CONVERGENCE OF THE FOLLOW-THE-LEADER SCHEME FOR SCALAR CONSERVATION LAWS WITH SPACE DEPENDENT FLUX

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ABSTRACT. This paper deals with the derivation of entropy solutions to Cauchy problems for a class of scalar conservation laws with space-density depending fluxes from systems of deterministic particles of follow-the-leader type. We consider fluxes which are product of a function of the density $v(\rho)$ and a function of the space variable $\phi(x)$. We cover four distinct cases in terms of the sign of ϕ, including cases in which the latter is not constant. The convergence result relies on a local maximum principle and on a uniform BV estimate for the approximating density.

1. Introduction

The approximation of nonlinear transport equations via follow-the-leader type schemes has attracted a lot of attention in the recent years. As a paradigm, consider Lighthill-Whitham-Richards' equation for traffic flow [21, 23]

$$\rho_t + (\rho v(\rho))_x = 0, \tag{1}$$

where ρ is the density of vehicles and $\rho \mapsto v(\rho)$ is a decreasing function modelling the Eulerian velocity of vehicles. As it is well known, in this model instantaneous response to the distance to the preceding vehicle is assumed by neglecting drivers' reaction time, whereas other models [2] take the latter into account. However, (1) is considered as a reliable model in several situations, for instance with low densities, see e.g. the recent book [24] and the references therein. Both approaches in [21, 23] and [2] treat the density of cars as a *continuum*, that is as a medium that can be divided into particles of arbitrary small mass without changing the physical nature of the system. On the other hand, the intrinsic nature of traffic flow is that of a discrete system of agents, each one of non trivial mass. Neglecting the driver's time reaction while adapting the speed to the distance to the preceding vehicle, the simplest (and most reasonable) law for the dynamics of $n+1$ drivers is provided by the *follow-the-leader* system

$$\begin{cases} \dot{x}_i(t) = v\left(\dfrac{\ell}{x_{i+1}(t) - x_i(t)}\right), & \text{for } i \in \{0, \dots, n-1\}, \\ \dot{x}_n(t) = v_{\max} = v(0), \end{cases} \tag{2}$$

where $x_0(t) < \ldots < x_n(t)$ denote the positions of the $n+1$ vehicles at time t, v is a given non-negative, non-increasing function on $[0, +\infty)$ with finite value $v_{\max}$ at 0 and ℓ is the (one-dimensional) mass of each vehicle. Typically, a maximum density $\rho_{\max}$ is prescribed in the model in order to avoid collisions, and the velocity v satisfies $v(\rho_{\max}) = 0$. The vehicle x_n, called 'leader', travels with maximum speed as no vehicles are ahead of it. The finite dimensional dynamical system (2) is usually coupled with $n+1$ initial conditions $x_i(0) = \bar{x}_i$, $i = 0, \dots, n$.

In a more general framework in which the dependence on ρ in the velocity term v in (1) includes possible diffusion terms, or external force fields, or nonlocal interaction terms, several results are available in the literature. We provide here a partial list of results. A probabilistic approach based on exclusion processes was developed in several works, we mention here [13, 14, 20]. When diffusion terms are included, we mention here the milestone results in [16, 27]. System (2) is a typical example of *deterministic* particle system, in that no stochastic effects are considered and the position of each particle is exactly computable for all times $t \geq 0$. A first attempt to detect diffusion effect via deterministic particles is due to [26]. The result in [15] extends this approach to nonlinear diffusions. A relevant recent result also involving external potentials is contained in [22]. Deterministic particle limits are also relevant in the literature of the modelling of swarming phenomena, see e.g. [5] and the references therein.

We observe at this stage that the quantity $\frac{\ell}{x_{i+1}(t)-x_i(t)}$ in (2) has the physical dimension of a one-dimensional density. Hence, the equation in (2) for $i < n$ can be considered as a Lagrangian discrete counterpart of the continuity equation (1). This motivates the mathematical interest for (2) as a possible many-particle approximation of (1), that is in the limit as $n \to +\infty$. While this fact has been largely known in the literature in the spirit of a 'formal limit', the result in [12] proved it as a rigorous result. More precisely, the result in [12] can be be stated as follows: take an arbitrary continuum initial condition $\overline{\rho} \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ with compact support, and consider a suitable *atomization* of ρ_0, for instance a set of $n+1$ particles $\bar{x}_0, \ldots, \bar{x}_n \in \mathbb{R}$ with the property that $\int_{\bar{x}_i}^{\bar{x}_{i+1}} \rho_0(x)dx = 1/n$ for all $i = 0, \ldots, n-1$. Now, consider the (unique) solution to (2) with initial condition $\bar{x}_i$, $i = 0, \ldots, n$ and the discrete piecewise reconstruction of the particles' density

$$\rho^n(x,t) = \sum_{i=0}^{n-1} \frac{1}{n(x_{i+1}(t) - x_i(t))} \mathbb{1}_{[x_i(t), x_{i+1}(t))}(x). \tag{3}$$

Then, ρ^n converges in $L^1_{loc}(\mathbb{R} \times \mathbb{R}_+)$ as $n \to +\infty$ to the unique entropy solution (in the sense of Kružkov [19]) ρ of the scalar conservation law (1) with initial datum $\overline{\rho}$. Such result was later extended to a larger class of initial conditions in [9].

Said results provide an abstract validation of the continuum approach (1) as a good approximation of the discrete model (2). In particular, the emergence of typical patterns such as rarefaction waves and shocks - that are easily computable in (1) - is established as a phenomenon that occurs also in the discrete setting in a 'coarse-grained' sense, that in the proper scaling regime in which the domain is large enough to include a very large number of vehicles and the total mass of the vehicles is normalized. We stress that said patterns are not detectable analytically in (2) for finite n. Moreover, this set of results is relevant also from the numerical point of view, as it allows to follow the movement of each vehicle unlike standard approaches such as classical Godunov type methods. Finally, these results hold without prescribing the initial condition to be far from the vacuum state (a restriction which would contradict the fact that the inertia-free approach of (1) is more suitable for low densities). We mention at this stage that the literature contains several results about the derivation of the second order ARZ model via deterministic follow-the-leader systems, see e.g. [1, 3].

Although traffic flow is a motivating example to justify (1) as a many-particle limit for (2), the results in [12, 9] hold under more general assumptions on the velocity map v: it suffices to assume that v is monotone (decreasing or increasing) and the monotonicity of v determines the proper upwind direction for the discrete density on the right-hand side of (2) (an increasing velocity v requires the use of the backward density $\frac{\ell}{x_i(t)-x_{i-1}(t)}$). Indeed, this deterministic particle approach to solving nonlinear continuity equations was later on extended to other models. In [11] the same approach was used to approximate solutions to (1) on a bounded domain with Dirichlet type conditions. In [10] a suitable modification of (2) was proven to converge in the many-particle limit to weak solutions to the Hughes model for pedestrian movements in one space dimension. Finally, a nonlocal version of (1) was considered in [8] as the many-particle limit of a suitable variant of (2) considering nonlocal interactions with all particles.

The present paper contributes to this line of research by considering the case of a scalar conservation law with space-dependent flux

$$\rho_t + \big(\rho v(\rho)\phi(x)\big)_x = 0, \tag{4}$$

where v is monotone and ϕ is a given external drift term depending on the position x. Besides being well motivated in the modelling context of traffic flow - for example in situations in which the speed of the vehicles is also affected by external factors (such as temporary road maintenance, or sudden turns or rises) - the equation (4) has a pretty wide range of potential applications in sedimentation processes [4], flow of glaciers [17], formation of Bose-Einstein condensates [28]. For a more general description of the applications of nonlinear scalar conservation laws we refer to [7] and the references therein.

Similarly to the approach of [12] and later results, we will assume throughout this paper that $v : [0, +\infty) \to [0, +\infty)$ is *monotone non-increasing* and non-negative, with $v(0) < +\infty$. A symmetric result could be stated in case of a non-decreasing v, we shall omit the details. As for the potential ϕ, we consider four cases:

(P1) $\phi(x) \geq 0$ for all $x \in \mathbb{R}$ (forward movement);

(P2) $\phi(x) \leq 0$ for all $x \in \mathbb{R}$ (backward movement);

(P3) $x\phi(x) \geq 0$ for all $x \in \mathbb{R}$ (repulsive movement);
(P4) $x\phi(x) \leq 0$ for all $x \in \mathbb{R}$ (attractive movement).

We refer to section 2 for the precise statement of all assumptions on v and ϕ. For each of the above four cases we shall provide an ad-hoc many-particle approximation result in the spirit of (2). For example, case (P1) requires the use of the *forward* follow-the-leader scheme

$$\begin{cases} \dot{x}_i(t) = v\left(\dfrac{\ell}{x_{i+1}(t) - x_i(t)}\right)\phi(x_i(t)), \quad \text{for } i = 0, \ldots, n-1, \\ \dot{x}_n(t) = v(0)\phi(x_n(t)). \end{cases} \tag{5}$$

The distinction between case (P1) and case (P2) is relevant in that it implies a change in the upwind direction of the scheme. More in detail, if $\phi \leq 0$ then all particles are subject to a drift directed towards the negative direction. Hence, it is reasonable to assume that each particle adjusts its speed by considering the distance to its *left* nearest neighbor, and the leftmost particle will be the leader travelling with $v = v(0)$. In case (P3), the drift direction changes at the origin $x = 0$, with a positive direction on $x \geq 0$ (non-negative ϕ) and a negative one on $x \leq 0$ (non-positive ϕ). We shall refer to this case as *repulsive* movement, since it implies a drift of all particles away from the origin. Two leaders (leftmost and rightmost particle) will travel with speed $v = v(0)$. Symmetrically, in case (P4) particles move towards the positive (negative respectively) direction on $x \leq 0$ (on $x \geq 0$ respectively). This implies that no actual 'leader' exists in the sense of the previous cases, and particles adapt their speed with respect to the relative position with their right (left respectively) nearest neighbor on $x \leq 0$ (on $x \geq 0$ respectively). This situation implies an *attractive* movement towards the origin, a phenomenon that could potentially imply collision between the two particles nearest to the origin in a finite time. To see this, consider the example $v(\rho) = (1+\rho)^{-1}$, $\phi(x) = -|x|^\alpha$ with $\alpha \in (0,1)$. Setting two particles at initial positions $-x_0, x_0$ with $x_0 > 0$, one can easily show that the two particles $-x(t)$ and $x(t)$ obeying

$$\dot{x}(t) = \phi(x(t))v\left(\frac{\ell}{2x(t)}\right), \qquad x(0) = x_0,$$

reach the origin in a finite time. Other significant examples originate in the continuum setting in the study of Bose-Einstein condensates, see [6], with $\phi(x) = -x$ and $v(\rho) = \rho^2$, in which the finite time blow-up in L^∞ of the density is proven. In order to bypass this problem, we shall require an additional assumption for case (P4), namely that the velocity map $v(\rho)$ vanishes at some prescribed maximal density value $R_{\max}$ and is equal to zero on $[R_{\max}, +\infty)$. Such assumption is reasonable in contexts such as traffic flow in a single lane, in which overtaking of vehicles is not allowed. Cases (P3) and (P4) are paradigmatic of sign changing ϕ: more general situations in which ϕ changes sign at more than one point can be treated via minor modifications with the strategy outlined above, we omit the details for simplicity.

In all the aforementioned four cases we are able to prove a convergence result in the spirit of [12]: given an initial condition $\bar{\rho} \in L^\infty(\mathbb{R}) \cap BV(\mathbb{R})$ non-negative and with compact support, we atomize $\bar{\rho}$ by a set of $n+1$ particles $\bar{x}_0, \ldots, \bar{x}_n$, we consider the piecewise constant density ρ^n as in (3) with $x_0(t), \ldots, x_n(t)$ solution to a suitable follow-the-leader scheme ((5) in case (P1) as an example) with initial datum $\bar{x}_0, \ldots, \bar{x}_n$, and prove that ρ^n converges locally in $L^1_{x,t}$ towards the unique entropy solution to (4) with $\bar{\rho}$ as initial condition. Such result requires as crucial steps:

- A local maximum principle showing that $\|\rho^n\|_{L^\infty(\mathbb{R})}$ is uniformly bounded with respect to n on arbitrary time intervals $[0,T]$;
- BV compactness estimates;
- Consistency with the definition of entropy solutions (in the Kružkov's sense [19]) in the $n \to +\infty$ limit.

Such a strategy requires an $L^1 \cap L^\infty$ setting. This is why we cannot consider case (P4) in presence of blow-up or concentration phenomena. This issue will be tackled in a future paper.

The paper is structured as follows. In section 2 we define our four approximation schemes and prove their main properties, including the maximum principle for all of them. We highlight that cases (P1)-(P2)-(P3) feature a maximum principle in terms of the initial L^∞ norm, whereas in case (P4) the uniform bound for ρ^n is provided in terms of the maximal density $R_{\max}$. In section 3 we prove the needed uniform BV estimate, as well as an equicontinuity property with respect to the Wasserstein distance that provide local

L^1 compactness in space and time. Finally, in section 4 we state and prove our main result in Theorem 4.1, that collects the convergence of the scheme in all four cases.

2. Statement of the problem and maximum principles

Let us consider the following Cauchy problem for a one-dimensional conservation law

$$\begin{cases} \rho_t + \big(\rho v(\rho)\phi(x)\big)_x = 0, & x \in \mathbb{R},\ t > 0, \\ \rho(x,0) = \overline{\rho}(x), & x \in \mathbb{R}, \end{cases} \tag{6}$$

where we assume that the function v and the initial datum $\overline{\rho}$ satisfy respectively

- (V) $v \in C^1(\mathbb{R}_+)$ is a non-negative function with $v'(\rho) \leq 0$ and $v(0) := v_{\max} < +\infty$;
- (I) $\overline{\rho} \in L^\infty(\mathbb{R}) \cap BV(\mathbb{R})$ is a non-negative, compactly supported function.

Concerning the potential ϕ, we shall deal with four different cases

- (P1) $\phi(x) \geq 0$ for all $x \in \mathbb{R}$ (forward movement);
- (P2) $\phi(x) \leq 0$ for all $x \in \mathbb{R}$ (backward movement);
- (P3) $x\phi(x) \geq 0$ for all $x \in \mathbb{R}$ (repulsive movement);
- (P4) $x\phi(x) \leq 0$ for all $x \in \mathbb{R}$ (attractive movement).

In all these cases, we assume the basic condition

- (P) $\phi \in W^{2,\infty}(\mathbb{R})$

and in the last one we add the following condition on the function v

- (V*) There exists $R_{\max} > 0$ such that $\overline{R} := ||\overline{\rho}||_{L^\infty(\mathbb{R})} \leq R_{\max}$, $v(\rho) > 0$ for $\rho < R_{\max}$ and $v(\rho) \equiv 0$ for $\rho \geq R_{\max}$.

For the sake of simplicity, we suppose that the initial mass is normalised, that is

$$||\overline{\rho}||_{L^1(\mathbb{R})} = 1.$$

Moreover, let us denote with

$$[\overline{x}_{\min}, \overline{x}_{\max}] = \mathrm{Conv}(\mathrm{supp}(\overline{\rho}))$$

the convex hull of the support of $\overline{\rho}$.

Our next goal is to provide an initial condition for the follow-the-leader systems. To perform this task, we split the interval $[\overline{x}_{\min}, \overline{x}_{\max}]$ into n sub-intervals having equal mass $\ell_n := 1/n$. So, for a fixed $n \in \mathbb{N}$ sufficiently large, we set $\overline{x}_0^n := \overline{x}_{\min}$, $\overline{x}_n^n := \overline{x}_{\max}$ and we define recursively

$$\overline{x}_i^n := \sup\left\{ x \in \mathbb{R} : \int_{\overline{x}_{i-1}^n}^{x} \overline{\rho}(x)dx < \ell_n \right\} \quad \text{for } i \in \{1, \dots, n-1\}.$$

From the previous definition we immediately have that $\overline{x}_0^n < \overline{x}_1^n < \cdots < \overline{x}_n^n$ and

$$\int_{\overline{x}_{i-1}^n}^{\overline{x}_i^n} \overline{\rho}(x)dx = \ell_n \quad \text{for } i \in \{1, \dots, n-1\}. \tag{7}$$

Next we introduce the follow-the-leader systems describing the evolution of the $n+1$ particles with initial positions $\overline{x}_i^n$, $i = 0, \dots, n$. The definition of the particle system depends on the cases (P1)-(P4) introduced above, hence we should introduce four different approximation schemes, nevertheless, as we will see in a moment, the latter two are strictly related to the former two. Cases (P1) and (P2) are the simplest ones, as the constant sign of ϕ does not affect the monotonicity of the velocity field $v(\rho)\phi(x)$. Consistently with the homogeneous case [12], when ϕ is non-negative the velocity field decreases with respect to the density ρ. Therefore, in case (P1) the follow-the-leader scheme should consider a *forward* finite-difference approximation of the density. Symmetrically, (P2) implies a *backward* approximation. Therefore, with the notation

$$R_i^n(t) := \frac{\ell_n}{x_{i+1}^n(t) - x_i^n(t)}, \qquad t \geq 0, \quad i \in \{0, \dots, n-1\},$$

in case (P1) we use the ODE system

$$\begin{cases} \dot{x}_i^n(t) = v(R_i^n(t))\phi(x_i^n(t)), & \text{for } i \in \{0, \dots, n-1\}, \\ \dot{x}_n^n(t) = v_{\max}\phi(x_n^n(t)), & \\ x_i^n(0) = \overline{x}_i^n, & \text{for } i \in \{0, \dots, n\} \end{cases} \tag{8}$$

and in case (P2) we use

$$\begin{cases} \dot{x}_i^n(t) = v(R_{i-1}^n(t))\phi(x_i^n(t)), & \text{for } i \in \{1, \dots, n\}, \\ \dot{x}_0^n(t) = v_{\max}\phi(x_0^n(t)), & \\ x_i^n(0) = \overline{x}_i^n, & \text{for } i \in \{0, \dots, n\}. \end{cases} \tag{9}$$

For cases (P3) and (P4) we consider a sort of combination of the previous two cases. With the notation

$$k_n := \max\big\{i \in \{0, \dots, n\} : \overline{x}_i^n \le 0\big\},$$

the ODE system in case (P3) is

$$\begin{cases} \dot{x}_i^n(t) = v(R_{i-1}^n(t))\phi(x_i^n(t)), & \text{for } i \in \{1, \dots, k_n\}, \\ \dot{x}_i^n(t) = v(R_i^n(t))\phi(x_i^n(t)), & \text{for } i \in \{k_n+1, \dots, n-1\}, \\ \dot{x}_0^n(t) = v_{\max}\phi(x_0^n(t)), & \\ \dot{x}_n^n(t) = v_{\max}\phi(x_n^n(t)), & \\ x_i^n(0) = \overline{x}_i^n, & \text{for } i \in \{0, \dots, n\}, \end{cases} \tag{10}$$

whereas in case (P4) we use

$$\begin{cases} \dot{x}_i^n(t) = v(R_i^n(t))\phi(x_i^n(t)), & \text{for } i \in \{0, \dots, k_n\}, \\ \dot{x}_i^n(t) = v(R_{i-1}^n(t))\phi(x_i^n(t)), & \text{for } i \in \{k_n+1, \dots, n\}, \\ x_i^n(0) = \overline{x}_i^n, & \text{for } i \in \{0, \dots, n\}. \end{cases} \tag{11}$$

For the sequel, we define the quantities

$$L := v_{\max}||\phi||_{L^\infty(\mathbb{R})} \quad \text{and} \quad L' := v_{\max}||\phi'||_{L^\infty(\mathbb{R})},$$

which are, thanks to assumption (P), two positive constant and, moreover, we drop the n-dependence for simplicity, whenever there is no ambiguity.

We remark that the Lipschitz conditions on v and ϕ ensure the local existence and uniqueness of solution to (8), (9), (10) and (11). In order to safeguard global existence, we need to prove three properties:

a) particles have a finite position and velocity on bounded time intervals,
b) particles always move in the same direction in (10) and (11),
c) particles never collide, consequently they always maintain the same order.

Remark 2.1 (Finite position and velocity on bounded time intervals)**.** *In all the four cases we have that the respective solution satisfies, for $t \ge 0$ and $i \in \{0, \dots, n\}$,*

$$|x_i^n(t)| < +\infty \quad \textit{and} \quad |\dot{x}_i^n(t)| \le L < +\infty. \tag{12}$$

Indeed, due to assumption (P) and (V) we immediately get $|\dot{x}_i(t)| \le L$ for all $t \ge 0$. Furthermore, integrating the ODE defining the evolution of $x_i(t)$, it follows that

$$|x_i(t)| = \left|\overline{x}_i + \int_0^t \dot{x}_i(s)ds\right| \le |\overline{x}_i| + \int_0^t |\dot{x}_i(s)|ds \le |\overline{x}_i| + Lt < +\infty.$$

Remark 2.2 (Upper bound for the distance of two consecutive particles)**.** *In all the four cases there exists a positive constant c, depending only on ϕ and v, such that the corresponding solution satisfies*

$$x_{i+1}^n(t) - x_i^n(t) \le \overline{x}_{\max} - \overline{x}_{\min} + ct \quad \textit{for } t \ge 0,\ i \in \{0, \dots, n-1\}, \tag{13}$$

indeed the previous remark implies that

$$|x_{i+1}(t) - x_i(t)| = \left|\overline{x}_{i+1} - \overline{x}_i + \int_0^t \big(\dot{x}_{i+1}(s) - \dot{x}_i(s)\big)ds\right| \le \overline{x}_{\max} - \overline{x}_{\min} + 2Lt.$$

Now we prove the following technical result regarding cases (P3)-(P4).

Proposition 2.1 (Preservation of the particles' sign in cases (P3)-(P4))**.** *Assume (V), (I) and (P) are satisfied and, moreover, assume (V*) is satisfied in case (P4). Then, as long as the solution to* (10) *and* (11) *exists, we have*

$$x_i(t) \le 0 \quad \textit{for } t \ge 0,\ i \in \{0, \dots, k_n\} \quad \textit{and} \quad x_i(t) \ge 0 \quad \textit{for } t \ge 0,\ i \in \{k_n + 1, \dots, n\}.$$

Proof. Concerning the case (P3), let us show the property for a fixed $i \in \{0, \dots, k_n\}$ (for the remaining indices we can follow a symmetric reasoning): we have to prove that $t^* := \inf\{t \ge 0 : x_i^n(t) > 0\} = +\infty$. Since $\overline{x}_i \le 0$, then $\dot{x}_i(0) = v(R_i(0))\phi(\overline{x}_i) \le 0$ and hence by continuity there exists $T_1 > 0$ such that $x_i(t) \le 0$ for all $t < T_1$. It follows that $\dot{x}_i(t) \le 0$ for all $t < T_1$ and thus

$$x_i(T_1) = \overline{x}_i + \int_0^{T_1} \dot{x}_i(s)ds \le \overline{x}_i \le 0.$$

Reasoning as before, we have the existence of a time $T_2 > T_1$ such that $x_i(t) \le 0$ for all $t < T_2$ and hence, since we can repeat this argument arbitrarily, we get $t^* = +\infty$ as we claimed.

As for the case (P4), let us consider as before only the indices $i \in \{0, \dots, k_n\}$. Then $\overline{x}_i < 0$ for $i < k_n$, while $\overline{x}_{k_n}$ can be either negative or zero and, in the latter case, we remark that $\dot{x}_{k_n}(0) = 0$, so by continuity there exists $T_1 > 0$ such that $x_{k_n}(t) = 0$ for all $t < T_1$. Therefore $\dot{x}_{k_n}(t) = 0$ for all $t < T_1$, so we get

$$x_{k_n}(T_1) = \overline{x}_{k_n} + \int_0^{T_1} \dot{x}_{k_n}(s)ds = \overline{x}_{k_n} = 0$$

and hence, arguing as before, by continuity we have that $x_{k_n}(t) = 0$ for all $t \ge 0$. As a consequence, we can assume without restriction that $\overline{x}_i < 0$ and, under this assumption, by continuity there exists $T > 0$ such that $x_i(t) \le 0$ for all $t \le T$. Suppose now by contradiction the existence of two times $T \le t_1 < t_2$ such that $x_i(t) \le 0$ for $t \le t_1$, $x_i(t) = 0$ for $t = t_1$ and $x_i(t) > 0$ for $t_1 < t \le t_2$. Then we have $\dot{x}_i(t) \le 0$ for $t_1 < t \le t_2$ and hence

$$x_i(t) = x_i(t_1) + \int_{t_1}^{t} \dot{x}_i(s)ds \le x_i(t_1) = 0 \quad \text{for } t_1 < t \le t_2,$$

which is a contradiction. □

The next proposition ensures that particles never collide in all the four cases: this gives the global existence of the solution for (8), (9), (10) and (11).

Proposition 2.2 (Discrete maximum principle)**.** *Assume that (V), (I) and (P) are satisfied.*

- *If one of (P1), (P2) or (P3) holds, then the respective solution to* (8), (9) *and* (10) *satisfies*

$$x_{i+1}^n(t) - x_i^n(t) \ge \frac{\ell_n}{\overline{\overline{R}}} e^{-L't} \quad \textit{for } t \ge 0,\ i \in \{0, \dots, n-1\}. \tag{14}$$

- *If (V*) and (P4) hold, then the solution to* (11) *satisfies*

$$x_{i+1}^n(t) - x_i^n(t) \ge \frac{\ell_n}{R_{\max}} \quad \textit{for } t \ge 0,\ i \in \{0, \dots, n-1\}. \tag{15}$$

Proof. We first observe that the statement is true for $t = 0$, indeed by (7) and (I) we have

$$\overline{x}_{i+1} - \overline{x}_i \ge \frac{\ell_n}{\overline{\overline{R}}} \ge \frac{\ell_n}{R_{\max}} \quad \text{for all } i \in \{0, \dots, n-1\}.$$

Now we should consider the four cases separately, but we are going to exploit in all the cases a recursive argument with suitable differences.

Let us suppose (P1) holds and let us take as basis of our recursive argument the index $i = n - 1$. Since the first order Taylor's expansion of ϕ at $x_{n-1}(t)$ is given by

$$\phi(x_n(t)) = \phi(x_{n-1}(t)) + \phi'(\tilde{x}(t))(x_n(t) - x_{n-1}(t))$$

for some $\tilde{x}(t) \in \big(x_{n-1}(t), x_n(t)\big)$, then it follows that

$$\begin{aligned}\frac{d}{dt}\big[x_n(t) - x_{n-1}(t)\big] =& v_{\max}\phi(x_n(t)) - v(R_{n-1}(t))\phi(x_{n-1}(t)) \\ =& \big(v_{\max} - v(R_{n-1}(t))\big)\phi(x_{n-1}(t)) + v_{\max}\phi'(\tilde{x}(t))\big(x_n(t) - x_{n-1}(t)\big) \\ \geq& v_{\max}\phi'(\tilde{x}(t))\big(x_n(t) - x_{n-1}(t)\big).\end{aligned}$$

Therefore, applying Gronwall lemma, we get that

$$x_n(t) - x_{n-1}(t) \geq \big(\overline{x}_n - \overline{x}_{n-1}\big)e^{\int_0^t v_{\max}\phi'(\tilde{x}(s))ds} \geq \frac{\ell_n}{\overline{R}}e^{\int_0^t v_{\max}\phi'(\tilde{x}(s))ds}$$

and finally, since the assumption (P) implies that ϕ' is bounded, from the previous inequality we easily get (14) for $i = n-1$.

Concerning the remaining indices, we argue by contradiction and assume (without restriction) the existence of an index $j \in \{0, \dots, n-2\}$ and of two times $0 \leq t_1 < t_2$ satisfying

$$x_{i+1}(t) - x_i(t) \geq \frac{\ell_n}{\overline{R}}e^{-L't} \quad \text{for } t \geq 0,\ i \in \{j+1, \dots, n-1\}$$

and

$$x_{j+1}(t) - x_j(t)\begin{cases} \geq \frac{\ell_n}{\overline{R}}e^{-L't} & \text{for } t < t_1, \\ = \frac{\ell_n}{\overline{R}}e^{-L't_1} & \text{for } t = t_1, \\ < \frac{\ell_n}{\overline{R}}e^{-L't} & \text{for } t_1 < t \leq t_2. \end{cases} \tag{16}$$

Using as before the first order Taylor's expansion of ϕ at $x_j(t)$, it holds

$$\begin{aligned}\frac{d}{dt}\big[x_{j+1}(t) - x_j(t)\big] =& v(R_{j+1}(t))\phi(x_{j+1}(t)) - v(R_j(t))\phi(x_j(t)) \\ =& \big(v(R_{j+1}(t)) - v(R_j(t))\big)\phi(x_j(t)) + v(R_{j+1}(t))\phi'(\tilde{x}(t))\big(x_{j+1}(t) - x_j(t)\big)\end{aligned} \tag{17}$$

for some $\tilde{x}(t) \in \big(x_j(t), x_{j+1}(t)\big)$. On the other hand, from the contradictory assumption it follows that

$$R_{j+1}(t) \leq \overline{R}e^{L't} < R_j(t)$$

and hence that

$$v(R_j(t)) \leq v\big(\overline{R}e^{L't}\big) \leq v(R_{j+1}(t))$$

for all $t_1 < t \leq t_2$. Using these estimates in (17), for all $t_1 < t \leq t_2$ we get that

$$\frac{d}{dt}\big[x_{j+1}(t) - x_j(t)\big] \geq v(R_{j+1}(t))\phi'(\tilde{x}(t))\big(x_{j+1}(t) - x_j(t)\big)$$

and finally, using again Gronwall lemma on the time interval (t_1, t) with $t \leq t_2$, it follows that

$$x_{j+1}(t) - x_j(t) \geq \big(x_{j+1}(t_1) - x_j(t_1)\big)e^{\int_{t_1}^t v(R_{j+1}(s))\phi'(\tilde{x}(s))ds} \geq \frac{\ell_n}{\overline{R}}e^{-L't},$$

which contradicts (16), since $t_1 < t \leq t_2$.

Concerning case (P2), we can reason in a symmetric way, taking the index $i = 0$ as basis of the recursive argument and assuming the existence of a 'first index j' at which the statement fails on a certain time interval (we omit the details).

In case (P3) holds, the proof of (14) for $i \in \{0, \dots, k_n - 1\}$ and $i \in \{k_n + 1, \dots, n\}$ is straightforward, indeed we can simply apply (rearranging the indices properly) the same arguments used in cases (P2) and (P1) respectively. Turning to the remaining case $i = k_n$, we first remark that

$$v(R_{k_n-1}(t)) \geq v(\overline{R}e^{L't}) \quad \text{and} \quad v(R_{k_n+1}(t)) \geq v(\overline{R}e^{L't}) \text{ for all } t \geq 0,$$

since (14) is valid for $i = k_n - 1$ and $i = k_n + 1$. As a consequence we have

$$\frac{d}{dt}\big[x_{k_n+1}(t) - x_{k_n}(t)\big] = v(R_{k_n+1}(t))\phi(x_{k_n+1}(t)) - v(R_{k_n-1}(t))\phi(x_{k_n}(t)) \geq v(\overline{R}e^{L't})\big(\phi(x_{k_n+1}(t)) - \phi(x_{k_n}(t))\big)$$

and, using the first order Taylor's expansion of ϕ, it follows that

$$\frac{d}{dt}\big[x_{k_n+1}(t) - x_{k_n}(t)\big] \geq v(\overline{R}e^{L't})\phi'(\tilde{x}(t))\big(x_{k_n+1}(t) - x_{k_n}(t)\big)$$

for some $\tilde{x}(t) \in \big(x_{k_n}(t), x_{k_n+1}(t)\big)$. Finally, applying again Gronwall lemma we get

$$x_{k_n+1}(t) - x_{k_n}(t) \geq \big(\overline{x}_{k_n+1} - \overline{x}_{k_n}\big) e^{\int_0^t v(\overline{R}e^{Ls})\phi'(\tilde{x}(s))ds} \geq \frac{\ell_n}{\overline{R}} e^{-L't}$$

and this concludes the proof for case (P3).

In case (P4), we prove (15) using a different recursive argument with respect to the previous cases. Let us first consider the indices $i \in \{0, \dots, k_n - 1\}$, let us take as base case the index $i = k_n - 1$ and suppose by contradiction the existence of $0 \leq t_1 < t_2$ such that

$$x_{k_n}(t) - x_{k_n-1}(t) \begin{cases} \geq \frac{\ell_n}{R_{\max}} & \text{for } t < t_1, \\ = \frac{\ell_n}{R_{\max}} & \text{for } t = t_1, \\ < \frac{\ell_n}{R_{\max}} & \text{for } t_1 < t \leq t_2. \end{cases}$$

Integrating the ODE in (11), for all $t_1 < t \leq t_2$ it follows that

$$x_{k_n}(t) - x_{k_n-1}(t) = x_{k_n}(t_1) - x_{k_n-1}(t_1) + \int_{t_1}^t \big[v(R_{k_n}(s))\phi(x_{k_n}(s)) - v(R_{k_n-1}(s))\phi(x_{k_n-1}(s))\big]ds,$$

where, due to the contradictory assumption, we have that $R_{k_n-1}(s) > R_{\max}$ and this implies, together with (V*), that $v(R_{k_n-1}(s)) = 0$. Hence we get, for $t_1 < t \leq t_2$, that

$$\int_{t_1}^t \big[v(R_{k_n}(s))\phi(x_{k_n}(s)) - v(R_{k_n-1}(s))\phi(x_{k_n-1}(s))\big]ds = \int_{t_1}^t v(R_{k_n}(s))\phi(x_{k_n}(s))ds \geq 0$$

and so it follows that

$$x_{k_n}(t) - x_{k_n-1}(t) \geq x_{k_n}(t_1) - x_{k_n-1}(t_1) = \frac{\ell_n}{R_{\max}},$$

which is a contradiction. For the remaining indices $i \in \{0, \dots, k_n - 2\}$, we can repeat a recursive argument similar to case (P1), while the validity of (15) for $i \in \{k_n + 1, \dots, n - 1\}$ can be proved in a symmetric way with respect to the previous indices: it is sufficient to take as base case the index $i = k_n + 1$ and then proceeding by contradiction (the details are left to the reader). As a consequence, it remains to show (15) for $i = k_n$ and for this purpose we argue again by contradiction, supposing the existence of $0 \leq t_1 < t_2$ such that

$$x_{k_n+1}(t) - x_{k_n}(t) \begin{cases} \geq \frac{\ell_n}{R_{\max}} & \text{for } t < t_1, \\ = \frac{\ell_n}{R_{\max}} & \text{for } t = t_1, \\ < \frac{\ell_n}{R_{\max}} & \text{for } t_1 < t \leq t_2. \end{cases}$$

On the other hand, for all $t_1 < t \leq t_2$ it holds that

$$x_{k_n+1}(t) - x_{k_n}(t) = x_{k_n+1}(t_1) - x_{k_n}(t_1) + \int_{t_1}^t v(R_{k_n}(s))\big[\phi(x_{k_n+1}(s)) - \phi(x_{k_n}(s))\big]ds,$$

where the contradictory assumption implies that $R_{k_n}(s) > R_{\max}$ and hence, due to (V*), that $v(R_{k_n}(s)) = 0$ for all $t_1 < t \leq t_2$. From this it follows that

$$x_{k_n+1}(t) - x_{k_n}(t) = x_{k_n+1}(t_1) - x_{k_n}(t_1) = \frac{\ell_n}{R_{\max}} \quad \text{for } t_1 < t \leq t_2,$$

which is a contradiction and this concludes the proof for case (P4). □

Remark 2.3. *The exponential rate in* (14) *is not optimal: using the same strategy, we can indeed prove that*

$$x_{i+1}(t) - x_i(t) \geq \frac{\ell_n}{\overline{R}} e^{v_{\max}\phi'_{\inf}t} \quad \textit{if } \phi'_{\inf} := \inf_{x\in\mathbb{R}} \phi'(x) \leq 0$$

and

$$x_{i+1}(t) - x_i(t) \geq \frac{\ell_n}{\overline{R}} e^{v_{\min}\phi'_{\inf}t} \quad \textit{if } \phi'_{\inf} \geq 0, \ \textit{with } v_{\min} := \min_{\eta\in\mathbb{R}^+} v(\eta).$$

Remark 2.4. *The discrete maximum principle proved above allows us to improve the upper bound for the distance of two consecutive particles exposed in remark 2.2 which becomes, for $t \geq 0$ and $i \in \{0, \dots, n-1\}$,*

$$x_{i+1}^n(t) - x_i^n(t) \leq x_n^n(t) - x_0^n(t) \leq \overline{x}_{\max} - \overline{x}_{\min} + ct, \tag{18}$$

where c is the same as in remark 2.2, unless in case (P4) where $c = 0$. More precisely, since $x_{i+1}(t) - x_i(t) \geq 0$ for all $t \geq 0$ and $i \in \{0, \dots, n-1\}$, then it holds

$$x_{i+1}(t) - x_i(t) \leq x_n(t) - x_0(t) \quad \textit{for } t \geq 0, \; i \in \{0, \dots, n-1\},$$

and so we get

$$x_{i+1}(t) - x_i(t) \leq \overline{x}_{\max} - \overline{x}_{\min} + \int_0^t \big(\dot{x}_n(s) - \dot{x}_0(s)\big) ds \quad \textit{for } t \geq 0, \; i \in \{0, \dots, n-1\}.$$

Finally, the fact that $c = 0$ in case (P4) is a consequence of the attractive movement of the two particles at the endpoints of the support.

Remark 2.5. *We can slightly weaken our assumption (P) in three of the cases examined, indeed it is sufficient to take $\phi \in W^{2,\infty}([\overline{x}_{\min}, +\infty))$ in case of forward movement, $\phi \in W^{2,\infty}((-\infty, \overline{x}_{\max}])$ in case of backward movement and $\phi \in W^{2,\infty}([\overline{x}_{\min}, \overline{x}_{\max}])$ in case of attractive movement.*

According to our construction and the previous propositions and remarks, the evolution of the $n+1$ particles $x_i(t)$ is well defined for all $t \geq 0$, hence we can introduce a time-depending piecewise constant density on the interval $[x_0(t), x_n(t)]$. Therefore, we set

$$\rho^n(x,t) := \sum_{i=0}^{n-1} R_i(t) \mathbb{1}_{[x_i(t), x_{i+1}(t))}(x) = \sum_{i=0}^{n-1} \frac{\ell_n}{x_{i+1}(t) - x_i(t)} \mathbb{1}_{[x_i(t), x_{i+1}(t))}(x). \tag{19}$$

3. BV estimate, time continuity and compactness

We first show a uniform control of the total variation of ρ^n which plays a key role in the proof of the convergence of our particle scheme. In the sequel, since we are interested on large values of $n \in \mathbb{N}$, without loss of generality we suppose n sufficiently large such that $2 \leq k_n \leq n-3$.

Proposition 3.1. *Assume (V), (I) and (P) are satisfied and, moreover, assume (V*) is satisfied in case (P4). If one of (P1), (P2), (P3) or (P4) holds, then there exist three positive constants α, β and γ, independent on n, such that*

$$\mathrm{TV}[\rho^n(\cdot,t)] \leq \alpha e^{[\beta t(1+t) + \gamma e^{L't}]} \quad \textit{for all} \;\; t \geq 0. \tag{20}$$

Proof. We start observing that

$$\mathrm{TV}[\rho^n(\cdot,t)] = R_0(t) + R_{n-1}(t) + \sum_{i=0}^{n-2} |R_i(t) - R_{i+1}(t)|.$$

Defining for brevity

$$\begin{aligned}
&\mu_0(t) := 1 + \mathrm{sign}\big(R_0(t) - R_1(t)\big), \\
&\mu_{n-1}(t) := 1 - \mathrm{sign}\big(R_{n-2}(t) - R_{n-1}(t)\big), \\
&\mu_i(t) := \mathrm{sign}\big(R_i(t) - R_{i+1}(t)\big) - \mathrm{sign}\big(R_{i-1}(t) - R_i(t)\big) \quad \text{for } i = 1, \dots, n-2, \\
&\omega_i(t) := \mathrm{sign}\big(R_i(t) - R_{i+1}(t)\big) \qquad\qquad\qquad\qquad\quad\; \text{for } i = 0, \dots, n-1,
\end{aligned}$$

we get

$$\mathrm{TV}[\rho^n(\cdot,t)] = \mu_0(t) R_0(t) + \mu_{n-1}(t) R_{n-1}(t) + \sum_{i=1}^{n-2} \mu_i(t) R_i(t)$$

and

$$\frac{d}{dt} \mathrm{TV}[\rho^n(\cdot,t)] := A(t) + B(t) + C(t),$$

with

$$A(t) := \mu_0(t)\dot{R}_0(t), \quad B(t) := \mu_{n-1}(t)\dot{R}_{n-1}(t) \quad \text{and} \quad C(t) := \sum_{i=1}^{n-2} \mu_i(t)\dot{R}_i(t).$$

Now we need to determine an upper bound for the functions $A(t)$, $B(t)$, $C(t)$ and we should treat as before the four cases separately.

Let us consider the first case (P1). For $A(t)$ and $B(t)$ we have, due to (P) and (14), the following three sub-cases:

$$A(t) \begin{cases} = 0 & \text{if } R_0(t) < R_1(t), \\ \leq L'\overline{R}e^{L't} & \text{if } R_0(t) = R_1(t), \\ \leq 2L'\overline{R}e^{L't} & \text{if } R_0(t) > R_1(t), \end{cases} \quad \text{and} \quad B(t) \begin{cases} = 0 & \text{if } R_0(t) < R_1(t), \\ \leq L'\overline{R}e^{L't} & \text{if } R_0(t) = R_1(t), \\ \leq 2L'\overline{R}e^{L't} & \text{if } R_0(t) > R_1(t). \end{cases}$$

Regarding the sum $C(t)$, we notice that, using (8) and the definition of $\dot{R}_i(t)$, we can split it as

$$C(t) := \sum_{i=1}^{n-2} I_i(t) + \sum_{i=1}^{n-2} \mu_i(t)R_i(t)I\!I_i(t),$$

where

$$I_i(t) := \mu_i(t)\frac{R_i(t)^2}{\ell_n}\phi(x_{i+1}(t))\big[v(R_i(t)) - v(R_{i+1}(t))\big] \quad \text{and} \quad I\!I_i(t) := \frac{R_i(t)}{\ell_n}v(R_i(t))\big[\phi(x_i(t)) - \phi(x_{i+1}(t))\big].$$

Furthermore we remark that each $I_i(t)$ is non-positive, indeed we have that

(1) If $R_{i-1}(t) > R_i(t) > R_{i+1}(t)$ or $R_{i+1}(t) > R_i(t) > R_{i-i}(t)$ or $R_{i+1}(t) = R_i(t) = R_{i-i}(t)$, then $\mu_i(t) = 0$ and so $I_i(t) = 0$;

(2) If $R_i(t) > R_{i+1}(t)$ and $R_i(t) > R_{i-i}(t)$, then $\mu_i(t) = 2$, $v(R_i(t)) \leq v(R_{i+1}(t))$ and hence

$$I_i(t) = 2\frac{R_i(t)^2}{\ell_n}\phi(x_{i+1}(t))\big[v(R_i(t)) - v(R_{i+1}(t))\big] \leq 0;$$

(3) If $R_i(t) < R_{i+1}(t)$ and $R_i(t) < R_{i-i}(t)$, then $\mu_i(t) = -2$, $v(R_{i+1}(t)) \leq v(R_i(t))$ and so

$$I_i(t) = -2\frac{R_i(t)^2}{\ell_n}\phi(x_{i+1}(t))\big[v(R_i(t)) - v(R_{i+1}(t))\big] \leq 0.$$

Therefore it follows that

$$\begin{aligned} C(t) &\leq \sum_{i=1}^{n-2} \omega_i(t)R_i(t)I\!I_i(t) - \sum_{i=0}^{n-3} \omega_i(t)R_{i+1}(t)I\!I_{i+1}(t) \\ &= \sum_{i=0}^{n-2} \omega_i(t)\big[R_i(t)I\!I_i(t) - R_{i+1}(t)I\!I_{i+1}(t)\big] - \omega_0(t)R_0(t)I\!I_0(t) + \omega_{n-2}(t)R_{n-1}(t)I\!I_{n-1}(t), \end{aligned} \tag{21}$$

where, since the assumption (P) implies

$$\big|I\!I_i(t)\big| = \frac{R_i(t)}{\ell_n}v(R_i(t))\big|\phi(x_i(t)) - \phi(x_{i+1}(t))\big| \leq L' \quad \text{for } i \in \{0, \ldots, n-1\},$$

then it holds

$$\big|-\omega_0(t)R_0(t)I\!I_0(t)\big| \leq L'\overline{R}e^{L't} \quad \text{and} \quad \big|\omega_{n-2}(t)R_{n-1}(t)I\!I_{n-1}(t)\big| \leq L'\overline{R}e^{L't}.$$

Moreover, we can rewrite the sum in the right-hand side of (21) as

$$\sum_{i=0}^{n-2} \omega_i(t)\big[R_i(t)I\!I_i(t) - R_{i+1}(t)I\!I_{i+1}(t)\big] = \sum_{i=0}^{n-2} \omega_i(t)R_i(t)\big(I\!I_i(t) - I\!I_{i+1}(t)\big) + \sum_{i=0}^{n-2} \omega_i(t)I\!I_{i+1}(t)\big(R_i(t) - R_{i+1}(t)\big),$$

where we immediately remark that

$$\sum_{i=0}^{n-2} \omega_i(t)I\!I_{i+1}(t)\big(R_i(t) - R_{i+1}(t)\big) = \sum_{i=0}^{n-2} I\!I_{i+1}(t)|R_i(t) - R_{i+1}(t)| \leq L' \, \mathrm{TV}[\rho^n(\cdot, t)].$$

Turning to the remaining sum, from the second order Taylor's expansion of ϕ at $x_{i+1}(t)$, we get

$$\phi(x_i(t)) = \phi(x_{i+1}(t)) - \phi'(x_{i+1}(t))\big(x_{i+1}(t) - x_i(t)\big) + \frac{\phi''(\tilde{x}_{i,i+1}(t))}{2}\big(x_{i+1}(t) - x_i(t)\big)^2$$

and

$$\phi(x_{i+2}(t)) = \phi(x_{i+1}(t)) + \phi'(x_{i+1}(t))\big(x_{i+2}(t) - x_{i+1}(t)\big) + \frac{\phi''(\tilde{y}_{i+1,i+2}(t))}{2}\big(x_{i+2}(t) - x_{i+1}(t)\big)^2$$

for some $\tilde{x}_{i,i+1}(t) \in \big(x_i(t), x_{i+1}(t)\big)$ and $\tilde{y}_{i+1,i+2}(t) \in \big(x_{i+1}(t), x_{i+2}(t)\big)$. Hence we can rewrite the sum as

$$\sum_{i=0}^{n-2} \omega_i(t) R_i(t)\big(II_i(t) - II_{i+1}(t)\big) := D_1(t) + D_2(t) + D_3(t),$$

with

$$D_1(t) := \sum_{i=0}^{n-2} \omega_i(t) R_i(t) \phi'(x_{i+1}(t))\big[v(R_{i+1}(t)) - v(R_i(t))\big],$$

$$D_2(t) := \sum_{i=0}^{n-2} \omega_i(t) \frac{R_i(t)^2}{2\ell_n} \phi''(\tilde{x}_{i,i+1}(t)) v(R_i(t))\big(x_{i+1}(t) - x_i(t)\big)^2$$

and

$$D_3(t) := \sum_{i=0}^{n-2} \omega_i(t) \frac{R_i(t) R_{i+1}(t)}{2\ell_n} \phi''(\tilde{x}_{i+1,i+2}(t)) v(R_{i+1}(t))\big(x_{i+2}(t) - x_{i+1}(t)\big)^2.$$

We first notice that $D_2(t)$ satisfies

$$D_2(t) = \frac{\ell_n}{2} \sum_{i=0}^{n-2} \omega_i(t) \phi''(\tilde{x}_{i,i+1}(t)) v(R_i(t)) \leq v_{\max} ||\phi''||_{L^\infty(\mathbb{R})}.$$

Concerning $D_1(t)$, since the Lagrange mean value theorem implies

$$D_1(t) = -\sum_{i=0}^{n-2} |R_i(t) - R_{i+1}(t)| R_i(t) \phi'(x_{i+1}(t)) v'(\tilde{\rho}_{i,i+1,i+2}(t))$$

for some $\tilde{\rho}_{i,i+1,i+2}(t)$ in the interval $(R_i(t), R_{i+1}(t))$ or $(R_{i+1}(t), R_i(t))$, then from (14) it follows that

$$D_1(t) \leq ||\phi'||_{L^\infty(\mathbb{R})} ||v'||_{L^\infty(\mathbb{R}_+)} \overline{R} e^{L't} \sum_{i=0}^{n-2} |R_i(t) - R_{i+1}(t)| \leq ||\phi'||_{L^\infty(\mathbb{R})} ||v'||_{L^\infty(\mathbb{R}_+)} \overline{R} e^{L't}\, \mathrm{TV}[\rho(\cdot, t)].$$

Turning to the last sum $D_3(t)$, defining for brevity

$$\gamma_i(t) := \frac{\phi''(\tilde{x}_{i,i+1}(t))}{2\ell_n} v(R_i)\big(x_{i+1}(t) - x_i(t)\big)^2 \quad \text{for } i \in \{1, \ldots, n-1\},$$

we can rewrite $D_3(t)$ as

$$D_3(t) = \sum_{i=0}^{n-2} \omega_i(t) R_{i+1}(t)^2 \gamma_{i+1}(t) + \sum_{i=0}^{n-2} \omega_i(t)\big(R_i(t) - R_{i+1}(t)\big) R_{i+1}(t) \gamma_{i+1}(t) := D_3^1(t) + D_3^2(t),$$

where $D_3^1(t)$ satisfies the same inequality as $D_2(t)$, while (13) implies that

$$\begin{aligned}
D_3^2(t) &= \frac{1}{2} \sum_{i=0}^{n-2} |R_i(t) - R_{i+1}(t)| \phi''(\tilde{x}_{i+1,i+2}(t)) v(R_{i+1}(t))\big(x_{i+2}(t) - x_{i+1}(t)\big) \\
&\leq ||\phi''||_{L^\infty(\mathbb{R})} v_{\max}\big(\overline{x}_{\max} - \overline{x}_{\min} + 2Lt\big) \sum_{i=0}^{n-2} \big|R_i(t) - R_{i+1}(t)\big| \\
&\leq ||\phi''||_{L^\infty(\mathbb{R})} v_{\max}\big(\overline{x}_{\max} - \overline{x}_{\min} + 2Lt\big)\, \mathrm{TV}[\rho(\cdot, t)].
\end{aligned}$$

Putting together all the previous estimates, we finally get the following differential inequality

$$\frac{d}{dt}\,\mathrm{TV}[\rho^n(\cdot,t)] \le c_1 + c_2 e^{L't} + \left(c_3 + c_4 t + c_5 e^{L't}\right)\mathrm{TV}[\rho^n(\cdot,t)] \quad \text{for all } t \ge 0, \tag{22}$$

where c_i, $i = 1,\dots,5$, are five positive constants depending only on ϕ, v and $\overline{\rho}$.

In case (P2) we can proceed in a symmetric way and we get also here (22) (the details are omitted).

Turning to case (P3), we first remark that the bounds for $A(t)$ and $B(t)$ are the same already pointed out in cases (P2) and (P1) respectively. Regarding $C(t)$, we split this sum as

$$C(t) = \sum_{i=1}^{k_n-1} \mu_i(t)\dot{R}_i(t) + \sum_{i=k_n+1}^{n-2} \mu_i(t)\dot{R}_i(t) + \mu_{k_n}(t)\dot{R}_{k_n}(t) := C_1(t) + C_2(t) + C_3(t), \tag{23}$$

where $C_1(t)$ and $C_2(t)$ can be estimated, rearranging the indices properly, using the same reasoning as in cases (P2) and (P1) respectively, while for $C_3(t)$ we need to consider three different sub-cases:

(1) If $R_{k_n-1}(t) > R_{k_n}(t) > R_{k_n+1}(t)$ or $R_{k_n+1}(t) > R_{k_n}(t) > R_{k_n-1}(t)$ or $R_{k_n+1}(t) = R_{k_n}(t) = R_{k_n-1}(t)$, then $\mu_{k_n}(t) = 0$ and hence $C_3(t) = 0$;

(2) If $R_{k_n}(t) > R_{k_n-1}(t)$ and $R_{k_n}(t) > R_{k_n+1}(t)$, then $\mu_{k_n}(t) = 2$ and so, since $\phi(x_{k_n}(t)) \le 0$ and $\phi(x_{k_n+1}(t)) \ge 0$, it holds

$$C_3(t) = 2\frac{R_{k_n}(t)^2}{\ell_n}\left[v(R_{k_n-1}(t))\phi(x_{k_n}(t)) - v(R_{k_n+1}(t))\phi(x_{k_n+1}(t))\right] \le 0;$$

(3) If $R_{k_n}(t) < R_{k_n-1}(t)$ and $R_{k_n}(t) < R_{k_n+1}(t)$, then it follows $\mu_{k_n}(t) = -2$, $v(R_{k_n-1}(t)) \le v(R_{k_n}(t))$, $v(R_{k_n+1}(t)) \le v(R_{k_n}(t))$ and therefore, since $\phi(x_{k_n}(t)) \le 0$ and $\phi(x_{k_n+1}(t)) \ge 0$, we get

$$C_3(t) = 2\frac{R_{k_n}(t)^2}{\ell_n}\left[v(R_{k_n+1}(t))\phi(x_{k_n+1}(t)) - v(R_{k_n-1}(t))\phi(x_{k_n}(t))\right] \le 2L'\overline{R}e^{L't}.$$

As a consequence, we hence have that (22) is valid also in case (P3).

In the last case (P4), we can estimate $A(t)$ and $B(t)$ in the same way as in cases (P1) and (P2) respectively, substituting $\overline{R}e^{L't}$ with $R_{\max}$ whenever it appears. Concerning $C(t)$, we can rewrite this sum using the same splitting (23) seen in case (P3), where $C_3(t)$ now satisfies $C_3(t) \le 2L'R_{\max}$. For $C_1(t)$ and $C_2(t)$, rearranging the indices properly and substituting again $\overline{R}e^{L't}$ with $R_{\max}$, we can follow the same reasoning as in cases (P1) and (P2) respectively: in this way, we obtain all the previous estimates with a slight difference only on the term $D_3^2(t)$. For this sum it holds, due to remark 2.4, that

$$\begin{aligned}
D_3^2(t) &= \frac{1}{2}\sum_{i=0}^{n-2} |R_i(t) - R_{i+1}(t)|\phi''(\tilde{x}_{i+1,i+2}(t))v(R_{i+1}(t))\big(x_{i+2}(t) - x_{i+1}(t)\big) \\
&\le ||\phi''||_{L^\infty(\mathbb{R})} v_{\max}(\overline{x}_{\max} - \overline{x}_{\min}) \sum_{i=0}^{n-2} |R_i(t) - R_{i+1}(t)| \\
&\le ||\phi''||_{L^\infty(\mathbb{R})} v_{\max}(\overline{x}_{\max} - \overline{x}_{\min})\, \mathrm{TV}[\rho(\cdot,t)]
\end{aligned}$$

and therefore we have that (22) is valid in this case with $c_4 = c_5 = 0$.

Applying now Gronwall lemma to (22), we get

$$\mathrm{TV}[\rho^n(\cdot,t)] \le \mathrm{TV}[\rho^n(\cdot,0)]e^{\int_0^t (c_3 + c_4\tau + c_5 e^{L'\tau})d\tau} + \int_0^t (c_1 + c_2 e^{-L's})e^{\int_s^t (c_3 + c_4\tau + c_5 e^{L'\tau})d\tau} ds, \tag{24}$$

where, since (7) and the mean value theorem imply

$$R_i(0) = \frac{\ell_n}{\overline{x}_{i+1} - \overline{x}_i} = \fint_{\overline{x}_i}^{\overline{x}_{i+1}} \overline{\rho}(x)dx = \overline{\rho}(z_i) \quad \text{for some } z_i \in (\overline{x}_i, \overline{x}_{i+1}),$$

then the total variation of $\rho^n(\cdot,0)$ satisfies

$$\mathrm{TV}[\rho^n(\cdot,0)] = R_0(0) + R_{n-1}(0) + \sum_{i=0}^{n-2} |R_i(0) - R_{i+1}(0)| = \overline{\rho}(z_0) + \overline{\rho}(z_{n-1}) + \sum_{i=0}^{n-2} |\overline{\rho}(z_i) - \overline{\rho}(z_{i+1})| \le \mathrm{TV}[\overline{\rho}].$$

Moreover, after a simple calculation it follows that

$$e^{\int_0^t (c_3+c_4\tau+c_5 e^{L'\tau})d\tau} \le e^{[\tilde c_1 t(1+t)+\tilde c_2 e^{L't}]}$$

and

$$\int_0^t (c_1 + c_2 e^{L's}) e^{\int_s^t (c_3+c_4\tau+c_5 e^{L'\tau})d\tau} ds \le \tilde c_3 e^{[\tilde c_4 t(1+t)+\tilde c_5 e^{L't}]},$$

therefore, combining together the previous three estimates in (24), we finally get (20). □

Remark 3.1. *The previous proposition gives us the needed compactness of the sequence $\{\rho^n\}_{n\in\mathbb{N}}$ with respect to space. Concerning the time variable, we are not able to obtain a L^1 uniform continuity estimate which would provide a sufficient control on the time oscillation. Nevertheless, we are going to prove a uniform time continuity estimate with respect to the 1-Wasserstein distance which is sufficient to get the required strong L^1 compactness with respect to space and time.*

We now recall the main properties on the one-dimensional 1-Wasserstein metric. Let μ be a probability measure on $\mathbb{R}$ with finite first moment, and let us denote with X_μ the pseudo-inverse of its cumulative distribution function, that is

$$X_\mu(z) := \inf\{x \in \mathbb{R} : \mu((-\infty, x]) > z\} \quad \text{for } z \in [0,1].$$

We first notice that $X_\mu \in L^1([0,1])$ (see [29]). Moreover, the 1-Wasserstein distance between two probability measures μ and ν on $\mathbb{R}$ can be defined as the L^1 distance of X_μ and X_ν, that is

$$W_1(\mu,\nu) := ||X_\mu - X_\nu||_{L^1([0,1])}. \tag{25}$$

In particular, starting from the definition (19) of ρ^n, we can explicitly compute the pseudo-inverse function X_{ρ^n} and we get

$$X_{\rho^n(\cdot,t)}(z) = \sum_{i=0}^{n-1} \big[x_i(t) + (z - i\ell_n) R_i(t)^{-1}\big] \mathbb{1}_{[i\ell_n,(i+1)\ell_n)}(z). \tag{26}$$

After this short preamble, we can prove the uniform time continuity estimate with respect to the 1-Wasserstein distance, which is stated in the following

Proposition 3.2. *Assume (V), (I) and (P) are satisfied and, moreover, assume (V*) is satisfied in case (P4). If one of (P1), (P2), (P3) or (P4) holds, then there exists a constant c, dependent only on v and ϕ, such that*

$$W_1\big(\rho^n(\cdot,t), \rho^n(\cdot,s)\big) \le c|t-s| \quad \textit{for all } t,s > 0. \tag{27}$$

Proof. Let $0 < s < t$ fixed. From (25), (26) and the triangular inequality, it follows that

$$W_1\big(\rho^n(\cdot,t),\rho^n(\cdot,s)\big) = \sum_{i=0}^{n-1} \int_{i\ell_n}^{(i+1)\ell_n} \big|x_i(t) - x_i(s) + (z - i\ell_n)\big(R_i(t)^{-1} - R_i(s)^{-1}\big)\big| dz \le A(s,t) + B(s,t),$$

with

$$A(s,t) := \sum_{i=0}^{n-1} \int_{i\ell_n}^{(i+1)\ell_n} |x_i(t) - x_i(s)| dz \quad \text{and} \quad B(s,t) := \sum_{i=0}^{n-1} \int_{i\ell_n}^{(i+1)\ell_n} (z - i\ell_n)\big|R_i(t)^{-1} - R_i(s)^{-1}\big| dz.$$

Now we should estimate $A(s,t)$ and $B(s,t)$ separately. Due to (12), we have that

$$A(s,t) = \ell_n \sum_{i=0}^{n-1} |x_i(t) - x_i(s)| = \ell_n \sum_{i=0}^{n-1} \left| \int_s^t \dot x_i(\tau) d\tau \right| \le \ell_n \sum_{i=0}^{n-1} \int_s^t |\dot x_i(\tau)| d\tau \le L(t-s),$$

while, turning to $B(s,t)$, we first notice that

$$B(s,t) = \sum_{i=0}^{n-1} \big|R_i(t)^{-1} - R_i(s)^{-1}\big| \int_{i\ell_n}^{(i+1)\ell_n} (z - i\ell_n) dz = \frac{\ell_n^2}{2} \sum_{i=0}^{n-1} \left| \int_s^t \frac{d}{d\tau} R_i(\tau)^{-1} d\tau \right| \le \frac{\ell_n^2}{2} \sum_{i=0}^{n-1} \int_s^t \frac{|\dot R_i(\tau)|}{R_i(\tau)^2} d\tau.$$

For clarity we should treat the four cases separately from now on. In case (P1), substituting the definition of $\dot{R}_i(t)$ we get

$$\begin{aligned}B(s,t) &\le \frac{\ell_n}{2}\sum_{i=0}^{n-2}\int_s^t \big|v(R_{i+1}(\tau))\phi(x_{i+1}(\tau)) - v(R_i(\tau))\phi(x_i(\tau))\big| d\tau \\ &+ \frac{\ell_n}{2}\int_s^t \big|v_{\max}\phi(x_n(\tau)) - v(R_{n-1}(\tau))\phi(x_{n-1}(\tau))\big| d\tau\end{aligned}$$

where, for all $\tau \ge 0$ and $i \in \{0, \dots, n-2\}$, it holds

$$\big|v(R_{i+1}(\tau))\phi(x_{i+1}(\tau)) - v(R_i(\tau))\phi(x_i(\tau))\big| \le \big|v(R_{i+1}(\tau))\phi(x_{i+1}(\tau))\big| \le L$$

and

$$\big|v_{\max}\phi(x_n(\tau)) - v(R_{n-1}(\tau))\phi(x_{n-1}(\tau))\big| \le L.$$

This implies that $B(s,t) \le \frac{L}{2}(t-s)$ and hence that $W_1\big(\rho^n(\cdot,t), \rho^n(\cdot,s)\big) \le \frac{3}{2}L(t-s)$, which concludes the proof of (27) in case (P1), since the calculation is still valid interchanging s and t.

In case (P2) we can reason in a symmetric way with respect to the previous case and we get the same estimate for $B(s,t)$. The details are left to the reader.

Turning to case (P3), we remark that $\dot{R}_i(\tau)$ has the same expression of cases (P2) and (P1) for $i \in \{0, \dots, k_n - 1\}$ and $i \in \{k_n + 1, \dots, n-1\}$ respectively, while

$$\dot{R}_{k_n}(\tau) = -\frac{R_{k_n}(\tau)^2}{\ell_n}\big[v(R_{k_n+1}(\tau))\phi(x_{k_n+1}(\tau)) - v(R_{k_n-1}(\tau))\phi(x_{k_n}(\tau))\big].$$

Since

$$\big|v(R_{k_n+1}(\tau))\phi(x_{k_n+1}(\tau)) - v(R_{k_n-1}(\tau))\phi(x_{k_n}(\tau))\big| \le 2L,$$

then it follows that $B(s,t) \le L(t-s)$ and hence also in this case (27) holds.

In the remaining case (P4), we have that $\dot{R}_i(\tau)$ has the same expression of cases (P1) and (P2) for $i \in \{0, \dots, k_n - 1\}$ and $i \in \{k_n + 1, \dots, n-1\}$ respectively, while

$$\dot{R}_{k_n}(\tau) = -\frac{R_{k_n}(\tau)^2}{\ell_n} v(R_{k_n}(\tau))\big[\phi(x_{k_n+1}(\tau)) - \phi(x_{k_n}(\tau))\big].$$

Since

$$v(R_{k_n}(\tau))\big|\phi(x_{k_n+1}(\tau)) - \phi(x_{k_n}(\tau))\big| \le 2L,$$

then $B(s,t)$ satisfies the same inequality seen in case (P3) and hence the validity of (27) is proved also in case (P4). □

Before passing to the main result of this paper, we recall a generalised version of Aubin-Lions lemma (see [25],[12], [9]) which has a key role in the sequel.

Theorem 3.1. *Let $T > 0$ fixed, $I \subset \mathbb{R}$ a bounded open interval (possibly depending on T), $\{\mu^n\}_{n\in\mathbb{N}}$ a sequence in $L^\infty((0,T); L^1(\mathbb{R}))$ such that $\mu^n(\cdot,t) \ge 0$ and $||\mu^n(\cdot,t)||_{L^1(\mathbb{R})} = 1$ for all $n \in \mathbb{N}$ and $t \in [0,T]$. If*

(A) *$\mathrm{supp}[\mu^n(\cdot,t)] \subseteq I$ for all $n \in \mathbb{N}$ and $t \in [0,T]$,*

(B) $\sup_{n\in\mathbb{N}} \int_0^T \Big[||\mu^n(\cdot,t)||_{L^1(I)} + \mathrm{TV}[\mu^n(\cdot,t); I]\Big] dt < \infty$,

(C) *There exists a constant c independent on n such that $W_1\big(\mu^n(\cdot,t), \mu^n(\cdot,s)\big) \le c|t-s|$ for all $s,t \in (0,T)$,*

then $\{\mu^n\}_{n\in\mathbb{N}}$ is strongly relatively compact in $L^1\big(\mathbb{R}\times[0,T]\big)$.

Furthermore, let us report an adapted version of the L^1 contraction property proved by Karlsen and Risebro in [18], which will be crucial in the proof of the uniqueness of the entropy solution to (6).

Theorem 3.2. *Let $T > 0$ fixed arbitrarily, let f a locally Lipschitz function on $\mathbb{R}$, let $\psi \in W^{1,1}_{loc}(\mathbb{R}) \cap C(\mathbb{R})$ such that $\psi, \psi' \in L^\infty(\mathbb{R})$ and consider the problem*

$$\begin{cases} w_t + \big(f(w)\psi(x)\big)_x = 0, & x \in \mathbb{R},\ t \in (0,T), \\ w(x,0) = \overline{w}(x), & x \in \mathbb{R}. \end{cases} \tag{28}$$

If $u, v \in L^\infty((0,T); BV(\mathbb{R}))$ are two entropy solutions of (28) *with respective initial datum u_0 and v_0, both in $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \cap BV(\mathbb{R})$, then, for almost every $t \in (0,T)$, it holds*

$$||u(\cdot,t) - v(\cdot,t)||_{L^1(\mathbb{R})} \leq ||u_0 - v_0||_{L^1(\mathbb{R})}.$$

In particular, this implies the existence of at most one entropy solution of (28).

4. Proof of the main results

Now we are ready to prove the main result of the paper. For clarity, in the sequel we drop the time dependence whenever it is clear from the context.

Theorem 4.1. *Let $T > 0$ fixed arbitrarily. Assume (V), (I) and (P) are satisfied and, moreover, assume (V*) is satisfied in case (P4). If one of (P1), (P2), (P3) or (P4) holds, then the approximated density $\{\rho^n\}_{n\in\mathbb{N}}$ defined in* (19) *converges, up to a subsequence, almost everywhere and in L^1 on $\mathbb{R} \times [0,T]$ to the unique entropy solution to the Cauchy problem* (6).

Proof. We first show that $\{\rho^n\}_{n\in\mathbb{N}}$ converges, up to a subsequence, almost everywhere and in L^1 on $\mathbb{R}\times[0,T]$. We notice that the support of $\rho^n(\cdot,t)$ is contained, for every $n \in \mathbb{N}$ and $t \in [0,T]$, in the closed interval

$$J := [a,b] = \begin{cases} \big[\overline{x}_{\min}, \overline{x}_{\max} + LT\big] & \text{in case (P1)}, \\ \big[\overline{x}_{\min} - LT, \overline{x}_{\max}\big] & \text{in case (P2)}, \\ \big[\overline{x}_{\min} - LT, \overline{x}_{\max} + LT\big] & \text{in case (P3)}, \\ \big[\overline{x}_{\min}, \overline{x}_{\max}\big] & \text{in case (P4)}. \end{cases}$$

Therefore, taking as I any open interval of the type $(a-c, b+d)$ with c and d arbitrary positive constants, we can apply theorem 3.1, indeed assumption (A) is valid due to the above construction, while assumptions (B) and (C) are a direct consequence of propositions 3.1 and 3.2 respectively. As a result, it follows that $\{\rho^n\}_{n\in\mathbb{N}}$ converges, up to a subsequence that we still denote in the sequel with $\{\rho^n\}_{n\in\mathbb{N}}$, almost everywhere and in L^1 on $\mathbb{R} \times [0,T]$ to a certain function ρ.

Now we show that $\{\rho^n\}_{n\in\mathbb{N}}$ satisfies, for every $k \geq 0$ and every non-negative $\varphi \in C_c^\infty(\mathbb{R} \times (0,T))$,

$$\begin{aligned} \liminf_{n\to+\infty} \int_0^T \int_{\mathbb{R}} \Big[&|\rho^n(x,t) - k|\varphi_t(x,t) + \operatorname{sign}(\rho^n(x,t) - k)\big(f(\rho^n(x,t)) - f(k)\big)\phi(x)\varphi_x(x,t) \\ &- \operatorname{sign}(\rho^n(x,t) - k) f(k)\phi'(x)\varphi(x,t)\Big] dxdt \geq 0, \end{aligned} \tag{29}$$

where we denote $f(\eta) := \eta v(\eta)$. Let us omit from now on also the x dependence whenever it is clear from the context. We first remark that, since $\operatorname{supp}[\varphi]$ is compact in $\mathbb{R} \times (0,T)$, then it holds

$$\int_0^T \int_{\mathbb{R}} \big[|\rho^n - k|\varphi_t + \operatorname{sign}(\rho^n - k)(f(\rho) - f(k))\phi\varphi_x - \operatorname{sign}(\rho^n - k)f(k)\phi'\varphi\big] dxdt := A + B + C,$$

where

$$A := \sum_{i=0}^{n-1} \int_0^T \int_{x_i}^{x_{i+1}} \big[|R_i - k|\varphi_t + \operatorname{sign}(R_i - k)(f(R_i) - f(k))\phi\varphi_x - \operatorname{sign}(R_i - k)f(k)\phi'\varphi\big] dxdt,$$

$$B := \int_0^T \int_{-\infty}^{x_0} \big[k\varphi_t + f(k)\phi\varphi_x + f(k)\phi'\varphi\big] dxdt \quad \text{and} \quad C := \int_0^T \int_{x_n}^{+\infty} \big[k\varphi_t + f(k)\phi\varphi_x + f(k)\phi'\varphi\big] dxdt.$$

Recalling that $\displaystyle\frac{d}{dt}\int_{\alpha(t)}^{\beta(t)} f(x,t)dx = \int_{\alpha(t)}^{\beta(t)} f_t(x,t)dx + f(\beta(t),t)\dot{\beta}(t) - f(\alpha(t),t)\dot{\alpha}(t)$, it follows that

$$B = k\int_0^T \left(\int_{-\infty}^{x_0} \varphi_t dx\right) dt + kv(k) \int_0^T \int_{-\infty}^{x_0} (\phi\varphi)_x dxdt = k\int_0^T \big(v(k)\phi(x_0) - \dot{x}_0\big)\varphi(x_0)dt$$

and analogously that

$$C = k\int_0^T \left(\int_{x_n}^{+\infty} \varphi_t dx\right) dt + kv(k) \int_0^T \int_{x_n}^{+\infty} (\phi\varphi)_x dxdt = k\int_0^T \big(\dot{x}_n - v(k)\phi(x_n)\big)\varphi(x_n)dt,$$

while we can rewrite A as

$$A = \sum_{i=0}^{n-1} \int_0^T \int_{x_i}^{x_{i+1}} \big[|R_i - k|\varphi_t + \mathrm{sign}(R_i - k) f(R_i)\phi\varphi_x - \mathrm{sign}(R_i - k) f(k)(\phi\varphi)_x\big] dxdt := A_1 + A_2 + A_3,$$

where

$$A_1 := \sum_{i=0}^{n-1} \int_0^T |R_i - k| \bigg(\int_{x_i}^{x_{i+1}} \varphi_t dx \bigg) dt, \quad A_2 := \sum_{i=0}^{n-1} \int_0^T \mathrm{sign}(R_i - k) f(R_i) \bigg(\int_{x_i}^{x_{i+1}} \phi\varphi_x dx \bigg) dt$$

and

$$A_3 := -\sum_{i=0}^{n-1} \int_0^T \mathrm{sign}(R_i - k) f(k) \big[\phi(x_{i+1})\varphi(x_{i+1}) - \phi(x_i)\varphi(x_i)\big] dxdt.$$

Integrating by parts and since $\mathrm{supp}[\varphi(x,\cdot)] \subseteq (0,T)$ for every $x \in \mathbb{R}$, we get that A_1 satisfies

$$\begin{aligned}
A_1 =& \sum_{i=0}^{n-1} \int_0^T |R_i - k| \bigg(\frac{d}{dt} \int_{x_i}^{x_{i+1}} \varphi dx \bigg) dt - \sum_{i=0}^{n-1} \int_0^T |R_i - k| \dot{x}_{i+1} \varphi(x_{i+1}) dt + \sum_{i=0}^{n-1} \int_0^T |R_i - k| \dot{x}_i \varphi(x_i) dt \\
=& - \sum_{i=0}^{n-1} \int_0^T \mathrm{sign}(R_i - k)(R_i - k) \dot{x}_{i+1} \varphi(x_{i+1}) dt - \sum_{i=0}^{n-1} \int_0^T \mathrm{sign}(R_i - k) \dot{R}_i \bigg(\int_{x_i}^{x_{i+1}} \varphi dx \bigg) dt \\
&+ \sum_{i=0}^{n-1} \int_0^T \mathrm{sign}(R_i - k)(R_i - k) \dot{x}_i \varphi(x_i) dt,
\end{aligned}$$

while A_2 has a different expression in the four cases, since we need to approximate the function ϕ differently according to its sign. In particular we have:

(P1) For case (P1), using the first order Taylor's expansion of ϕ at x_i in the interval (x_i, x_{i+1}), which is given, for all $x \in (x_i, x_{i+1})$, by

$$\phi(x) = \phi(x_i) + \phi'(\tilde{x}_{i,i+1})(x - x_i) \quad \text{for some} \quad \tilde{x}_{i,i+1} \in (x_i, x_{i+1}), \tag{30}$$

we can rewrite A_2 as

$$\begin{aligned}
A_2 =& \sum_{i=0}^{n-1} \int_0^T \mathrm{sign}(R_i - k) R_i v(R_i) \phi(x_i) \big(\varphi(x_{i+1}) - \varphi(x_i) \big) dt \\
&+ \sum_{i=0}^{n-1} \int_0^T \mathrm{sign}(R_i - k) R_i v(R_i) \bigg(\int_{x_i}^{x_{i+1}} \phi'(\tilde{x}_{i,i+1})(x - x_i) \varphi_x dx \bigg) dt.
\end{aligned}$$

(P2) In case (P2), we use instead the first order Taylor's expansions of ϕ at x_{i+i}, that is

$$\phi(x) = \phi(x_{i+1}) + \phi'(\tilde{y}_{i,i+1})(x - x_{i+1}) \quad \text{for some } \tilde{y}_{i,i+1} \in (x_i, x_{i+1}) \tag{31}$$

and in this way we get that

$$\begin{aligned}
A_2 =& \sum_{i=0}^{n-1} \int_0^T \mathrm{sign}(R_i - k) R_i v(R_i) \phi(x_{i+1}) \big(\varphi(x_{i+1}) - \varphi(x_i) \big) dt \\
&+ \sum_{i=0}^{n-1} \int_0^T \mathrm{sign}(R_i - k) R_i v(R_i) \bigg(\int_{x_i}^{x_{i+1}} \phi'(\tilde{y}_{i,i+1})(x - x_{i+1}) \varphi_x dx \bigg) dt.
\end{aligned}$$

(P3) For case (P3), we use the Taylor's expansions (30) and (31) of ϕ respectively for $i \in \{k_n+1, \dots, n-1\}$ and $i \in \{0, \dots, k_n - 1\}$, in order to have that

$$
\begin{aligned}
A_2 =& \sum_{i=0}^{k_n-1} \int_0^T \operatorname{sign}(R_i - k) R_i v(R_i) \phi(x_{i+1}) \big(\varphi(x_{i+1}) - \varphi(x_i)\big) dt \\
&+ \sum_{i=k_n+1}^{n-1} \int_0^T \operatorname{sign}(R_i - k) R_i v(R_i) \phi(x_i) \big(\varphi(x_{i+1}) - \varphi(x_i)\big) dt \\
&+ \int_0^T \operatorname{sign}(R_{k_n} - k) R_{k_n} v(R_{k_n}) \bigg(\int_{x_{k_n}}^{x_{k_n+1}} \phi \varphi_x dx\bigg) dt \\
&+ \sum_{i=0}^{k_n-1} \int_0^T \operatorname{sign}(R_i - k) R_i v(R_i) \bigg(\int_{x_i}^{x_{i+1}} \phi'(\tilde{y}_{i,i+1})(x - x_{i+1}) \varphi_x dx\bigg) dt \\
&+ \sum_{i=k_n+1}^{n-1} \int_0^T \operatorname{sign}(R_i - k) R_i v(R_i) \bigg(\int_{x_i}^{x_{i+1}} \phi'(\tilde{x}_{i,i+1})(x - x_i) \varphi_x dx\bigg) dt.
\end{aligned}
$$

(P4) In case (P4), we combine again cases (P1) and (P2), namely we use the Taylor's expansions (30) and (31) of ϕ respectively for $i \in \{0, \dots, k_n - 1\}$ and $i \in \{k_n + 1, \dots, n - 1\}$. Then it follows that

$$
\begin{aligned}
A_2 =& \sum_{i=0}^{k_n-1} \int_0^T \operatorname{sign}(R_i - k) R_i v(R_i) \phi(x_i) \big(\varphi(x_{i+1}) - \varphi(x_i)\big) dt \\
&+ \sum_{i=k_n+1}^{n-1} \int_0^T \operatorname{sign}(R_i - k) R_i v(R_i) \phi(x_{i+1}) \big(\varphi(x_{i+1}) - \varphi(x_i)\big) dt \\
&+ \int_0^T \operatorname{sign}(R_{k_n} - k) R_{k_n} v(R_{k_n}) \bigg(\int_{x_{k_n}}^{x_{k_n+1}} \phi \varphi_x dx\bigg) dt \\
&+ \sum_{i=0}^{k_n-1} \int_0^T \operatorname{sign}(R_i - k) R_i v(R_i) \bigg(\int_{x_i}^{x_{i+1}} \phi'(\tilde{x}_{i,i+1})(x - x_i) \varphi_x dx\bigg) dt \\
&+ \sum_{i=k_n+1}^{n-1} \int_0^T \operatorname{sign}(R_i - k) R_i v(R_i) \bigg(\int_{x_i}^{x_{i+1}} \phi'(\tilde{y}_{i,i+1})(x - x_{i+1}) \varphi_x dx\bigg) dt.
\end{aligned}
$$

From now on, let us consider the four cases separately. Putting together all the previous identities, we get that in case (P1) it holds

$$
\begin{aligned}
&\int_0^T \int_{\mathbb{R}} \big[|\rho^n - k| \varphi_t + \operatorname{sign}(\rho^n - k) \big(f(\rho) - f(k)\big) \phi \varphi_x - \operatorname{sign}(\rho^n - k) f(k) \phi' \varphi\big] dx dt \\
=& k \int_0^T \big(v(k) - v(R_0)\big) \phi(x_0) \varphi(x_0) dt + k \int_0^T \big(v_{\max} - v(k)\big) \phi(x_n) \varphi(x_n) dt \\
&+ \sum_{i=0}^{n-1} \int_0^T \operatorname{sign}(R_i - k) \bigg[-\dot{R}_i \bigg(\int_{x_i}^{x_{i+1}} \varphi dx\bigg) - k \big[\dot{x}_i - v(k) \phi(x_i)\big] \varphi(x_i) \\
&\qquad\qquad - \big[R_i(\dot{x}_{i+1} - \dot{x}_i) - k\big(\dot{x}_{i+1} - v(k)\phi(x_{i+1})\big)\big] \varphi(x_{i+1})\bigg] dt \\
&+ \sum_{i=0}^{n-1} \int_0^T \operatorname{sign}(R_i - k) R_i v(R_i) \bigg(\int_{x_i}^{x_{i+1}} \phi'(\tilde{x}_{i,i+1})(x - x_i) \varphi_x dx\bigg) dt,
\end{aligned}
$$

where

$$
-R_i \big(\dot{x}_{i+1} - \dot{x}_i\big) \varphi(x_{i+1}) = -R_i \frac{\dot{x}_{i+1} - \dot{x}_i}{x_{i+1} - x_i} \bigg(\int_{x_i}^{x_{i+1}} \varphi(x_{i+1}) dx\bigg) = \dot{R}_i \bigg(\int_{x_i}^{x_{i+1}} \varphi(x_{i+1}) dx\bigg).
$$

Moreover, using the definition of $\dot{R}_i(t)$, the previous identity becomes

$$\int_0^T \int_{\mathbb{R}} \big[|\rho^n - k|\varphi_t + \mathrm{sign}(\rho^n - k)\big(f(\rho) - f(k)\big)\phi\varphi_x - \mathrm{sign}(\rho^n - k)f(k)\phi'\varphi\big]dxdt := kD + E_1 + E_2,$$

where

$$\begin{aligned} D := & \int_0^T \big(v(k) - v(R_0)\big)\phi(x_0)\varphi(x_0)dt + \int_0^T \big(v_{\max} - v(k)\big)\phi(x_n)\varphi(x_n)dt \\ & + \sum_{i=0}^{n-1}\int_0^T \mathrm{sign}(R_i - k)\Big[\big[\dot{x}_{i+1} - v(k)\phi(x_{i+1})\big]\varphi(x_{i+1}) - \big[\dot{x}_i - v(k)\phi(x_i)\big]\varphi(x_i)\Big]dt, \end{aligned}$$

$$E_1 := \sum_{i=0}^{n-1}\int_0^T \mathrm{sign}(R_i - k)R_i v(R_i)\left(\int_{x_i}^{x_{i+1}} \phi'(\tilde{x}_{i,i+1})(x - x_i)\varphi_x dx\right)dt$$

and

$$E_2 := \sum_{i=0}^{n-1}\int_0^T \mathrm{sign}(R_i - k)\frac{R_i^2}{\ell_n}(\dot{x}_{i+1} - \dot{x}_i)\left(\int_{x_i}^{x_{i+1}} \big(\varphi(x) - \varphi(x_{i+1})\big)dx\right)dt.$$

Regarding E_1, since $\varphi(x,\cdot)$ is a Lipschitz function for every $x \in \mathbb{R}$ and due to (18), we have that

$$E_1 \geq -\ell_n L' \sum_{i=0}^{n-1}\int_0^T \big|\varphi(x_{i+1}) - \varphi(x_i)\big|dt \geq -\ell_n L'\mathscr{L}_\varphi \int_0^T (x_n - x_0)dt \geq -\ell_n L'\mathscr{L}_\varphi T\Big[\overline{x}_{\max} - \overline{x}_{\min} + 2LT\Big], \tag{32}$$

while, using again the Lipschitz condition of φ and due to (18) and (20), we get that E_2 satisfies

$$\begin{aligned} E_2 \geq & -\sum_{i=0}^{n-2}\int_0^T \frac{R_i^2}{\ell_n}\big|v(R_i)\phi(x_i) - v(R_{i+1})\phi(x_{i+1})\big|\left(\int_{x_i}^{x_{i+1}} \big|\varphi(x) - \varphi(x_{i+1})\big|dx\right)dt \\ & - \int_0^T \frac{R_{n-1}^2}{\ell_n}\big|v_{\max}\phi(x_n) - v(R_{n-1})\phi(x_{n-1})\big|\left(\int_{x_{n-1}}^{x_n} \big|\varphi(x) - \varphi(x_n)\big|dx\right)dt \\ \geq & -\ell_n\mathscr{L}_\varphi\left[\sum_{i=0}^{n-2}\int_0^T \big|v(R_i)\phi(x_i) - v(R_{i+1})\phi(x_{i+1})\big|dt + \int_0^T \big|v_{\max}\phi(x_n) - v(R_{n-1})\phi(x_{n-1})\big|dt\right] \\ \geq & -\ell_n\mathscr{L}_\varphi\left[\sum_{i=0}^{n-2}\int_0^T v(R_i)\big|\phi(x_i) - \phi(x_{i+1})\big|dt + \sum_{i=0}^{n-2}\int_0^T \phi(x_{i+1})\big|v(R_i) - v(R_{i+1})\big|dt + LT\right] \\ \geq & -\ell_n\mathscr{L}_\varphi\left[L'\int_0^T (x_n - x_0)dt + \phi_{\max}||v'||_{L^\infty(\mathbb{R}_+)}\int_0^T\sum_{i=0}^{n-2}|R_i - R_{i+1}|dt + LT\right] \\ \geq & -\ell_n\mathscr{L}_\varphi\left[L'T\big[\overline{x}_{\max} - \overline{x}_{\min} + 2LT\big] + \phi_{\max}||v'||_{L^\infty(\mathbb{R}_+)}\alpha e^{[\beta T(1+T) + \gamma e^{L'T}]}T + LT\right]. \end{aligned} \tag{33}$$

Putting together all the previous estimates, we hence get that $E_1 + E_2 \geq -c\,\ell_n$ for some constant c independent on n and so, since the right-hand side of this inequality tends to zero as $n \to +\infty$, to conclude the proof it is sufficient to show that D is non-negative. From a direct calculation we remark that

$$\begin{aligned} & \sum_{i=0}^{n-1}\mathrm{sign}(R_i - k)\Big[\big[\dot{x}_{i+1} - v(k)\phi(x_{i+1})\big]\varphi(x_{i+1}) - \big[\dot{x}_i - v(k)\phi(x_i)\big]\varphi(x_i)\Big] \\ & = \sum_{i=1}^{n-1}\big[\mathrm{sign}(R_{i-1} - k) - \mathrm{sign}(R_i - k)\big]\big(v(R_i) - v(k)\big)\phi(x_i)\varphi(x_i) \\ & \quad - \mathrm{sign}(R_0 - k)\big(v(R_0) - v(k)\big)\phi(x_0)\varphi(x_0) + \mathrm{sign}(R_{n-1} - k)\big(v_{\max} - v(k)\big)\phi(x_n)\varphi(x_n), \end{aligned} \tag{34}$$

therefore D can be rewritten as

$$D := \int_0^T D_0 dt + \sum_{i=1}^{n-1} \int_0^T D_i dt + \int_0^T D_n dt,$$

where

$$D_0 := \big(1 + \mathrm{sign}(R_0 - k)\big)\big(v(k) - v(R_0)\big)\phi(x_0)\varphi(x_0), \quad D_n := \big(1 + \mathrm{sign}(R_{n-1} - k)\big)\big(v_{\max} - v(k)\big)\phi(x_n)\varphi(x_n)$$

and

$$D_i := \big[\mathrm{sign}(R_{i-1} - k) - \mathrm{sign}(R_i - k)\big]\big(v(R_i) - v(k)\big)\phi(x_i)\varphi(x_i) \quad \text{for } i \in \{1, \dots, n-1\}.$$

Concerning D_0 and D_n, we have two different sub-cases, namely

$$D_0 \begin{cases} = 0 & \text{if } R_0 \leq k, \\ \geq 0 & \text{if } R_0 < k, \end{cases} \quad \text{and} \quad D_n \begin{cases} = 0 & \text{if } R_{n-1} < k, \\ \geq 0 & \text{if } R_{n-1} \geq k. \end{cases}$$

Turning to D_i with $i \in \{1, \dots, n-1\}$, since v is non-increasing and ϕ, φ are non-negative, after a simple calculation we get that

$$D_i \begin{cases} = 0 & \text{if } R_{i-1} > k \text{ and } R_i > k \text{ or } R_{i-1} < k \text{ and } R_i < k \text{ or } R_i = k, \\ \geq 0 & \text{otherwise} \end{cases}$$

and this concludes the proof of (29) in case (P1).

For case (P2), using instead the first order Taylor's expansion (31), we can proceed in a symmetric way to get the validity of (29) also in this case (the details are left to the reader).

Let us now consider the case (P3). Proceeding as in the previous cases we get

$$\int_0^T \int_{\mathbb{R}} \big[|\rho^n - k|\varphi_t + \mathrm{sign}(\rho^n - k)\big(f(\rho) - f(k)\big)\phi\varphi_x - \mathrm{sign}(\rho^n - k) f(k)\phi'\varphi\big] dx dt := kD + E_1^3 + E_2^3 + F,$$

where

$$\begin{aligned} D := & \int_0^T \big(v(k) - v_{\max}\big)\phi(x_0)\varphi(x_0) dt + \int_0^T \big(v_{\max} - v(k)\big)\phi(x_n)\varphi(x_n) dt \\ & + \sum_{i=0}^{n-1} \int_0^T \mathrm{sign}(R_i - k)\Big[\big[\dot{x}_{i+1} - v(k)\phi(x_{i+1})\big]\varphi(x_{i+1}) - \big[\dot{x}_i - v(k)\phi(x_i)\big]\varphi(x_i)\Big] dt, \\ E_1^3 := & \sum_{i=0}^{k_n-1} \int_0^T \mathrm{sign}(R_i - k) R_i v(R_i) \bigg(\int_{x_i}^{x_{i+1}} \phi'(\tilde{y}_{i,i+1})(x - x_{i+1})\varphi_x dx\bigg) dt \\ & + \sum_{i=k_n+1}^{n-1} \int_0^T \mathrm{sign}(R_i - k) R_i v(R_i) \bigg(\int_{x_i}^{x_{i+1}} \phi'(\tilde{x}_{i,i+1})(x - x_i)\varphi_x dx\bigg) dt, \end{aligned}$$

$$E_2^3 := -\sum_{i=0}^{k_n-1} \int_0^T \mathrm{sign}(R_i - k)\dot{R}_i \bigg(\int_{x_i}^{x_{i+1}} \big(\varphi - \varphi(x_i)\big) dx\bigg) dt - \sum_{i=k_n+1}^{n-1} \int_0^T \mathrm{sign}(R_i - k)\dot{R}_i \bigg(\int_{x_i}^{x_{i+1}} \big(\varphi - \varphi(x_{i+1})\big) dx\bigg) dt$$

and

$$\begin{aligned} F := & -\int_0^T \mathrm{sign}(R_{k_n} - k)\dot{R}_{k_n} \bigg(\int_{x_{k_n}}^{x_{k_n+1}} \varphi dx\bigg) dt - \int_0^T \mathrm{sign}(R_{k_n} - k) R_{k_n} \big[\dot{x}_{k_n+1}\varphi(x_{k_n+1}) - \dot{x}_{k_n}\varphi(x_{k_n})\big] dt \\ & + \int_0^T \mathrm{sign}(R_{k_n} - k) R_{k_n} v(R_{k_n}) \bigg(\int_{x_{k_n}}^{x_{k_n+1}} \phi(x)\varphi_x(x) dx\bigg) dt. \end{aligned}$$

Rearranging the indices properly, we can prove that E_1^3 and E_2^3 satisfy (32) and (33) respectively. Concerning F, we first remark that it holds

$$\begin{aligned} -R_{k_n}\big[\dot{x}_{k_n+1}\varphi(x_{k_n+1}) - \dot{x}_{k_n}\varphi(x_{k_n})\big] = & -R_{k_n}\dot{x}_{k_n+1}\big(\varphi(x_{k_n+1}) - \varphi(x_{k_n})\big) - R_{k_n}\varphi(x_{k_n})(\dot{x}_{k_n+1} - \dot{x}_{k_n}) \\ = & -R_{k_n}\dot{x}_{k_n+1}\big(\varphi(x_{k_n+1}) - \varphi(x_{k_n})\big) + \dot{R}_{k_n}\bigg(\int_{x_{k_n}}^{x_{k_{n+1}}} \varphi(x_{k_n}) dx\bigg). \end{aligned}$$

Moreover, since the first order Taylor's expansion of ϕ at 0 in the intervals $(x_{k_n}, 0)$ and $(0, x_{k_n+1})$ implies

$$\int_{x_{k_n}}^{x_{k_n+1}} \phi(x)\varphi_x(x)dx = \phi(0)\big(\varphi(x_{k_n+1}) - \varphi(x_{k_n})\big) + \int_{x_{k_n}}^{0} \phi'(\tilde{y}_{k_n})x\varphi_x(x)dx + \int_{0}^{x_{k_n+1}} \phi'(\tilde{x}_{k_n+1})x\varphi_x(x)dx$$

for some $\tilde{y}_{k_n} \in (x_{k_n}, 0)$ and $\tilde{x}_{k_n+1} \in (0, x_{k_n+1})$, then, using the Lipschitz condition on φ and (13), we get

$$\begin{aligned}
F \geq & -\int_0^T |\dot{R}_{k_n}| \left(\int_{x_{k_n}}^{x_{k_n+1}} \big|\varphi - \varphi(x_{k_n})\big| dx \right) dt - \int_0^T R_{k_n} \dot{x}_{k_n+1} \big|\varphi(x_{k_n+1}) - \varphi(x_{k_n})\big| dt \\
& - \int_0^T R_{k_n} v(R_{k_n}) |\phi(0)| \big|\varphi(x_{k_n+1}) - \varphi(x_{k_n})\big| dt - \int_0^T R_{k_n} v(R_{k_n}) \left| \int_{x_{k_n}}^{0} \phi'(\tilde{y}_{k_n}) x\varphi_x dx \right| dt \\
& - \int_0^T R_{k_n} v(R_{k_n}) \left| \int_0^{x_{k_n+1}} \phi'(\tilde{x}_{k_n+1}) x\varphi_x dx \right| dt \\
\geq & -\ell_n \mathscr{L}_\varphi T \big[4L + L'\big(\overline{x}_{\max} - \overline{x}_{\min} + 2LT\big)\big].
\end{aligned}$$

As a consequence it follows that $E_1^3 + E_2^3 + F \geq -c\,\ell_n$ for some constant c independent on n and hence it remains to prove as before that D is non-negative. From (34) we get that

$$\begin{aligned}
D = & \int_0^T \big(1 + \operatorname{sign}(R_0 - k)\big)\big(v(k) - v_{\max}\big)\phi(x_0)\varphi(x_0)dt \\
& + \sum_{i=1}^{k_n} \int_0^T \big[\operatorname{sign}(R_{i-1} - k) - \operatorname{sign}(R_i - k)\big]\big(v(R_{i-1}) - v(k)\big)\phi(x_i)\varphi(x_i)dt \\
& + \sum_{i=k_n+1}^{n-1} \int_0^T \big[\operatorname{sign}(R_{i-1} - k) - \operatorname{sign}(R_i - k)\big]\big(v(R_i) - v(k)\big)\phi(x_i)\varphi(x_i)dt \\
& + \int_0^T \big(1 + \operatorname{sign}(R_{n-1} - k)\big)\big(v_{\max} - v(k)\big)\phi(x_n)\varphi(x_n)dt
\end{aligned}$$

and this implies that $D \geq 0$, since we can estimate each term as we did in the previous cases.

Turning to the last case (P4), we combine again cases (P1) and (P2), namely we use the Taylor's expansions (30) and (31) of ϕ respectively for $i \in \{0, \ldots, k_n - 1\}$ and $i \in \{k_n + 1, \ldots, n-1\}$. This implies, arguing as before, that

$$\int_0^T \int_{\mathbb{R}} \big[|\rho^n - k|\varphi_t + \operatorname{sign}(\rho^n - k)\big(f(\rho) - f(k)\big)\phi\varphi_x - \operatorname{sign}(\rho^n - k)f(k)\phi'\varphi\big]dxdt := kD + E_1^4 + E_2^4 + F,$$

where

$$\begin{aligned}
D := & \int_0^T \big(v(k) - v(R_0)\big)\phi(x_0)\varphi(x_0)dt + \int_0^T \big(v(R_{n-1}) - v(k)\big)\phi(x_n)\varphi(x_n)dt \\
& + \sum_{i=0}^{n-1} \int_0^T \operatorname{sign}(R_i - k) \Big[\big[\dot{x}_{i+1} - v(k)\phi(x_{i+1})\big]\varphi(x_{i+1}) - \big[\dot{x}_i - v(k)\phi(x_i)\big]\varphi(x_i) \Big] dt,
\end{aligned}$$

$$\begin{aligned}
E_1^4 := & \sum_{i=0}^{k_n-1} \int_0^T \operatorname{sign}(R_i - k) R_i v(R_i) \left(\int_{x_i}^{x_{i+1}} \phi'(\tilde{x}_{i,i+1})(x - x_i)\varphi_x dx \right) dt \\
& + \sum_{i=k_n+1}^{n-1} \int_0^T \operatorname{sign}(R_i - k) R_i v(R_i) \left(\int_{x_i}^{x_{i+1}} \phi'(\tilde{y}_{i,i+1})(x - x_{i+1})\varphi_x dx \right) dt,
\end{aligned}$$

$$E_2^4 := -\sum_{i=0}^{k_n-1} \int_0^T \operatorname{sign}(R_i - k)\dot{R}_i \left(\int_{x_i}^{x_{i+1}} \big(\varphi - \varphi(x_{i+1})\big)dx \right) dt - \sum_{i=k_n+1}^{n-1} \int_0^T \operatorname{sign}(R_i - k)\dot{R}_i \left(\int_{x_i}^{x_{i+1}} \big(\varphi - \varphi(x_i)\big)dx \right) dt,$$

while F is the same term defined in case (P3). Arguing as in the previous cases, we can prove that E_1^4 and E_2^4 satisfy respectively (32) and (33), so it follows that $E_1^4 + E_2^4 + F \geq -c\,\ell_n$ for some constant c independent on n and hence to conclude the proof we need to show as before that D is non-negative. Using again (34) we get that

$$\begin{aligned} D = &\int_0^T \big(1+\operatorname{sign}(R_0-k)\big)\big(v(k)-v(R_0)\big)\phi(x_0)\varphi(x_0)dt \\ &+\sum_{i=1}^{k_n}\int_0^T \big[\operatorname{sign}(R_{i-1}-k)-\operatorname{sign}(R_i-k)\big]\big(v(R_i)-v(k)\big)\phi(x_i)\varphi(x_i)dt \\ &+\sum_{i=k_n+1}^{n-1}\int_0^T \big[\operatorname{sign}(R_{i-1}-k)-\operatorname{sign}(R_i-k)\big]\big(v(R_{i-1})-v(k)\big)\phi(x_i)\varphi(x_i)dt \\ &+\int_0^T \big(1+\operatorname{sign}(R_{n-1}-k)\big)\big(v(R_{n-1})-v(k)\big)\phi(x_n)\varphi(x_n)dt \end{aligned}$$

and hence it follows that $D \geq 0$ also in this last case, since we can estimate each term as we did before.

Now it remains to prove that ρ satisfies the entropy condition, that is

$$\begin{aligned} \int_0^T\int_{\mathbb{R}}\Big[&|\rho(x,t)-k|\varphi_t(x,t)+\operatorname{sign}(\rho(x,t)-k)\big(f(\rho(x,t))-f(k)\big)\phi(x)\varphi_x(x,t) \\ &-\operatorname{sign}(\rho(x,t)-k)f(k)\phi'(x)\varphi(x,t)\Big]dxdt \geq 0 \end{aligned}$$

for every $k \geq 0$ and every non-negative $\varphi \in C_c^\infty(\mathbb{R}\times(0,T))$.

We first notice that the previous inequality is a direct consequence of (29): we need only to show that it is possible to interchange the limit and the integrals. The convergence of $\{\rho^n\}_{n\in\mathbb{N}}$ to ρ almost everywhere and in L^1 on $\mathbb{R}\times[0,T]$ implies that

$$\lim_{n\to+\infty}\int_0^T\int_{\mathbb{R}}|\rho^n-k|\varphi_t dxdt = \int_0^T\int_{\mathbb{R}}|\rho-k|\varphi_t dxdt$$

and moreover, since $f(\mu)=\operatorname{sign}(\mu-k)\big(f(\mu)-f(k)\big)$ is a continuous function, we also have that

$$\lim_{n\to+\infty}\int_0^T\int_{\mathbb{R}}\operatorname{sign}(\rho^n-k)\big(f(\rho^n)-f(k)\big)\phi\varphi_x dxdt = \int_0^T\int_{\mathbb{R}}\operatorname{sign}(\rho-k)\big(f(\rho)-f(k)\big)\phi\varphi_x dxdt.$$

Therefore it remains to show that

$$\lim_{n\to+\infty}\int_0^T\int_{\mathbb{R}}\operatorname{sign}(\rho^n-k)f(k)\phi'\varphi dxdt = \int_0^T\int_{\mathbb{R}}\operatorname{sign}(\rho-k)f(k)\phi'\varphi dxdt, \tag{35}$$

where, since $f(\mu)=\operatorname{sign}(\mu-k)$ is a discontinuous function, we can't interchange the limit and the integrals directly. To overcome this problem, we need to consider two smooth approximations of the sign function $\eta_\varepsilon^\pm$ such that

$$\operatorname{sign}(z)-\eta_\varepsilon^+(z)\geq 0 \quad \text{and} \quad \operatorname{sign}(z)-\eta_\varepsilon^-(z)\leq 0, \tag{36}$$

for instance

$$\eta_\varepsilon^+(z) := \begin{cases} -1 & \text{for } z<0, \\ \frac{2z}{\varepsilon}-1 & \text{for } 0\leq z\leq\varepsilon, \\ 1 & \text{for } z>\varepsilon, \end{cases} \quad \text{and} \quad \eta_\varepsilon^-(z) := \begin{cases} -1 & \text{for } z<-\varepsilon, \\ \frac{2z}{\varepsilon}+1 & \text{for } -\varepsilon\leq z\leq 0, \\ 1 & \text{for } z>0. \end{cases}$$

Let us denote $M := ||\phi'||_{L^\infty(\mathbb{R})}$ from now on. We first remark that (36) implies

$$\begin{aligned} \int_0^T\int_{\mathbb{R}}\operatorname{sign}(\rho^n-k)f(k)\phi'\varphi dxdt &= \int_0^T\int_{\mathbb{R}}\operatorname{sign}(\rho^n-k)f(k)(\phi'-M)\varphi dxdt + \int_0^T\int_{\mathbb{R}}\operatorname{sign}(\rho^n-k)f(k)M\varphi dxdt \\ &\leq \int_0^T\int_{\mathbb{R}}\eta_\varepsilon^+(\rho^n-k)f(k)(\phi'-M)\varphi dxdt + \int_0^T\int_{\mathbb{R}}\eta_\varepsilon^-(\rho^n-k)f(k)M\varphi dxdt. \end{aligned}$$

On the other hand, from the Lipschitz condition of $\eta_\varepsilon^{\pm}$ and using again the convergence of $\{\rho^n\}_{n\in\mathbb{N}}$ to ρ, it follows that

$$\begin{aligned}\lim_{n\to+\infty}\int_0^T\int_{\mathbb{R}}&\big[\eta_\varepsilon^+(\rho^n-k)-\eta_\varepsilon^+(\rho-k)\big]f(k)(\phi'-M)\varphi dxdt\\&\le f(k)||\varphi||_{L^\infty(\mathbb{R}\times[0,T])}\lim_{n\to+\infty}\int_0^T\int_{\mathbb{R}}|\eta_\varepsilon^+(\rho^n-k)-\eta_\varepsilon^+(\rho-k)||\phi'-M|dxdt\\&\le 2f(k)M||\varphi||_{L^\infty(\mathbb{R}\times[0,T])}\mathscr{L}_{\eta_\varepsilon^+}\lim_{n\to+\infty}\int_0^T\int_{\mathbb{R}}|\rho^n-\rho|dxdt=0\end{aligned}\tag{37}$$

and analogously that

$$\begin{aligned}\lim_{n\to+\infty}\int_0^T\int_{\mathbb{R}}&\big[\eta_\varepsilon^-(\rho^n-k)-\eta_\varepsilon^-(\rho-k)\big]f(k)M\varphi dxdt\\&\le f(k)M||\varphi||_{L^\infty(\mathbb{R}\times[0,T])}\lim_{n\to+\infty}\int_0^T\int_{\mathbb{R}}|\eta_\varepsilon^-(\rho^n-k)-\eta_\varepsilon^-(\rho-k)|dxdt\\&\le f(k)M||\varphi||_{L^\infty(\mathbb{R}\times[0,T])}\mathscr{L}_{\eta_\varepsilon^-}\lim_{n\to+\infty}\int_0^T\int_{\mathbb{R}}|\rho^n-\rho|dxdt=0.\end{aligned}\tag{38}$$

Combining the previous three estimates, we hence get

$$\limsup_{n\to+\infty}\int_0^T\int_{\mathbb{R}}\operatorname{sign}(\rho^n-k)f(k)\phi'\varphi dxdt\le\int_0^T\int_{\mathbb{R}}\big[\eta_\varepsilon^+(\rho-k)(\phi'-M)+\eta_\varepsilon^-(\rho-k)M\big]f(k)\varphi dxdt$$

and, since it holds

$$\big[\eta_\varepsilon^+(\rho-k)(\phi'-M)+\eta_\varepsilon^-(\rho-k)M\big]f(k)\varphi\le 3f(k)M\varphi\in L^1(\mathbb{R}\times[0,T]),$$

then we can apply the dominated convergence theorem and pass to the limit in ε, which implies

$$\limsup_{n\to+\infty}\int_0^T\int_{\mathbb{R}}\operatorname{sign}(\rho^n-k)f(k)\phi'\varphi dxdt\le\int_0^T\int_{\mathbb{R}}\operatorname{sign}(\rho-k)f(k)\phi'\varphi dxdt.\tag{39}$$

Proceeding in a symmetric way we notice that

$$\begin{aligned}\int_0^T\int_{\mathbb{R}}\operatorname{sign}(\rho^n-k)f(k)\phi'\varphi dxdt\ge&\int_0^T\int_{\mathbb{R}}\eta_\varepsilon^-(\rho^n-k)f(k)(\phi'-M)\varphi dxdt\\&+\int_0^T\int_{\mathbb{R}}\eta_\varepsilon^+(\rho^n-k)f(k)M\varphi dxdt\end{aligned}$$

and, using again (37), (38) and since

$$\big[\eta_\varepsilon^-(\rho-k)(\phi'-M)+\eta_\varepsilon^+(\rho-k)M\big]f(k)\varphi\le 3f(k)M\varphi\in L^1(\mathbb{R}\times[0,T]),$$

therefore we can apply as before the dominated convergence theorem in ε and we get

$$\liminf_{n\to+\infty}\int_0^T\int_{\mathbb{R}}\operatorname{sign}(\rho^n-k)f(k)\phi'\varphi dxdt\ge\int_0^T\int_{\mathbb{R}}\operatorname{sign}(\rho-k)f(k)\phi'\varphi dxdt.$$

Combining (39) with the last inequality, we hence get (35) and this implies that ρ is a weak solution to (6) satisfying the entropy condition.

Finally, to conclude that ρ is the unique entropy solution we notice that $\overline{\rho}\in L^1(\mathbb{R})\cap L^\infty(\mathbb{R})\cap BV(\mathbb{R})$ due to assumption (I), $f:=\rho v(\rho)\in Lip_{loc}(\mathbb{R}_+)$ since $v\in C^1(\mathbb{R}_+)$ by assumption (V), $\psi:=\phi$ is in $W^{1,1}_{loc}(\mathbb{R})\cap C(\mathbb{R})$ and satisfies $\psi,\psi\in L^\infty(\mathbb{R})$ due to assumption (P). As a consequence, we can apply theorem 3.2 and this concludes the proof of our main result. □

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References

[1] A. Aw, A. Klar, T. Materne, and M. Rascle. Derivation of continuum traffic flow models from microscopic follow-the-leader models. *SIAM Journal on Applied Mathematics*, 63(1): 259–278, 2002.

[2] A. Aw and M. Rascle. Resurrection of 'second order' models of traffic flow. *SIAM Journal on Applied Mathematics*, 60(3): 916–938, 2000.

[3] F. Berthelin, P. Degond, M. Delitala, and M. Rascle. A model for the formation and evolution of traffic jams. *Archive for Rational Mechanics and Analysis*, 187(2): 185–220, 2008.

[4] F. Betancourt, R. Bürger, K. H. Karlsen, and E. M. Tory. On nonlocal conservation laws modelling sedimentation. *Nonlinearity*, 24(3): 855–885, 2011.

[5] J. A. Carrillo, Y.-P. Choi, and M. Hauray. The derivation of swarming models: mean-field limit and Wasserstein distances. *Collective Dynamics from Bacteria to Crowds. CISM International Centre for Mechanical Sciences*, 553: 1–46, 2014.

[6] J. A. Carrillo, M. Di Francesco, and G. Toscani. Condensation phenomena in nonlinear drift equations. *Annali della Scuola Normale Superiore di Pisa, Classe di Scienze*, 5(15): 145–171, 2016.

[7] C. M. Dafermos. *Hyperbolic conservation laws in continuum physics*, volume 325 of *Grundlehrender Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2010.

[8] M. Di Francesco, S. Fagioli, and E. Radici. Deterministic particle approximation for nonlocal transport equations with nonlinear mobility. *Journal of Differential Equations*, 266(5): 2830–2868, 2019.

[9] M. Di Francesco, S. Fagioli, and M. D. Rosini. Deterministic particle approximation of scalar conservation laws. *Bollettino dell'Unione Matematica Italiana*, 10(3): 487–501, 2017.

[10] M. Di Francesco, S. Fagioli, M. D. Rosini, and G. Russo. Deterministic particle approximation of the Hughes model in one space dimension. *Kinetic and Related Models*, 10(1): 215–237, 2017.

[11] M. Di Francesco, S. Fagioli, M. D. Rosini, and G. Russo. Follow-the-leader approximations of macroscopic models for vehicular and pedestrian flows. *Active Particles*, 1: 333–378, 2017.

[12] M. Di Francesco and M. D. Rosini. Rigorous derivation of nonlinear scalar conservation laws from follow-the-leader type models via many particle limit. *Archive for Rational Mechanics and Analysis*, 217(3): 831–871, 2015.

[13] P. A. Ferrari. Shock fluctuations in asymmetric simple exclusion. *Probabilty Theory and Related Fields*, 91(1): 81–101, 1992.

[14] P. L. Ferrari and P. Nejjar. Shock fluctuations in flat tasep under critical scaling. *Journal of Statistical Physics*, 160(4): 985–1004, 2015.

[15] L. Gosse and G. Toscani. Identification of asymptotic decay to self-similarity for one-dimensional filtration equations. *SIAM Journal on Numerical Analysis*, 43(6): 2590–2606, 2006.

[16] M. Z. Guo, G. C. Papanicolaou, and S. R. S. Varadhan. Nonlinear diffusion limit for a system with nearest neighbor interactions. *Communications in Mathematical Physics*, 118(1): 31–59, 1988.

[17] K. H. Karlsen and K.-A. Lie. An unconditionally stable splitting scheme for a class of nonlinear parabolic equations. *IMA Journal of Numerical Analysis*, 19(4): 609–635.

[18] K. H. Karlsen and N. H. Risebro. On the uniqueness and stability of entropy solutions of nonlinear degenerate parabolic equations with rough coefficients. *Discrete and Continuous Dynamical Systems*, 9(5): 1081–1104, 2003.

[19] S.N. Kružkov. First order quasilinear equations in several independent variables. *Mathematics of the USSR-Sbornik*, 10(2): 217–243, 1970.

[20] T. M. Liggett. *Interacting particle systems*, volume 276 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, 1985.

[21] M.J. Lighthill and G.B. Whitham. On kinematic waves. II. A theory of traffic flow on long crowded roads. *Proceedings of the Royal Society. London. Series A. Mathematical, Physical and Engineering Sciences*, 229(1178): 317–345, 1955.

[22] D. Matthes and H. Osberger. Convergence of a variational Lagrangian scheme for a nonlinear drift diffusion equation. *ESAIM: Mathematical Modelling and Numerical Analysis*, 48(3): 697–726, 2014.

[23] P.I. Richards. Shock waves on the highway. *Operations Research*, 4(1): 42–51, 1956.

[24] M.D. Rosini. *Macroscopic models for vehicular flows and crowd dynamics: theory and applications*. Understanding Complex Systems. Springer, Heidelberg, 2013.

[25] R. Rossi and G. Savaré. Tightness, integral equicontinuity and compactness for evolution problems in banach spaces. *Annali della Scuola Normale Superiore di Pisa. Classe di Scienze*, 5(2): 395–431, 2003.

[26] G. Russo. Deterministic diffusion of particles. *Communications on Pure and Applied Mathematics*, 43(6): 697–733, 1990.

[27] D. W. Stroock and S. R. S. Varadhan. *Multidimensional diffusion processes*, volume 233 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin-New York, 1979.

[28] G. Toscani. Finite time blow up in Kaniadakis-Quarati model of Bose-Einstein particles. *Communications in Partial Differential Equations*, 37(1): 77–87, 2012.

[29] C. Villani. *Optimal transport. Old and new*, volume 338 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2009.

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