Functional Analysis in Applied Mathematics and Engineering:

Final exam: part 3 - 20/02/2017

FULL NAME:____

MATRICOLA:____

(1) (i) Let P be a bounded linear operator on a Hilbert space.

(a) When is P called an *orthogonal projection*? [2]

(b) Prove that if P is an orthogonal projection, then $\|P\| = 1$. [3]

(ii) State (without proof) *Riesz' representation theorem.* [3]

(iii) Let H be a Hilbert space and let A be a bounded linear operator on H.

(a) Define the notion of adjoint operator A^*. [1]

(b) Prove that $\mathrm{Ran}A \subset (\mathrm{Ker}A^*)^\perp$. [3]

(c) Prove that $(\mathrm{Ker}A^*)^\perp \subset \overline{\mathrm{Ran}A}$ (Hint: use that if V is a linear subspace, than $V^{\perp\perp} = \overline{V}$). [3]

(2) (i) Let A be a bounded linear operator on a Hilbert space H.

(a) Define the *resolvent set* and the *spectrum* of A. [1]

(b) Define the *point spectrum*, the *continuous spectrum*, and the *residual spectrum* of A. [2]

(ii) Let $T : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ be the left-shift operator

$$(Tx)_k = x_{k+1} \qquad \text{for all } k \geq 1.$$

(a) Prove that $\|T\| = 1$. [3]

(b) Prove that the point spectrum of T coincides with $(-1, 1)$. [3]

(c) Prove that the the spectrum of T coincides with $[-1, 1]$. [2]

(iii) Let $g \in C([0, 1])$. Consider the operator $T : L^2([0, 1]) \to L^2([0, 1])$ defined by

$$(Tf)(x) = \int_0^x g(y)f(y)dy.$$

Prove that T has no eigenvalues. [4]