sigma(1)} \cdots a_{m\sigma(m)}$$

for the determinant of $m \times m$ matrices. In fact we are still far from fully understanding the combinatorial properties of the monomials occurring in Det (A).

In order to overcome this problem, various homological approaches to hyperdeterminants have been developed.

For example, Boffi and Weyman, [BW], analyzed a particular Koszul complex whose homology is affected by the vanishing of Det (A). The 0-th homology of this complex is determined by the Jacobian ideal of A. Knowing the homological properties of such an ideal could shed light on this Koszul complex and, in turn, on Det (A).

In conclusion, the theory of hyperdeterminants, starting from the definitions to arrive to the latest homological approaches, suggests that interesting information could be trapped in the ideal generated by all the first partial derivatives of a trilinear form, the *Jacobian* ideal of the form.

In our work, we analyze the information coded in the possible primary decompositions of the Jacobian ideal of trilinear forms whose coefficients satisfy a weak kind of genericity.

Whatever it has been just said about hyperdeterminants can be phrased in multilinear terms. The restriction to the trilinear case is just a matter of semplicity.

In the work with Swanson, however, we only deal with the three-dimensional case. In fact the extension to a multidimensional case looks far from trivial. However we arrive to characterize the class of three-dimensional arrays (with dimensions $n = m + p - 1$) with nonzero hyperdeterminant. We call these forms *trilinear forms in general position.*

3. Trilinear forms in general position

Let $R = [X_1, \ldots, X_n, Y_1, \ldots, Y_m, Z_1, \ldots, Z_p]$ with K a field. Assume $n \geq m \geq p$. Let

$$A = \sum_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m \\ 1 \leq k \leq p}} a_{ijk} X_i Y_j Z_k$$

be a trilinear form in R and let J_A be the ideal of R generated by the partial derivatives of A.

Together with Boffi and Bruns, I studied the associated primes of J_A when the coefficients of A identified a particular class of three-dimensional arrays with non-zero hyperdeterminant: $n = m+p-1$, (*boundary condition*) and $a_{ijk} \neq 0$ if and only if $i = j+k-1$, (*diagonal condition*).

It was then natural to ask if it was possible to relax these assumptions.

In the work with Swanson, we answer positively to this question. In fact we show that the structure described in the previous work holds in a larger context.

The critical idea behind our results is the new concept of *general position* which we abstract from the proofs of Boffi, Bruns and myself. Thanks to this notion we eliminate the diagonal condition and relax the condition $n = m+p-1$ to $n \geq m+p-1$.

Remarkably, when the equality holds, the class of trilinear forms we analyze identifies that of the three-dimensional arrays for which the hyperdeterminant is non-zero.

Our methods fully exploit the interplay of three matrices of linear forms obtained by taking opportune second derivatives of A.

More precisely, let X be the $p \times m$ matrix $(A_{Y_j Z_i})$, Y be the $p \times n$ matrix $(A_{X_j Z_i})$ and Z be the $m \times n$ matrix $(A_{X_j Y_i})$. The entries of X are linear forms in $K[X_1, \ldots, X_n]$, those of Y are linear forms in $K[Y_1, \ldots, Y_m]$ and those of Z are linear forms in $K[Z_1, \ldots, Z_p]$.

Here and in what follows we adopt the following notation: $\underline{X}, \underline{Y}$ and $\underline{Z}$ denote $(X_1, \ldots, X_n), (Y_1, \ldots, Y_m)$ and $(Z_1, \ldots, Z_p)$. Depending on the context, these stand for either the ideals or the row vectors. A_X stands for either the ideal or the vector $(A_{X_1}, \ldots, A_{X_n})$. A_Y and A_Z are defined similarly. With this notation we can express the first partial derivatives of A as products of matrices: $A_X = \underline{Z}Y = \underline{Y}Z$, $A_Y = \underline{Z}X = \underline{X}Z^t$, $A_Z = \underline{X}Y^t = \underline{Y}X^t$.

Assume, as usual, that for any matrix M and any integer $q \geq 0$, $I_q(M)$ stands for the ideal generated by the $q \times q$ minors of M. Thus we have $A_X = I_1(\underline{Z}Y) = I_1(\underline{Y}Z)$, $A_Y = I_1(\underline{Z}X) = I_1(\underline{X}Z^t)$, $A_Z = I_1(\underline{X}Y^t) = I_1(\underline{Y}X^t)$, and some particular ideal of minors enter in the game since $\underline{Z}I_p(X) \subseteq I_1(\underline{Z}X) = A_Y$, and $\underline{Z}I_p(Y), \underline{Y}I_m(Z) \subseteq I_1(\underline{Z}Y) = I_1(\underline{Y}Z) = A_X$.

Before proceeding further let me give you a simple example. Let

$$A = X_1Y_1Z_1 + X_2Y_1Z_2 + X_2Y_2Z_1 + X_3Y_2Z_2.$$

In this case we have

$$Y = \begin{bmatrix} Y_1 & Y_2 & 0 \\ 0 & Y_1 & Y_2 \end{bmatrix}, \quad X = \begin{bmatrix} X_1 & X_2 \\ X_2 & X_3 \end{bmatrix}, \quad Z = \begin{bmatrix} Z_1 & Z_2 & 0 \\ 0 & Z_1 & Z_2 \end{bmatrix}$$

and the first partial derivatives are

$$A_{X_1} = Y_1Z_1, \quad A_{X_2} = Y_1Z_2 + Y_2Z_1, \quad A_{X_3} = Y_2Z_2$$

$$A_{Y_1} = X_1Z_1 + X_2Z_2, \quad A_{Y_2} = X_2Z_1 + X_3Z_2$$

$$A_{Z_1} = X_1Y_1 + X_2Y_2, \quad A_{Z_2} = X_2Y_1 + X_3Y_2$$

Definition 3.1: A trilinear form A is said to be in *general position* when the conditions below are satisfied.
• $n \geq m \geq p$ and $n \geq m + p - 1$,
• $I_p(X)$ has height $m - p + 1$ (maximal possible),
• $I_p(Y)$ has height m (maximal possible),
• $T^{-1}A_Y$ is prime of height m where $T = \{Z_l^n\}_{n\geq 0}$, $1 \leq l \leq p$,
• $T_Z^{-1}(A_X) = T_Z^{-1}(\underline{Y})$, with $T_Z = K[Z_1, \ldots, Z_p] \setminus \{0\}$.

It is therefore natural to ask: Which trilinear forms are in general position? Is this notion linked to other already known concepts? What is the structure of J_A in this generalized setting?

The notion to which the concept of general position is linked is that of 1-genericity.

Definition 3.2: Let $W_1, \ldots, W_s$ be indeterminates over a field K. Let M be a $q \times r$ matrix whose entries are linear forms in $K[W_1, \ldots, W_s]$. We say that M is 1-*generic* if for any invertible row operation on M, the entries of each row generate an ideal of height $\min\{r, s\}$.

This concept extends that introduced by Eisenbud in *Linear sections of determinantal varieties*, when $s \geq q + r - 1$. In that case his definition and ours agree. However, the $1 \times r$ matrix $[W_1 \cdots W_s 0 \cdots 0]$, where $s < r$, is 1-generic in our sense but not in Eisenbud's

It is now natural to ask if there is an effective criterion to decide if a matrix is 1-generic or not?

Assume that K is algebraically closed. Eisenbud proves that if M is a 1-generic $q \times r$ matrix in s variables $W_1, \ldots, W_s$ and $s \geq q + r - 1$, $q \leq r$, then the height of $I_q(M)$ is $r - q + 1$.

We prove that if W is a $q \times r$ matrix whose entries are linear forms in the variables $W_1, \ldots, W_s$ and $s, q \leq r$, then W is 1-generic if and only if the height of $I_q(W)$ is maximal possible, namely s. Also, W is 1-generic if and only if the radical of $I_q(W)$ is $(W_1, \ldots, W_s)$.

Notice that the field K needs to be algebraically closed, as shown by the following example. Let $K = \mathbb{Q}$ and Y_1, Y_2 variables. Let Y be the 2×3 matrix

$$Y = \begin{bmatrix} Y_1 & Y_2 & 0 \\ Y_2 & Y_1 + Y_2 & 0 \end{bmatrix}.$$

Each of the two rows generates (Y_1, Y_2), and for every $b \in \mathbb{Q}$, the entries of (row 1) + b(row 2) generate

$$(Y_1 + bY_2, Y_2 + bY_1 + bY_2) = (Y_1 + bY_2, Y_2(1 - b^2 + b)).$$

As there is no rational number b for which $1 - b^2 + b = 0$, this last ideal also has height 2 and Y is 1-generic. However, $I_2(Y)$ is principal, so it cannot have height 2.

Coming back to the trilinear form A and to the three matrices of linear forms X, Y and Z obtained by taking the second partial derivatives of A, the following result highlights the link between the notion of general position and that of 1-genericity and the interplay of X, Y and Z.

Theorem 3.3: *Let $n \geq m + p - 1$, $n \geq m \geq p$, and K an algebraically closed field. The following are equivalent: Let $n \geq m + p - 1$ and $n \geq m \geq p$. Let K be an algebraically closed field. Then the following are equivalent:*

(i) X is a 1-generic matrix.
(ii) The transpose of X is a 1-generic matrix.
(iii) Y is a 1-generic matrix.
(iv) $I_p(Y)$ has height m.
(v) The radical of $I_p(Y)$ is $(Y_1, \ldots, Y_m)$.
(vi) Z is a 1-generic matrix.
(vii) $I_m(Z)$ has height p.
(viii) The radical of $I_m(Z)$ is $(Z_1, \ldots, Z_p)$.
(ix) A is a trilinear form in general position.

As a consequence we obtain an effective criterion to detect the 1-genericity of matrix $p \times n$ X of linear forms in n variables where $n \geq m + p - 1$. In fact having such a matrix X one may build a trilinear form A by multypling X on the right and on the left by the appropriate vectors in th Y_j's and the Z_k's. Hence one may construct the Y matrix and just check the height of its maximal minors.

Also, we can now answer to the question: Which trilinear forms are in general position?

In fact we prove that if K is an algebraically closed field there exists a non-empty Zariski-open subset U of K^{nmp} such that if $(a_{ijk}) \in U$, then the corresponding A is in general position.

More precisely, when $n = m + p - 1$, A is in general position if and only if the hyperdeterminant of A is different from 0.

In particular the examples studied by Boffi, Bruns and myself are in general position.

Eventually, we describe the structure of J_A in this wider setting as shown by the following results.

Theorem 3.4: *Assume that A is in general position, and that $m > p$. Then the minimal primes of J_A are $(Z_1, \ldots, Z_p, A_Z)$ and $(Y_1, \ldots, Y_m, A_Y) + I_p(X)$.*

Theorem 3.5: *Assume that A is in general position and that $m = p$. Then the minimal primes of J_A are*

$$(Z_1, \ldots, Z_p, A_Z) + I_p(X), (Y_1, \ldots, Y_m, A_Y) + I_p(X), \quad \textit{and} \quad (Y_1, \ldots, Y_m, Z_1, \ldots, Z_p).$$

Theorem 3.6: *If A is a trilinear form in general position,*

$$\sqrt{J_A} = J_A + \underline{Y}\,\underline{Z}.$$

On the primary components of J_A we have the following information:

Theorem 3.7: *Let A be in general position and let P be a prime ideal minimal over J_A. Then the P-primary component of J_A is P.*

Proposition 3.8: *Assume that A is in general position. Then*

$$J_A = \sqrt{J_A} \cap \big(J_A + I_p(X) + I_p(Y) + I_m(Z)\big).$$

Thus every embedded component of J_A contains $\big(J_A + I_p(X) + I_p(Y) + I_m(Z)\big)$.

We next introduce several admissible changes of variables, admissible in the sense that the primary decompositions stay the same. We admit linear changes of variables among the X_i, the Y_j and the Z_k separately. Such a change is an automorphism of $K[\underline{X}, \underline{Y}, \underline{Z}]$ and it preserves the Jacobian ideal J_A. Thus the primary decomposition of J_A is unaffected by these changes.

Some specific changes we can use are as follows:

1. Renaming of the X_i, in other words, a linear change of variables among the X_i,
2. Elementary row operation on X: this corresponds to a linear change of variables among the Z_k and an elementary row operation on Y,
3. Elementary column operation on X: this corresponds to a linear change of variables among the Y_j and an elementary row operation on Z.

Note that none of these changes affects the 1-genericity of X, Y and Z.

When $p = 2$, by Eisenbud [E2, Theorem 5.1 iii)] these admissible changes transform X into the scrollar space form $\mathbf{M}(a_1, \ldots, a_d)$ with $a_1 \geq a_2 \geq \cdots \geq a_d \geq 1$, $\sum_i a_i = n$, $d = n - m$. Explicitly, X has the form:

$$\left[\begin{array}{cccc|ccc|c|ccc} X_1 & X_2 & \cdots & X_{a_1-1} & X_{a_1+1} & \cdots & X_{a_1+a_2-1} & \cdots & X_{a_1+\cdots+a_{d-1}+1} & \cdots & X_{a_1+\cdots+a_d-1} \\ X_2 & X_3 & \cdots & X_{a_1} & X_{a_1+2} & \cdots & X_{a_1+a_2} & & X_{a_1+\cdots+a_{d-1}+2} & \cdots & X_{a_1+\cdots+a_d} \end{array}\right].$$

For example, when $n = m + 1$ (smallest possible),

$$X = \mathbf{M}(m+1) = \begin{bmatrix} X_1 & X_2 & \cdots & X_{m-1} & X_m \\ X_2 & X_3 & \cdots & X_m & X_{m+1} \end{bmatrix},$$

and when $n > m + 1$, X is a juxtaposition of $d = n - m$ such matrices, with no overlaps among the variables in these submatrices. We use the name *scroll* to indicate a single block of $\mathbf{M}(a_1, \ldots, a_d)$.

We calculate the primary decomposition of J_A when $p = 2$. We first explicitly do the case $m = p = 2$. We do this case separately for the sake of clarity. Here, $n \geq m+p-1 = 3$, and as every variable X_i is used, necessarily $4 = mp \geq n$.

Theorem 3.9: *Let $m = p = 2$. If $n = 4$, J_A has no embedded components. If $n = 3$, an irredundant primary decomposition is*

$$J_A = \left((\underline{Y}, A_Y) + I_2(X)\right) \cap \left((\underline{Z}, A_Z) + I_2(X)\right) \cap (\underline{Y}, \underline{Z}) \cap (\underline{X}, A_X, I_2(Y), I_2(Z)).$$

Theorem 3.10: *Suppose that $p = 2$, $n < 2p$, and $X = \mathbf{M}(a_1, \ldots, a_d)$ with $a_1, \ldots, a_d \in \{a, a+1, a+2\}$ for some integer a. Then J_A has exactly one embedded prime, namely the maximal homogeneous ideal. As the embedded component one can take $J_A + (X_1, \ldots, X_n) + I_2(Y) + I_m(Z)$.*

Theorem 3.11: *Let $p = 2$, m and n arbitrary. If $n = 2m$, J_A has no embedded components. When instead $n < 2m$, then J_A has only one embedded component, and that one is primary to the maximal homogeneous ideal. The embedded component may be taken to be*

$$J_A + (X_1, \ldots, X_n)^{m-1} + I_2(Y) + I_m(Z).$$

The class of Jacobian ideals of trilinear forms considered in [BBG] always had the maximal irrelevant ideal as an associated prime. The previous theorems are further evidence of this behavior. We do not know if the same hold more generally for arbitrary $n \geq m \geq p$:

Question 3.12: *Is the maximal irrelevant ideal an associated prime whenever K is algebraically closed and A is a trilinear form in general position?*

We also compute the primary decomposition of various cases in which $m = p = 3$ and the matrix X is symmetric. The embedded primes all involve the ideals $I_t(X)$ for $t = 1, 2$. We are also able to obtain further information on the embedded associated primes in the general case $m = p$ when the matrix X is a Hankel matrix.

The previous results brought us to analyze the structure of the ideal of submaximal minors of a 1-generic matrix in general. Eisenbud proved that the maximal minors generate a prime ideal. The same is not true for the ideals generated by the minors of smaller size.

Since admissible changes of variables transform any 1-generic $2 \times n$ matrix of linear forms into the scrollar space form $\mathbf{M}(a_1, \ldots, a_d)$, we concentrate on matrices $3 \times n$. For example, we are able to find the primary decomposition of the submaximal minors of the matrix

$$\begin{bmatrix} X_1 & X_2 & \cdots & X_{n-1} & X_n \\ X_2 & X_3 & \cdots & X_n & X_{n+1} \\ r_3 X_3 & r_4 X_4 & \cdots & r_{n+1} X_{n+1} & r_{n+2} X_{n+2} \end{bmatrix}$$

when $n \geq 3$ and $r_i \neq 0$.

Question 3.13: *Let X be a $p \times m$ 1-generic symmetric matrix in n variables, $n \geq m+p-1$, is the ideal generated by the submaximal minors of X a prime ideal?*

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