Functional Analysis in Applied Mathematics and Engineering:

Final exam: part 2 - 20/02/2017

Full name:____

Matricola:____

(3) (i) Let X be a Banach space.

(a) Given a bounded linear operator T on X, define the operator norm of T. [1]

(b) Provide the definition of *compact operator* T on X. [2]

(c) Given a sequence of compact operators T_n on X such that T_n converges to a bounded linear operator T in the operator norm, prove that T is compact. [3]

(ii) Let X be a Banach space.

(a) Define the notion of *topological dual space* X^*. [1]

(b) Given a sequence x_n in X, define the notion of *weak convergence* for x_n. [2]

(iii) Let $p \in (1, +\infty)$. Let $f_n : \mathbb{R} \to \mathbb{R}$ be the functions sequence defined by

$$f_n(x) = \begin{cases} n^{1/p} & \text{if } x \in [0, 1/n] \\ 0 & \text{otherwise.} \end{cases}$$

(a) Prove that f_n does not converge to zero strongly in $L^p(\mathbb{R})$. [3]

(b) Prove that f_n converges weakly to zero in $L^p(\mathbb{R})$. [3]

(4) (i) State (without proof) the parallelogram law on a Hilbert space. [2]

(ii) Let M be a closed subspace of a Hilbert space H. Prove that for each $x \in X$ there exists a unique $y \in M$ such that $\|x - y\| = \inf_{z \in M} \|x - z\|$. [5]

(iii) Let H be a Hilbert space.

(a) Define the notion of orthonormal basis for H. [2]

(b) Prove that if H is separable (i.e. H has a dense countable subset) and infinite-dimensional then H has a countable orthonormal basis. [4]

(c) Write an orthonormal basis for the Hilbert space $\ell^2(\mathbb{N})$. [2]