Functional Analysis in Applied Mathematics and Engineering:
Final exam: part 1 - 20/02/2017

Full name:____

Matricola:____

(1) (i) Let (K, d) be a compact metric space and let $C(K)$ be the linear space of continuous functions $f : K \to \mathbb{R}$.

(a) Define the $\|\cdot\|_\infty$ norm on $C(K)$. [1]

(b) Let f_n be a sequence in $C(K)$ and assume $\lim_{n\to+\infty} \|f_n - f\|_\infty = 0$ for some $f : K \to \mathbb{R}$. Prove that $f \in C(K)$. [4]

(ii) (a) State (without proof) the *Arzelá-Ascoli Theorem.* [2]

(b) Let $\mathcal{F}_M = \{f \in C([0,1]) \ : \ f \text{ is differentiable and } \|f'\|_\infty \le M\}$. Prove that the set

$$\{f \in \mathcal{F}_M \ : \ f(0) = 1\}$$

is precompact in $C([0,1])$. [3]

(iii) For any $n \in \mathbb{N}$, let $f_n : [0,1] \to \mathbb{R}$ be defined by

$$f_n(x) = (1 - x)x^n$$

(a) Show that $f_n \to 0$ pointwise in $(0,1)$ as $n \to +\infty$. [2]

(b) Is $\{f_n\}_n$ uniformly convergent on $(0,1)$? Justify your answer in detail. [3]

(2) (i) Given a measurable function $f : \mathbb{R}^d \to [0, +\infty]$, provide the definition of the Lebesgue integral of f. [2]

(ii) State (without proof) Lebesgue dominated convergence theorem. [3]

(iii) Prove *Minkowski's inequality*

$$\|f + g\|_{L^p(\mathbb{R}^d)} \le \|f\|_{L^p(\mathbb{R}^d)} + \|g\|_{L^p(\mathbb{R}^d)}$$

for all measurable functions f, g on $\mathbb{R}^d$ and for all $p \in (1, +\infty)$. [5]

(iv) Let $f : [0,1] \to \mathbb{R}$ be the measurable function defined by

$$f(x) = \begin{cases} \frac{\sin|x|}{|x|\sqrt{|x|}} & \text{if } x \in (0,1] \\ 0 & \text{if } x = 0. \end{cases}$$

Determine all the $p \in [1, +\infty)$ such that $f \in L^p([0,1])$. [5]