

Operator semigroups and applications

Main reference: Engel-Nagel: One-parameter semigroups for linear evolution equations, GTM 194, 2000.

- abstract theory + many applications
- philosophical + historical background

Definition: A family $(T(t))_{t \geq 0}$ of bounded linear operators on a Banach space X is called a strongly continuous (or briefly C_0) semigroup if

- $T(0) = Id$ operators commute! } semigroup
- $T(t) \cdot T(s) = T(t+s)$ $\forall t, s \geq 0$ } law
- the map $\mathbb{R}_+ := [0, +\infty) \ni t \mapsto T(t)x \in X$ is continuous $\forall x \in X$. (strong continuity)

Why one would introduce and study C_0 -semigroups?

Some motivation: Consider an autonomous system evolving in time with state space X . ↑
time invariant

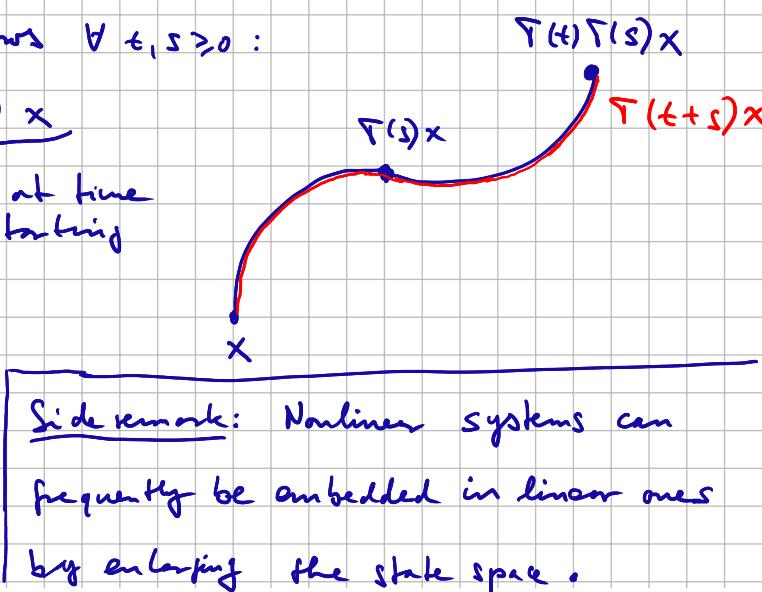
Define $T(t): X \rightarrow X$ by

$T(t)x :=$ state of the system at time t
starting at the initial state x .

Then by determinism it follows $\forall t, s \geq 0$:

- $T(t) \underbrace{T(s)x}_{\substack{= \text{state at} \\ \text{time } s \\ \text{starting} \\ \text{at } x}} = \underbrace{T(t+s)x}_{\substack{= \text{state at time} \\ t+s \text{ starting} \\ \text{at } x}}$
- Clearly $T(0) = Id$

semigroup law



- $\Gamma(t)$ linear: Superposition of solutions
- $t \mapsto \Gamma(t)x$ continuous: No jumps in evolution in time
- $\Gamma(t)$ bounded (\equiv continuous): $\Gamma(t)x =$ state at time t
depends continuously on the initial state x .

Example: Let A be a complex $n \times n$ matrix (or, more general, a bounded linear operator on a Banach space X).

Then

$$\Gamma(t) := e^{tA} := \sum_{n=0}^{+\infty} \frac{t^n A^n}{n!}$$

in $M_n(\mathbb{C})$ (or $\mathcal{L}(X)$)

converges if $t \geq 0$ (in fact $t \in \mathbb{C}$)

and defines a C_0 -semigroup on \mathbb{C}^n (or X).

Remarks. • In this case $(\Gamma(t))_{t \geq 0}$ is even uniformly continuous (or, even more, holomorphic if one considers $t \in \mathbb{C}$), i.e.

$\mathbb{R}_+, t \mapsto \Gamma(t) \in \mathcal{L}(X)$ = space of bounded linear operators on X

is continuous (or, if $t \in \mathbb{C}$, holomorphic).

- $\forall x \in X$ the function $u(t) := \Gamma(t)x$ is the unique solution of the Cauchy (or initial value) problem

$$(CP) \quad \begin{cases} u'(t) = Au(t), & t \geq 0 \\ u(0) = x. \end{cases} \quad (\text{here } u = \frac{d}{dt}u)$$

Hence, in this case, one can interpret $(\Gamma(t))_{t \geq 0}$ as the solution family of (CP).

Let's return to general C_0 -semigroups.

What is the corresponding operator A in this case?

In the above example

$$A = \dot{\Gamma}(0) = \lim_{t \rightarrow 0} \frac{\Gamma(t) - \Gamma(0)}{t} \in \mathcal{L}(X)$$

This motivates the following

Definition: If $(T(t))_{t \geq 0}$ is a C_0 -semigroup on a Banach space X , the operator $(A, D(A))$ defined by

$$D(A) := \left\{ x \in X \mid \lim_{t \rightarrow 0^+} \frac{T(t)x - x}{t} \text{ converges in } X \right\}$$

$$Ax := \lim_{t \rightarrow 0^+} \frac{T(t)x - x}{t} \quad \text{for } x \in D(A)$$

is called the (infinitesimal) generator of $(T(t))_{t \geq 0}$.

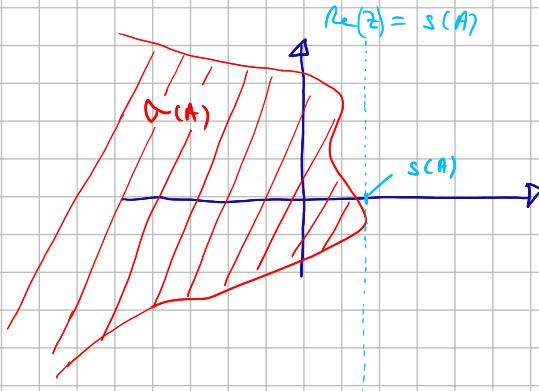
Formally: " $A = \dot{T}(0)$ ", " $T(t) = e^{tA}$ "

Some facts about generators.

Let $(A, D(A))$ be the generator of a C_0 -semigroup on a Banach space X . Then:

- $(A, D(A))$ is a closed, densely defined (i.e., $\overline{D(A)} = X$) linear operator on X .
 - A is bounded \Leftrightarrow by the closed graph theorem $D(A) = X$ sufficient int $t=0$
 $\Leftrightarrow (T(t))_{t \geq 0}$ is uniformly continuous for $t \geq 0$.
 - A uniquely determines $(T(t))_{t \geq 0}$, i.e., two semigroups having the same generator coincide.
 - $T(t)D(A) \subseteq D(A)$ $\forall t \geq 0$, i.e., a semigroup leaves the domain of its generator invariant.
 - Let $\sigma(A) := \left\{ \lambda \in \mathbb{C} \mid \lambda - A : D(A) \rightarrow X \text{ is not invertible} \right\} \subseteq \mathbb{C}$
 - \uparrow algebraically
 - \uparrow spectrum of A
 - \uparrow spectral value of A
- Then $s(A) := \sup_{\substack{\uparrow \\ \text{spectral bound of } A}} \{ \operatorname{Re}(\lambda) \mid \lambda \in \sigma(A) \} < +\infty$
- \uparrow see below

Hence: The spectrum of a generator is always contained in some left half-plane of \mathbb{C} .



by the closed graph theorem
(if A is closed)

- Let $\sigma(A) := \mathbb{C} \setminus \rho(A)$
"resolvent set of A "

 $i.e., \lambda \in \sigma(A) \Leftrightarrow (\lambda - A)^{-1} = R(\lambda, A) \in \mathcal{L}(X)$
exists "resolvent operator"
- $\sigma(A)$ is always closed, hence $\rho(A)$ is always open in \mathbb{C} .

let us now generalize the Cauchy problem (CP):

- If $(A, D(A))$ is a linear operator on a Banach space X ,
the initial value problem

$$(ACP) \quad \begin{cases} u'(t) = Au(t), t \geq 0 \\ u(0) = x \end{cases}$$

is called the Abstract Cauchy Problem with initial value x associated to A .

- $u: \mathbb{R}_+ \rightarrow X$ is called a (classical) solution of (ACP) if it is continuously differentiable, $u(t) \in D(A)$ $\forall t \geq 0$ and (ACP) holds.
- We call (ACP) well-posed if $\forall x \in D(A)$ (ACP) has a unique solution $u = u(\cdot, x)$ depending continuously on the initial value x . More precisely,

$$D(A) \ni x_n \rightarrow 0 \Rightarrow u(t, x_n) \xrightarrow{(n \rightarrow \infty)} 0 \text{ uniformly for } t \text{ in compact intervals } [0, t_0]$$

Briefly:

Well-posedness of (ACP) means

existence + uniqueness + continuous dependence on initial data
of solutions

Now the following important result holds which also justifies and motivates the study of C_0 -semigroups.

Theorem: Let $(A, D(A))$ be a closed, densely defined linear operator on a Banach space X . Then

(ACP) is well-posed $\Leftrightarrow (A, D(A))$ generates a C_0 -semigroup $(\Gamma(t))_{t \geq 0}$ on X .

Moreover, in this case the unique solution of (ACP) for $x \in D(A)$ is given by $u(t) = \Gamma(t)x$.

Hence we arrive at the following important

Question: Which operators $(A, D(A))$ are generators of C_0 -semigroups, i.e., for which A one can define " e^{tA} "?

→ Lecture 2

Before we study this problem we mention

Some facts about C_0 -semigroups

In order to check strong continuity of a operator semigroup the following is quite useful:

Lemma: For $(\Gamma(t))_{t \geq 0} \subset \mathcal{L}(X)$ satisfying the semigroup law the following conditions are equivalent:

a) $(\Gamma(t))_{t \geq 0}$ is strongly continuous

b) $\lim_{t \rightarrow 0^+} T(t)x = x \quad \forall x \in X$ (strong right-continuity at $t=0$)

c) There exist $M \geq 1$, $t_0 > 0$ and a dense set $D \subseteq X$ such that

$$\text{i)} \|T(t)\| \leq M \quad \forall t \in [0, t_0] \quad (\text{uniform boundedness})$$

$$\text{ii)} \lim_{t \rightarrow 0^+} T(t)x = x \quad \forall x \in D. \quad (\text{right continuity at } t=0 \text{ on a dense set})$$

Ideas of the proof: a) \Rightarrow c) : uniform boundedness theorem

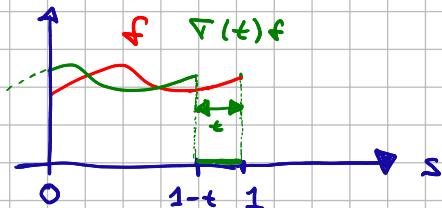
c) \Rightarrow b) : $3-\varepsilon$ argument

b) \Rightarrow a) : semigroup law.

Example: For $f \in X = L^1[0,1]$ and $t \geq 0$ define

$$(T(t)f)(s) := \begin{cases} f(s+t) & \text{if } s+t \leq 1 \\ 0 & \text{if } s+t > 1 \end{cases}$$

left shift semigroup



Obviously, $(T(t))_{t \geq 0}$ satisfies the semigroup law.

Then $(T(f))_{t \geq 0}$ defines a C_0 -semigroup on X (or, more generally, on $L^p[0,1] \quad \forall 1 \leq p < \infty$)

Proof: • $\|T(t)\| \leq 1 \quad \forall t \geq 0$.

not for $p = \infty$

• $D := C[0,1]$ is dense in $L^1[0,1]$.

• Let $f \in D$ and $\varepsilon > 0$. Then by uniform continuity

there exist $\delta > 0$ s.t. $|f(r) - f(s)| < \varepsilon \quad \forall r, s \in [0,1], |r-s| < \delta$.

Hence, for $0 \leq t < 1$

$$\begin{aligned} \|T(t)f - f\|_1 &\leq \underbrace{(1-t)}_{\leq 1} \cdot \underbrace{\|f(\cdot + t) - f(\cdot)\|}_{\infty, [0,1-t]} + t \cdot \|f\|_\infty \\ &\leq \varepsilon + \varepsilon \cdot \|f\|_\infty = \varepsilon \cdot (1 + \|f\|_\infty) \end{aligned}$$

if $0 \leq t < \min\{\varepsilon, \delta, 1\}$ $\Rightarrow (T(f))_{t \geq 0}$ is strongly continuous.

Remark: $T(t) = 0 \quad \forall t \geq 1$, i.e., $T(t)$ is nilpotent.

Problem: What is the generator of $(\Gamma(t))_{t \geq 0}$?

Rule of thumb: The generator of a left-shift semigroup on a function space like $L^p(I)$ or $C(I)$ is always given by the first derivative, i.e.,

$$A = \frac{d}{ds}$$

defined on a domain of "differentiable" functions verifying a boundary condition on the right end point of the interval (if there is one) which depends on what is shifted inside on the right.

In the above example one obtains $\subset C[0,1]$

$$A = \frac{d}{ds}, \quad D(A) = \left\{ f \in W^{1,1}[0,1] \mid f(1) = 0 \right\}$$

Recall that $\Gamma(t)D(A) \subseteq D(A)$ which gives a hint how $D(A)$ has to look like!

Remark: Nevertheless in a certain sense

$$\Gamma(t) = e^{t \cdot \frac{d}{ds}}$$

we have $\Gamma(t) = 0 \quad \forall t \geq 1$. This is in sharp contrast to the familiar properties of the exponential function which doesn't have zeros in \mathbb{C} !

Recall that by the above lemma for every C_0 -semigroup $(\Gamma(t))_{t \geq 0}$ there exist $M \geq 1$ and $t_0 > 0$ such that

$$\|\Gamma(t)\| \leq M \quad \forall t \in [0, t_0]$$

By the semigroup law this implies that every C_0 -semigroup is exponentially bounded.

More precisely, we have the following

Corollary: If $(\Gamma(t))_{t \geq 0}$ is a C_0 -semigroup then there exist

$M > 1$ and $\omega \in \mathbb{R}$ such that

$$\|\Gamma(t)\| \leq M \cdot e^{\omega t} \quad \forall t \geq 0$$

Definition: For a C_0 -semigroup $(\Gamma(t))_{t \geq 0}$ with generator A we define

$$\omega_0(A) := \inf \left\{ \omega \in \mathbb{R} \mid \text{there exists } M_\omega > 1 \text{ s.t. } \|\Gamma(t)\| \leq M_\omega \cdot e^{\omega t} \forall t \geq 0 \right\}$$

||

growth bound of $(\Gamma(t))_{t \geq 0}$

Remarks: • If $\omega_0(A) < 0$ then $(\Gamma(t))_{t \geq 0}$ is called uniformly exponentially stable. For the solutions of $(A \in \mathbb{P})$ this means that $u(t, x) \rightarrow 0$ as $t \rightarrow +\infty$ uniformly for initial values x in the unit ball of X .

- In general, $\omega_0(A)$ is not a minimum, even if $\dim(X) < +\infty$: let $X = \mathbb{C}^2$, $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then

$\Gamma(t) = e^{tA} = \begin{pmatrix} e^{t0} & t \\ 0 & e^{t0} \end{pmatrix}$ is unbounded as $t \rightarrow +\infty$ nevertheless $\omega_0(A) = 0$.

- For the nilpotent left-shift semigroup one has $\omega_0(A) = -\infty$.
- There are semigroups $(\Gamma(t))_{t \geq 0}$ (see below) such that

$$\|\Gamma(t)\| \geq 2 \quad \forall t \geq 0,$$

i.e., in general one cannot expect an estimate like

$$\|\Gamma(t)\| \leq e^{\omega \cdot t} \quad \text{with } \omega \in \mathbb{R}$$

$\rightarrow 1 \text{ as } t \rightarrow 0^+$

even by increasing ω .

Example: On $X = L^1[0, 1]$ define for $0 \leq r < 1$

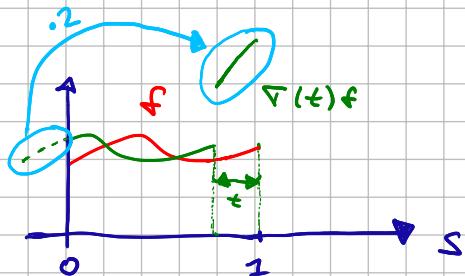
$$(\Gamma(r)f)(s) := \begin{cases} f(r+s) & \text{if } r+s \leq 1 \\ 2f(r+s-1) & \text{if } r+s > 1 \end{cases} \quad (\Gamma(1) = 2\text{-Id})$$

left-shift semigroup with jump

Then $\Gamma(t) := 2^n \cdot \Gamma(r)$ if $t = nr$, $0 \leq r < 1$, $n \in \mathbb{N}$, defines a C_0 -semigroup on X with generator

$$A = \frac{d}{ds}, \quad D(A) = \{ f \in W^{1,1}[0, 1] \mid 2 \cdot f(0) = f(1) \}$$

Moreover, $\|\Gamma(t)\| \geq 2 \quad \forall t > 0$.



Proposition: If $(\Gamma(t))_{t \geq 0}$ is a C_0 -semigroup with generator A

then

$$\Im(A) \leq \omega_0(A)$$

$$\left(\int_0^{+\infty} e^{-\lambda t} \cdot e^{\alpha t} dt = (\lambda - \alpha)^{-1} \right)$$

Moreover, $\forall x \in X$

$$R(\lambda, A)x = \int_0^{+\infty} e^{-\lambda t} \Gamma(t)x dt \quad \forall \lambda \in \mathbb{C} \text{ satisfying } \operatorname{Re} \lambda > \omega_0(A)$$

= laplace transform of $\Gamma(t)x$

Reminder: Lyapunov stability theorem for linear systems.

Let $A \in M_n(\mathbb{C})$ be a complex $n \times n$ matrix.

Then the solutions $u(\cdot, x)$ of the associated Cauchy problem satisfy

$$u(t, x) \rightarrow 0 \text{ as } t \rightarrow +\infty \quad \forall x \in \mathbb{C}^n$$

$$\Leftrightarrow \operatorname{Re} \lambda < 0 \text{ if eigenvalue } \lambda \text{ of } A \quad (\Leftrightarrow \lambda \in \sigma(A))$$

$$\Leftrightarrow \Im(A) < 0.$$

Hence: The location of the spectrum $\sigma(A)$ (= set of eigenvalues) in the complex plane determines the asymptotic behaviour of the solutions of the associated ($A \in \mathbb{P}$).

In the general case of semigroup generators this is not true anymore! In fact, there exist generators A of C_0 -semigroups such that

$$\mathfrak{s}(A) < 0 < \omega_0(A)$$

To obtain the equality " $\text{scal} = \omega_0(A)$ " one has to make some additional assumption, e.g. in form of some regularity condition for the map $t \mapsto T(t)$.

Definition: A C_0 -semigroup $(T(t))_{t \geq 0}$ is called eventually norm continuous, if there exists $t_0 \geq 0$ such that

$$\|T(t) - T(t_0)\| \rightarrow 0 \quad \text{as } t \rightarrow t_0^+.$$

Theorem: Let $(T(t))_{t \geq 0}$ be a eventually norm continuous semigroup then

$$\mathfrak{s}(A) = \omega_0(A),$$

i.e., the location of $\sigma(A)$ in \mathbb{C} determines the asymptotic behaviour of the solutions of the associated ($A \in \mathbb{P}$).

Example: The nilpotent left-shift semigroup is eventually norm continuous (in particular this shows that this property does not imply boundedness of the generator A). Hence, $\mathfrak{s}(A) = \omega_0(A) = -\infty \Rightarrow \sigma(A) = \emptyset$ (this fact can also be easily verified directly!).