A NONLOCAL SWARM MODEL FOR PREDATORS-PREY INTERACTIONS

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Abstract. We consider a two-species system of nonlocal interaction PDEs modeling the swarming dynamics of predators and prey, in which all agents interact through attractive/repulsive forces of gradient type. In order to model the preator-prey interaction, we prescribed proportional potentials (with opposite signs) for the cross-interaction part. The model has a particle-based discrete (ODE) version and a continuum PDE version. We investigate the structure of particle stationary solution and their stability in the ODE system in a systematic form, and then consider simple examples. We then prove that the stable particle steady states are locally stable for the fully nonlinear continuum model, provided a slight reinforcement of the particle condition is required. The latter result holds in one space dimension. We complement all the particle examples with simple numerical simulations, and we provide some two-dimensional examples to highlight the complexity in the large time behaviour of the system.

1. Introduction

The mathematical modeling of the collective motion for multi-agent aggregates arises in various research fields such as biology, ecology, robotics, and control theory, as well as sociology and economics. This subject has attracted a lot of attention in the recent years, and the modeling of bird flocks, fish schools, and insect swarming in general has been deeply investigated by several applied mathematicians, see for instance [10, 22, 25, 27–31]. Among the others, the problem of predator-prey interactions has been formulated in the context of swarming models and collective behaviour, see e.g. [22, 25, 28], with the goal of catching the typical *spatial patterns* that occur in practical situations, see also a related contribution in [3]. A significant example is the formation of empty space surrounding a group of predators, with aggregates of prey all around, which is usually observed in fish schools or in flock of sheeps - as a special case of self-emerging patterns that typically arise in swarming models.

The dynamics of predators and prey in animal and social biology has attracted the interest of many applied mathematicians in the last century, since the pioneering works of Lotka [24] and Volterra [33]. In the classical literature, the predator-prey interaction is typically described via reaction terms in a set of differential equations, possibly combined with diffusion terms, we refer to the classical book [26] and the references therein.

The main idea behind the new swarming modelling approach is that the presence of each prey biases the movement of each predator via a *transport* term rather than a reaction term (e.g. similarly to the transport term in a chemotaxis model), and a symmetric mechanisms activates the movement of the prey away from the predators. Consequently, an *interacting particle* approach to this problem can be formulated as follows.

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Let $X_1, \dots, X_N$ be the spatial positions of N predators, and let $Y_1, \dots, Y_M$ be the positions of M prey. Each predator X_i has a *strength* coefficient $m_X^i > 0$, whereas each prey Y_i has an *attractiveness* coefficient $m_Y^i > 0$. Such coefficients may be considered as proportional to the single masses of the individuals, and we shall therefore refer to them as *masses*. We assume that predator and prey are subject to a *self-organising force* in absence of interaction with the other species. More precisely, we assume that each predator moves under the effect of a radial nonlocal force directed towards all other predators, and a similar phenomenon holds for the prey. Whether the self-organising forces are attractive or repulsive depends on the situation under study. We then assume that the predators are attracted by the prey, where the prey are subject to a repulsive force with respect to all the predators. All the above mentioned forces are considered as gradients of potential quantities depending on distances between the agents under consideration. The social mechanisms resulting into social attraction of the predators, social attraction of the prey, and predator-prey interaction respectively are, in principle, independent. On the other hand, the predator-prey interaction is assumed to take place via the same force field in our aproach, both in the attractive drift that makes the predator move towards a prey and in the repulsive drift that drive the prey away from the predators.

The resulting system is

$$\begin{cases} \dot{X}_i(t) = -\displaystyle\sum_{k=1}^{N} m_X^k \nabla S_1(X_i(t) - X_k(t)) - \sum_{h=1}^{M} m_Y^h \nabla K(X_i(t) - Y_h(t)), \\ \dot{Y}_j(t) = -\displaystyle\sum_{h=1}^{M} m_Y^h \nabla S_2(Y_j(t) - Y_h(t)) + \alpha \sum_{k=1}^{M} m_X^k \nabla K(Y_j(t) - X_k(t)), \end{cases} \tag{1}$$

with $i = 1, \dots, N$ and $j = 1, \dots, M$. The coefficient $\alpha > 0$ is the result of a mass normalization of both species, after which $m_X^1 + \dots + m_X^N = m_Y^1 + \cdots + m_Y^M = 1$, which simplifies the analysis of the model. All the potentials in (1) incorporate such normalization process, and the constant α expresses the ratio between the total mass of the predators and the total mass of the prey. The potentials S_1 and S_2 are called *self-interaction* potentials, they express the social tendency of each single species to form social aggregates without the effect of the other species. The potential K is responsible for the predator-prey interaction, and is called *cross-interaction* potential. We observe that our model does not take into account the annihilation rate of the prey once they are caught by a predator. This issue will be tackled in future studies.

We recall that an interaction potential G is said to be radial if $G(x) = g(|x|)$ for some $g : [0, +\infty) \to \mathbb{R}$. G is attractive if $g'(r) > 0$ for $r > 0$, G is repulsive if $g'(r) < 0$ for $r > 0$. We shall assume that all the potential S_1, S_2, K are radial. No assumptions are required on the attractiveness and repulsiveness of S_1 and S_2, whereas we shall always require that K is attractive. More precise assumptions on the three potentials will be stated in Section 2. The constant α can be interpreted as a *mobility* coefficient for the prey.

A minimal version of (1) in two dimensions, with only one predator and arbitrarily many prey, has been considered in [17, 29]. In both papers, the self-interaction of the prey is driven by a potential S_2 with a short range (singular) repulsion and a long range attraction, whereas $S_1 \equiv 0$, i. e. no self-interaction among the predators. Two distinct potential are used in [17, 29] to model predator-prey interaction. In particular, the attractive force driving the predators towards the prey is singular at zero, and is supposed to be stronger than the repulsive force pushing the prey away from the predators at small distances. By incorporating a singular short range repulsion in the self-interaction of the prey, the approach in [17, 29] induces the formation of nontrivial patterns in some way to prevent the action of the predators. Our model tries to catch a similar behaviour as a direct consequence of the predator-prey interaction. Moreover, we require that the two predator-prey interaction potentials are proportional in order to have a suitable linear combination of first momenta preserved in time, which possibly allows to select a unique equilibrium for fixed initial data.

The model (1) has a continuum counterpart, namely

$$\begin{cases} \partial_t \mu_1 = \mathrm{div}(\mu_1(\nabla S_1 * \mu_1 + \nabla K * \mu_2)) \\ \partial_t \mu_2 = \mathrm{div}(\mu_2(\nabla S_2 * \mu_2 - \alpha \nabla K * \mu_1)), \end{cases} \tag{2}$$

where μ_1 and μ_2 are time dependent probability measures denoting the (normalized) concentration of predators and prey respectively. This system is a two-species generalization of the nonlocal interaction equation

$$\partial_t \mu = \mathrm{div}\left(\mu \nabla K * \mu\right). \tag{3}$$

The main issue of (3) is the possibility of developing blowup in finite time, even for solutions which are initially very smooth. This characteristic is mainly due to the possibility that the interaction kernel K is attractive close to the origin and that it has a certain singularity in the second derivative (for example $K(x) \approx |x|^{1+\alpha}$, with $\alpha \in [0, 1)$ has such a Lipschitz singularity). After the pioneering works [9, 23], a fairly complete L^p-theory has been developed in [4–8], also for potentials producing blowup in finite time. Application of the Wasserstein distance and of the gradient flows theory for smooth kernels was introduced in [1, 14] and extended to singular kernels in [12, 13, 16].

A systematic mathematical theory for a system of the form (2) has been performed in [18]. In particular, system (2) should be framed in the context of non symmetrizable systems, for which the Wasserstein gradient flow theory developed in [1] and adapted to systems in [18] does not work. For a class of potentials with regularity less that C^2, [18] proves existence via an implicit-explicit version of the JKO scheme [21]. For C^2 potentials, existence and uniqueness can be proved by the method of characteristics.

Despite its rather simple structure, and despite being quite similar to the so called 2×2 symmetrizable systems of nonlocal interaction equations (with a gradient flow structure) considered in [18], the discrete system (1) and the continuum counterpart (2) turn out to feature a rather complex dynamics. First of all, in order to simplify the problem, we shall work with C^2 potentials, which implies authomatically that all the steady states will be combinations of Dirac's deltas. Then, we start exploring steady states, first in a systematic form and then by considering simple examples with at most two agents per species. Hence, we investigate the structure of the linearised system around particle steady states at the discrete level.

We show that, in a suitable discrete norm adapted to the steady state under consideration, such system features a simple linear structure, which simplifies the study of the spectrum, and therefore allows to detect relatively simple stability conditions. We then turn back to the simple examples, and provide simple stability conditions. In order to simplify the coverage of the paper, most of the computations are performed in one space dimension. Some computations, such as stability conditions in the particle framework, can be easily generalised to the multi dimensional case. All the examples are complemented with simple numerical simulations.

The main result concerns with the local nonlinear stability vs. instability of steady states for the continuum system of nonlocal PDEs (2) in one space dimension. Basically we prove that a simple sufficient condition for the stability can be obtained by a slightly stronger requirement of the stability condition for the particle system (1). The stability result holds in the ∞-Wasserstein distance, and is contained in Theorem 4.1. We conclude the paper with some simulations in two space dimensions, which highlight the complexity of the system, and suggest the existence of more complex patterns in case of singular potentials.

The paper is structured as follows. In Section 2 we recall the basics on p-Wasserstein distances in spaces of probability measures, with the natural extension to two species Cartesian products proposed in [18]. Moreover, we briefly recall the existing theory on two species systems of the form (2). In Section 3 we perform a systematic study of the particle steady states and their linear stability under the particle system (1), with some examples involving at most two particles per species. In Section 4 we prove our main result in Theorem 4.1, which provides a reasonable sufficient condition for the nonlinear local stability of the particle steady states for the continuum model (2) in one space dimension. Finally, in Section 5 we propose some numerical examples in two dimensions, which emphasize the asymptotic complexity of the model under study.

2. Preliminaries

In this section we collect preliminary concepts on the Wasserstein distance and we state the assumptions on our model.

We recall that a function $G:\mathbb{R}^d\to\mathbb{R}$ is a *radial* potential if

(Rad) $G(x)=g(|x|)$ for some function $g:[0,+\infty)\to\mathbb{R}$.

A radial potential is called an attractive potential if

(Attr) $G(x)=g(|x|)$ with $g\in C^1((0,+\infty)$ and $g'(r)>0$ if $r>0$.

Throughout the whole paper we shall require

(A1) $S_1,S_2,K\in C^{2,\gamma}(\mathbb{R}^d)$ for a Hölder exponent $0<\gamma\leq 1$.
(A2) S_1, S_2, and K satisfy (Rad).
(A3) K satisfies (Attr).

We are interested in the stationary states of (2), namely with the measures μ_1 and μ_2 satisfying

$$\begin{cases}\mathrm{div}(\mu_1(\nabla S_1*\mu_1+\nabla K*\mu_2))=0\\ \mathrm{div}(\mu_2(\nabla S_2*\mu_2-\alpha\nabla K*\mu_1))=0.\end{cases}\tag{4}$$

Before defining our notion of solution for (2), we recall the basics tools in optimal transport and review the theory for (2) developed in literature.

2.1. **Wasserstein distances.** Let $p\in[1,+\infty)$. We recall the definition of the p-Wasserstein space of probability measures with finite p-moment

$$\mathscr{P}_p(\mathbb{R}^d)=\left\{\mu\in\mathscr{P}(\mathbb{R}^d):m_p(\mu)=\int_{\mathbb{R}^n}|x|^p\,d\mu(x)<\infty\right\}.$$

For a measure $\mu\in\mathscr{P}(\mathbb{R}^d)$ and a Borel map $T:\mathbb{R}^d\to\mathbb{R}^k$, denote with $T_\#\mu\in\mathscr{P}(\mathbb{R}^d)$ the push-forward of μ through T, defined by

$$\int_{\mathbb{R}^k}f(y)dT_\#\mu(y)=\int_{\mathbb{R}^d}f(T(x))d\mu(x)\qquad\text{for all } f \text{ Borel functions on } \mathbb{R}^k.$$

We endow the space $\mathscr{P}_p(\mathbb{R}^d)$ with the Wasserstein distance, see for instance [1,2,32]

$$W_p^p(\mu,\nu)=\inf_{\gamma\in\Gamma(\mu,\nu)}\left\{\int_{\mathbb{R}^{2d}}|x-y|^p d\gamma(x,y)\right\},\tag{5}$$

where $\Gamma(\mu_1,\mu_2)$ is the class of transport plans between μ and ν, that is the class of measures $\gamma \in \mathscr{P}(\mathbb{R}^{2d})$ such that, denoting by π^i the projection operator on the i-th component of the product space, the marginality condition

$$\pi^i_{\#}\gamma = \mu_i \quad i = 1,2$$

is satisfied. By introducing $\Gamma_o(\mu,\nu)$ as the class of optimal plans, in which the minimum in (5) is achieved, we can rewrite the Wasserstein distance as

$$W_p^p(\mu,\nu) = \int_{\mathbb{R}^{2d}} |x-y|^p d\gamma(x,y), \qquad \gamma \in \Gamma_o(\mu,\nu).$$

The ∞-Wasserstein distance is defined on the space of compactly supported probability measures as

$$W_\infty(\mu,\nu) = \lim_{p\to+\infty} W_p^p(\mu,\nu) = \inf\{\operatorname{ess\,sup}_\gamma |x-y| \,:\, \gamma \in \Gamma(\mu,\nu)\}.$$

Since we are working in a 'multi-species' structure, we consider the product space $\mathscr{P}_p(\mathbb{R}^d) \times \mathscr{P}_p(\mathbb{R}^d)$ endowed with a product structure. In the following we shall use bold symbols to denote elements in a product space. For a $p \in [1,+\infty]$, we use the notation

$$\mathcal{W}_p^p(\boldsymbol{\mu},\boldsymbol{\nu}) = W_p^p(\mu_1,\nu_1) + W_p^p(\mu_2,\nu_2),$$

with $\boldsymbol{\mu} = (\mu_1,\mu_2), \boldsymbol{\nu} = (\nu_1,\nu_2) \in \mathscr{P}_p(\mathbb{R}^d) \times \mathscr{P}_p(\mathbb{R}^d)$.

2.2. **Existence theory for (2).** Concerning the well-posedness of (2) in a measure sense, in [18] the authors present a systematic existence and uniqueness theory for the general system of nonlocal interaction equations with two species

$$\begin{cases} \partial_t \mu_1 = \operatorname{div}\left(\mu_1 \nabla S_1 * \mu_1 + \mu_1 \nabla K_1 * \mu_2\right) \\ \partial_t \mu_2 = \operatorname{div}\left(\mu_2 \nabla S_2 * \mu_2 + \mu_2 \nabla K_2 * \mu_1\right) \end{cases}, \tag{6}$$

in which S_1, S_2, K_1, K_2 are even, locally Lipschitz continuous potentials with a gradient which is continuous except possibly at the origin. We state our definition of weak measure solution for (6), which uses the space $\mathscr{P}_2(\mathbb{R}^d)$.

Definition 2.1. *A curve $\boldsymbol{\mu}(\cdot) = (\mu_1(\cdot),\mu_2(\cdot)) : [0,+\infty) \to \mathscr{P}_2(\mathbb{R}^d)^2$ is a* weak measure solution *to (6) is, for all $\phi,\psi \in C_c^\infty(\mathbb{R}^d)$, we have*

$$\begin{aligned} \frac{d}{dt}\int \phi(x) d\mu_1(x,t) &= -\frac{1}{2}\iint \nabla H_1(x-y)\cdot(\nabla\phi(x) - \nabla\phi(y)) d\mu_1(x) d\mu_1(y) \\ &\quad - \iint \nabla K_1(x-y)\cdot \nabla\phi(x) d\mu_1(x) d\mu_2(y) \\ \frac{d}{dt}\int \psi(x) d\mu_2(x,t) &= -\frac{1}{2}\iint \nabla H_2(x-y)\cdot(\nabla\psi(x) - \nabla\psi(y)) d\mu_2(x) d\mu_2(y) \\ &\quad - \iint \nabla K_2(x-y)\cdot \nabla\psi(x) d\mu_2(x) d\mu_1(y). \end{aligned}$$

We recall that in the case of atomic initial conditions, namely with

$$\boldsymbol{\mu}(0) = (\mu_1(0),\mu_2(0)) = \left(\sum_{i=1}^N m_i^X \delta_{X_i(0)}, \sum_{j=1}^M m_j^Y \delta_{Y_j(0)}\right),$$

with $m_1^X + \ldots + m_N^X = m_1^Y + \ldots + m_M^Y = 1$, the *moving particle solution*

$$\boldsymbol{\mu}(t) = (\mu_1(t),\mu_2(t)) = \left(\sum_{i=1}^N m_i^X \delta_{X_i(t)}, \sum_{j=1}^M m_j^Y \delta_{Y_j(t)}\right),$$

with

$$\begin{cases} \dot{X}_i(t) = -\sum_{k=1}^{N} m_k^X \nabla S_1(X_i(t) - X_k(t)) - \sum_{k=1}^{M} m_k^Y \nabla K_1(X_i(t) - Y_k(t)) \\ \dot{Y}_j(t) = -\sum_{k=1}^{M} m_k^Y \nabla S_2(Y_j(t) - Y_k(t)) - \sum_{k=1}^{N} m_k^X \nabla K_2(Y_j(t) - X_k(t)) \end{cases}, \tag{7}$$

for $i \in \{1, \dots, N\}$ and $j \in \{1, \dots, M\}$, formally satisfies (6) (rigorously when $S_1, S_2, K_1, K_2 \in C^1(\mathbb{R}^d)$).

The following theorem states existence of weak measure solutions for (6). In such a general setting, and in particular without requiring any algebraic constraint for the four interaction potentials, the existence of weak measure solutions is provided using an implicit-explicit version of the Jordan-Kinderlehrer-Otto (JKO) scheme [21], provided ∇K_1 and ∇K_2 are continuous at the origin.

Theorem 2.1 (Existence of weak measure solutions [18])**.** *Assume S_1, S_2, K_1, K_2 are even, locally Lipschitz continuous potentials, and assume that ∇K_1 and ∇K_2 are continuous on $\mathbb{R}^d$ and ∇S_1 and ∇S_2 are continuous on $\mathbb{R}^d \setminus \{0\}$. Let $\boldsymbol{\mu}_0 \in \mathscr{P}_2(\mathbb{R}^d)^2$ be fixed. Then, there exists an absolutely continuous curve $\boldsymbol{\mu}(\cdot) : [0, +\infty) \to \mathscr{P}_2(\mathbb{R}^d)^2$ such that $\boldsymbol{\mu}(0) = \boldsymbol{\mu}_0$ and $\boldsymbol{\mu}(t)$ is a weak measure solution to (6) in the sense of Definition 2.1.*

Under the present assumptions on the interaction potentials, no uniqueness result is proven in [18]. However, for C^2 potentials, using a variant of the method of characteristics, the following result holds.

Theorem 2.2 (Stability [18])**.** *Let $p \in [1, +\infty]$. Assume that S_1, S_2, K_1, K_2 are C^2 and consider two initial measures $\boldsymbol{\mu}_0, \boldsymbol{\nu}_0 \in \mathscr{P}_p(\mathbb{R}^d)^2$ with compact support and the corresponding weak measure solutions of (6) $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$. Then, there exists a constant $\tilde{C} > 0$ such that*

$$\mathcal{W}_p(\boldsymbol{\mu}_t, \boldsymbol{\nu}_t) \leq e^{\tilde{C}t} \mathcal{W}_p(\boldsymbol{\mu}_0, \boldsymbol{\nu}_0) \quad t \geq 0. \tag{8}$$

Consequently, for a given initial condition $\boldsymbol{\mu_0} \in \mathscr{P}(\mathbb{R}^d)^2_p$, there exists a unique weak measure solution to (6).

Proof. The proof in the case $p = 2$ is contained in [18, Theorem 4.2]. The general case $p \in [1, +\infty)$ can be performed by following exactly the same procedure, and will therefore be omitted. The proof for $p = +\infty$ can be obtained by sending $p \to +\infty$ in the previous step. $\square$

In the case of symmetrizable system, i.e. with cross-interaction potentials such that

$$K_1 = \beta K_2 \quad \beta > 0, \tag{9}$$

the same results are proved through a generalization of the Wasserstein gradient flow theory approach [1, 2]. The relationship between the potential, in fact, brings out an important property of the system: the system (6) can in fact be associated with the energy functional

$$\mathcal{F}(\mu_1, \mu_2) = \frac{1}{2} \int_{\mathbb{R}^d} H_1 * \mu_1 d\mu_1 + \frac{1}{2} \int_{\mathbb{R}^d} H_2 * \mu_2 d\mu_2 + \int_{\mathbb{R}^d} K * \mu_2 d\mu_1,$$

and rewritten, at least formally, as

$$\begin{cases} \partial_t \mu_1 = \operatorname{div}\left(\mu_1 \nabla \frac{\delta \mathcal{F}}{\delta \mu_1}\right) \\ \partial_t \mu_2 = \alpha \operatorname{div}\left(\mu_2 \nabla \frac{\delta \mathcal{F}}{\delta \mu_2}\right) \end{cases}, \tag{10}$$

where the terms $\frac{\delta \mathcal{F}}{\delta \mu_1}$ and $\frac{\delta \mathcal{F}}{\delta \mu_2}$ can be interpreted at this stage as functional derivative in the spirit of Frechét derivative. The theory developed in [18] under the assumption (9) requires the additional assumption that all the interaction kernels S_1, S_2, K_1, K_2 are λ-convex for some $\lambda \in \mathbb{R}$ (i. e. convex up to quadratic perturbations in a suitable weighed metric structure on $\mathscr{P}_2(\mathbb{R}^d) \times \mathscr{P}_2(\mathbb{R}^d)$), but the regularity of all the kernels can be relaxed to assuming that the gradient of all kernels is continuous except at the origin.

It has been pointed out in [18] that the particle system (7) is a gradient flow in a suitable weighted finite dimensional norm when $K_1 = \beta K_2$ for some $\beta > 0$, and it is well known that the gradient flow structure can partly compensate the lack of regularity of the functional (i. e. of the interaction kernels) at the origin, in a way to get uniqueness of solutions. Such a remark explains why the uniqueness for (6) is a difficult task with mildly singular potentials and when the symmetry assumption (9) is dropped. Another problem that one has to tackle when (9) does not hold is the fact that (6) has, in general, no other conserved quantities except the total mass, whereas (9) implies that the joint center of mass

$$\beta \int x d\mu_1(x) + \int x d\mu_2(x)$$

is constant for all times.

Our system (2) does not fall into the class of symmetrizable systems (9), because the two cross-interaction potentials are proportional via a *negative* constant in our case. Hence, no gradient flow structure can be used. Hence, the general results in Theorem 2.1 and 2.2 apply. However, (2) has essentially a similar algebraic structure to that of a symmetrizable system. Indeed, it is easy to show that the joint center of mass

$$C_\alpha(t) := \alpha \int x d\mu_1(x) - \int x d\mu_2(x) \tag{11}$$

is constant in time in (2). We shall see that such property is crucial in the study of stationary states later on in the paper. For future use we recall the particle version of (2):

$$\begin{cases} \dot{X}_i(t) = -\sum_{k=1}^{N} m_X^k \nabla S_1(X_i(t) - X_k(t)) - \sum_{k=1}^{M} m_Y^k \nabla K(X_i(t) - Y_k(t)) \\ \dot{Y}_j(t) = -\sum_{k=1}^{M} m_Y^k \nabla S_2(Y_j(t) - Y_k(t)) + \alpha \sum_{k=1}^{N} m_X^k \nabla K(Y_j(t) - X_k(t)) \end{cases}, \tag{12}$$

for $i \in \{1, \dots, N\}$ and $j \in \{1, \dots, M\}$.

2.3. **The one dimensional case.** We shall focus most of our attention on the one dimensional case for (2)

$$\begin{cases} \partial_t \mu_1 = \partial_x (\mu_1 (S_1' * \mu_1 + K' * \mu_2)) \\ \partial_t \mu_2 = \partial_x (\mu_2 (S_2' * \mu_2 - \alpha K' * \mu_1)) \end{cases}, \tag{13}$$

In one space dimension, the Wasserstein distance can be expressed in terms of the so called *pseudo-inverse functions* [15]. Consider a measure $\mu \in \mathscr{P}(\mathbb{R})$ and let $R : \mathbb{R} \to [0,1]$ the distribution function associated to μ,

$$R(x) = \int_{-\infty}^{x} d\mu(x) = \mu\left((-\infty, x]\right).$$

Define the the *pseudo-inverse function* of the distribution function R on $[0,1)$ by

$$u(z) := \inf \{x \in \mathbb{R} \,|\, R(x) > z\} \qquad z \in [0,1).$$

Then, the $p-$Wasserstein distance can be expressed as the L^p-norm of the pseudo-inverse function by

$$W_p(\mu_1,\mu_2) = \|u_1 - u_2\|_{L^p([0,1])}, \tag{14}$$

where u_1 and u_2 are constructed from μ_1 and μ_2 respectively as above, see [32]. Clearly, u_1 and u_2 are non-decreasing functions.

The main advantage in introducing the pseudo-inverse functions is that we can rewrite the system (2) in a convenient way to deal with convolutions, cf. for instance [11, 15, 19, 20, 23]. Indeed, in terms of the pseudo-inverses u_1 and u_2, (13) can be re-written as the system of integro-differential equations

$$\begin{cases} \partial_t u_1(z,t) = \displaystyle\int_0^1 S_1'(u_1(\zeta,t) - u_1(z,t))d\zeta + \int_0^1 K'(u_2(\zeta,t) - u_1(z,t))d\zeta \\ \partial_t u_2(z,t) = \displaystyle\int_0^1 S_2'(u_2(\zeta,t) - u_2(z,t))d\zeta - \alpha\int_0^1 K'(u_1(\zeta,t) - u_2(z,t))d\zeta \end{cases}. \tag{15}$$

Another key advantage in using this formulation is that the atomic parts of a probability measure μ correspond to piecewise constant regions of the pseudo-inverse function u. For example let $\mu = \delta_{x_0}$, then the corresponding pseudo-inverse variable is $u \equiv x_0$ on $[0,1)$. If $\mu = \gamma\delta_{x_1} + (1-\gamma)\delta_{x_2}$ with $\gamma \in (0,1)$, then the corresponding pseudo-inverse is $u = x_1\mathbb{1}_{[0,\gamma)} + x_2\mathbb{1}_{[\gamma,1)}$.

3. A systematic study of particle steady states

3.1. **Particle steady states.** In this section we focus on steady states for (2) which are linear combinations of Dirac's deltas, namely $\bar\mu_1, \bar\mu_2 \in \mathscr{P}(\mathbb{R}^d)$, with

$$(\bar\mu_1,\bar\mu_2) = \left(\sum_{k=1}^N m_X^k \delta_{\bar X_k}(x), \sum_{h=1}^M m_Y^h \delta_{\bar Y_h}(x)\right). \tag{16}$$

Hence, (12) implies the following steady state condition

$$\begin{cases} 0 = \displaystyle\sum_{k=1}^N \nabla S_1(\bar X_k - \bar X_i)m_X^k + \sum_{h=1}^M \nabla K(\bar Y_h - \bar X_i)m_Y^h \\ 0 = \displaystyle\sum_{h=1}^M \nabla S_2(\bar Y_h - \bar Y_j)m_Y^h - \alpha\sum_{k=1}^N \nabla K(\bar X_k - \bar Y_j)m_X^k \end{cases}, \tag{17}$$

for $i \in \{1,\dots,N\}$ and $j \in \{1,\dots.M\}$. By taking the linear combination of the first N equations in (17) with coefficients m_X^i for $i = 1,\dots,N$, since the assumptions (A1) and (A2) imply $\nabla S_1(0) = 0$, we get the necessary condition

$$\sum_{i=1}^N \sum_{h=1}^M \nabla K(\bar Y_h - \bar X_i)m_Y^h m_X^i = 0, \tag{18}$$

which is due to the fact that ∇S_1 is an odd vector field.

We notice that d out of the $d(N+M)$ conditions in (17) are redundant. Indeed, by taking the linear combination of the first N equations (in $\mathbb{R}^d$) with coefficients αm_X^i with $i = 1,\dots,N$, plus the linear combination of the final M equations (in $\mathbb{R}^d$) with coefficients $-m_Y^j$ with $j = 1,\dots,M$, we get the trivial identity $0 = 0$ in $\mathbb{R}^d$. Hence, system (17) alone is not enough to determine a unique steady state. This is not surprising, since we know that the d-dimensional quantity

$$C_\alpha = \alpha\sum_{i=1}^N m_X^i X_i - \sum_{j=1}^M m_Y^j Y_j$$

is an invariant of the time-dependent problem, and therefore one would like to produce a unique steady state once the quantity C_α is prescribed. Let us analyse this issue more in detail. Under the notation $\Omega = (X, Y) \in \mathbb{R}^{d(N+M)}$ with $X \in (\mathbb{R}^d)^N$ and $Y \in (\mathbb{R}^d)^M$, consider the nonlinear mapping $\mathcal{A} : \mathbb{R}^{d(M+N)} \to \mathbb{R}^{d(M+N)}$ defined by $\mathcal{A}(\Omega) = (a_X^1(\Omega), \dots, a_X^N(\Omega), a_Y^1(\Omega), \dots, a_Y^M(\Omega))$ and

$$a_X^i(\Omega) = \sum_{k=1}^{N} \nabla S_1(X_k - X_i) m_X^k + \sum_{h=1}^{M} \nabla K(Y_h - X_i) m_Y^h, \qquad i = 1, \dots, N,$$

$$a_Y^j(\Omega) = \sum_{h=1}^{M} \nabla S_2(Y_h - Y_j) m_Y^h - \alpha \sum_{k=1}^{N} \nabla K(X_k - Y_j) m_X^k, \qquad j = 1, \dots, M.$$

We know that

$$\alpha \sum_{i=1}^{N} m_X^i a_X^i(\Omega) - \sum_{j=1}^{M} m_Y^j a_Y^j(\Omega) = 0, \qquad \text{for all } \Omega \in \mathbb{R}^{d(M-N)}. \tag{19}$$

Since (19) is a set of d equations, this implies that the image of $\mathbb{R}^{d(M+N)}$ through $\mathcal{A}$ is a manifold of at most dimension $d(M + N - 1)$. Let us assume for simplicity that such dimension is *exactly* $d(M + N - 1)$. Set

$$\Lambda = (\alpha m_X^1 \mathbb{I}_d, \dots, \alpha m_X^N \mathbb{I}_d, -m_Y^1 \mathbb{I}_d, \dots, -m_Y^M \mathbb{I}_d) \in \mathcal{M}_{d\times d(N+M)},$$

with $\mathbb{I}_d$ being the identity matrix in $\mathcal{M}_{d\times d}$, and consider the joint system

$$\mathcal{A}(\Omega) = 0, \quad \Lambda\Omega = c, \tag{20}$$

for some $c \in \mathbb{R}^d$. Let us denote by Λ_i the i-th row of Λ, $i = 1, \dots, d$. By the implicit function theorem, a sufficient condition for (20) to be locally uniquely solvable around some steady state $\bar{\Omega}$ is that the matrix

$$\begin{pmatrix} D\mathcal{A}(\bar{\Omega}) \\ \Lambda \end{pmatrix} \tag{21}$$

has rank $d(N + M)$. Under the assumption that $\mathcal{A}$ has a rank of dimension $d(M + N - 1)$, this is equivalent to requiring that no vector in $\mathrm{span}(\Lambda_1, \dots, \Lambda_d)$ belongs to $\mathcal{R}(D\mathcal{A}(\bar{\Omega})^T) = \mathrm{Ker}(D\mathcal{A}(\bar{\Omega}))^\perp$. On the other hand, condition (19) gives

$$\Lambda_i \in \mathrm{Ker}(D\mathcal{A}(\bar{\Omega})^T) = \mathcal{R}(D\mathcal{A}(\bar{\Omega}))^\perp, \qquad \text{for all } i = 1, \dots, d.$$

Hence, by requiring the orthogonality condition

$$\mathrm{Ker}(D\mathcal{A}(\bar{\Omega})^T) \cap \mathrm{Ker}(D\mathcal{A}(\bar{\Omega}))^\perp = \{0\}, \tag{22}$$

we immediately obtain that the matrix in (21) has maximal rank, and therefore (20) is locally solvable. If condition (22) is not satisfied, then (20) is solvable only for a restricted class of constants c.

In order to deepen the understanding of the above structure, we consider some simple examples.

3.1.1. *Example: one predator and one prey.* In the case of one predator and one prey in one space dimension, i. e. with a single Dirac mass in both components, with the unique predator positioned in $\bar{X}$ and the unique prey in $\bar{Y}$. Then, condition (18) reduces to

$$K'(\bar{X} - \bar{Y}) = 0,$$

which is satisfied if and only if $\bar{X} = \bar{Y}$. Actually, this is the only condition that arises from system (17). In order to detect a unique steady state, one has to use the constraint

$$\alpha\bar{X} - \bar{Y} = c,$$

which gives, for $\alpha \neq 1$, the unique solution

$$(\bar{X}, \bar{Y}) = \left(\frac{c}{\alpha - 1}, \frac{c}{\alpha - 1}\right).$$

For $\alpha = 1$ the above solution is inconsistent, and one only has a solution if $c = 0$, in which case every solution of the form $(\bar{X}, \bar{X})$ is a steady state.

In this case the matrix $D\mathcal{A}(\bar{X}, \bar{Y})$ is given by

$$D\mathcal{A}(\bar{X}, \bar{Y}) = \begin{pmatrix} -K''(\bar{Y} - \bar{X}) & K''(\bar{Y} - \bar{X}) \\ -\alpha K''(\bar{Y} - \bar{X}) & \alpha K''(\bar{Y} - \bar{X}) \end{pmatrix},$$

and the vector $\Lambda = (\alpha, -1)$. It is immediately clear that, assuming $K''(\bar{Y} - \bar{X}) \neq 0$, we have $\mathrm{Ker}(D(\mathcal{A}(\bar{X}, \bar{Y}))^T) = [\Lambda]$, whereas $\mathrm{Ker}(D(\mathcal{A}(\bar{X}, \bar{Y}))) = [(1, 1)]$. Hence, $\mathrm{Ker}(D(\mathcal{A}(\bar{X}, \bar{Y})))^\perp = [(1, -1)]$. Since

$$[(\alpha, -1)] = [(1, -1)]$$

if and only if $\alpha = 1$, we see the consistence with the general structure seen above.

3.1.2. *One predator and two prey.* Consider now the one dimensional configuration of one predator in $\bar{X}$ and two prey in $\bar{Y}_1$ and $\bar{Y}_2$ with masses m and $1-m$ respectively, with $m \in (0, 1)$. Assume without restriction $\bar{Y}_1 < \bar{Y}_2$. Then, a necessary condition for $(\bar{X}, \bar{Y}_1, \bar{Y}_2)$ to be a steady state is

$$mK'(\bar{X} - \bar{Y}_1) = (1 - m)K'(\bar{Y}_2 - \bar{X}).$$

The above condition can be satisfied only if $\bar{Y}_1 < \bar{X} < \bar{Y}_2$, in particular, if $m = \frac{1}{2}$, thanks to the symmetry of the interaction kernel K, we have that the steady state has the following symmetry property

$$\bar{Y}_2 - \bar{X} = \bar{X} - \bar{Y}_1.$$

Using the stationary equation for $\bar{Y}_1$, one obtains the second condition

$$\frac{S_2'(\bar{Y}_2 - \bar{Y}_1)}{2} = \alpha K'\left(\frac{\bar{Y}_2 - \bar{Y}_1}{2}\right).$$

We have two conditions for three unknowns. Once again, we need to invoke the extra condition

$$\alpha \bar{X} - \frac{1}{2}(\bar{Y}_1 + \bar{Y}_2) = c.$$

With the notation above, and with the notation $K_1 = K''(\bar{Y}_1 - \bar{X})$, $K_2 = K''(\bar{Y}_2 - \bar{X})$, $B = S_2''(\bar{Y}_2 - \bar{Y}_1)$, we have

$$\mathcal{D}A(\bar{X}, \bar{Y}_1, \bar{Y}_2) = \begin{pmatrix} -mK_1 - (1 - m)K_2 & mK_1 & (1 - m)K_2 \\ -\alpha K_1 & \alpha K_1 - (1 - m)B & (1 - m)B \\ -\alpha K_2 & mB & \alpha K_2 - mB \end{pmatrix},$$

and

$$\Lambda = (\alpha, -m, -(1 - m)).$$

Assuming $\mathrm{Ker}(D\mathcal{A}(\bar{X}, \bar{Y}_1, \bar{Y}_2)) \cap (\mathrm{Ker}(D\mathcal{A}(\bar{X}, \bar{Y}_1, \bar{Y}_2))^T)^\perp \neq \{0\}$ implies, for some $\mu, \nu_1, \nu_2 \in \mathbb{R}$,

$$\mu \begin{pmatrix} \alpha \\ -m \\ -(1 - m) \end{pmatrix} = \nu_1 \begin{pmatrix} -\alpha K_1 \\ \alpha K_1 - (1 - m)B \\ (1 - m)B \end{pmatrix} + \nu_2 \begin{pmatrix} -\alpha K_2 \\ mB \\ \alpha K_2 - mB \end{pmatrix}$$

and a simple computation implies $\mu = 0$ if $\alpha = 1$. Hence, once again, a unique steady state exists if $\alpha \neq 1$ provided the matrix $D\mathcal{A}(\bar{X}, \bar{Y}_1, \bar{Y}_2)$ has rank at least two. Such condition is satisfied e.g. when

$$\alpha K_1 K_2 \neq B(K_1 m - K_2(1 - m)).$$

3.2. **A general stability condition.** We now turn to the linearised stability of the steady states under the particle system (7). For simplicity, we shall now analyse the one dimensional ODEs

$$\begin{cases} \dot{X}_i(t) = \sum_{k=1}^{N} m_X^k S_1'(X_k(t) - X_i(t)) + \sum_{h=1}^{M} m_Y^h K'(Y_h(t) - X_i(t)) \quad i = 1,..,N \\ \dot{Y}_j(t) = \sum_{h=1}^{M} m_Y^h S_2'(Y_h(t) - Y_j(t)) - \alpha \sum_{k=1}^{N} m_X^k K'(X_k(t) - Y_j(t)) \quad j = 1,..,M \end{cases}, \tag{23}$$

with

$$\sum_{k=1}^{N} m_X^k = \sum_{h=1}^{M} m_Y^h = 1.$$

A preliminary analysis on the linearised stability of stationary states for the ODE system (23) can be an useful guideline for the analysis of the continuum model.

Assume that the masses $m_X^1, \ldots, m_X^N, m_Y^1, \ldots, m_Y^M$ are fixed. Assume a steady state $\bar{\Omega} = (\bar{X}_1, \ldots, \bar{X}_N, \bar{Y}_1, \ldots, \bar{Y}_M)$ is given. We recall that the quantity

$$M_\alpha(t) = \alpha \sum_{i=1}^{N} m_X^i X_i(t) - \sum_{j=1}^{M} m_Y^j Y_j(t), \tag{24}$$

is preserved in time. We recall that the constraint

$$\alpha \sum_{i=1}^{N} m_X^i X_i - \sum_{j=1}^{M} m_Y^j Y_j = c \tag{25}$$

for some $c \in \mathbb{R}$ is needed in order to select a unique steady state.

We now analyse the stability of the above steady state for (23). The linearised system for (23) around $\bar{\Omega}$, with $\Omega(t) = \bar{\Omega} + \delta\Omega(t)$ and

$$\delta\Omega(t) = (\delta X_1, ..., \delta X_N, \delta Y_1, ..., \delta Y_M),$$

reads, for $i = 1,..N$ and $j = 1,..,M$:

$$\begin{cases} \delta\dot{X}_i = -\sum_{k=1}^{N} m_X^k S_1''(\bar{X}_i - \bar{X}_k)(\delta X_i - \delta X_k) - \sum_{h=1}^{M} m_Y^h K''(\bar{X}_i - \bar{Y}_h)(\delta X_i - \delta Y_h) \\ \delta\dot{Y}_j = -\sum_{h=1}^{M} m_Y^h S_2''(\bar{Y}_j - \bar{Y}_h)(\delta Y_j - \delta Y_h) + \alpha \sum_{k=1}^{N} m_X^k K''(\bar{Y}_j - \bar{X}_k)(\delta Y_j - \delta X_k) \end{cases}. \tag{26}$$

Clearly, the perturbed state must satisfy the constraint $M_\alpha(t) = c$. Hence

$$(\alpha m_X^1, \ldots, \alpha m_X^N, -m_Y^1, \ldots, m_Y^M) \cdot \delta\Omega = 0. \tag{27}$$

By introducing the following quantities

$$d_X^i = \left(\sum_{k=1,\, k\neq i}^{N} m_X^k S_1''(\bar{X}_i - \bar{X}_k) + \sum_{h=1}^{M} m_Y^h K''(\bar{X}_i - \bar{Y}_h) \right) \quad i = 1,..,N,$$

$$d_Y^j = \left(\sum_{h=1,\, h\neq j}^{M} m_Y^h S_2''(\bar{Y}_j - \bar{Y}_h) - \alpha \sum_{k=1}^{N} m_X^k K''(\bar{Y}_j - \bar{X}_k) \right) \quad j = 1,..,M,$$

the system (26) reads

$$
\begin{cases}
\delta\dot{X}_i = -d_X^i\delta X_i + \displaystyle\sum_{k=1,\,k\neq i}^{N} m_X^k S_1''(\bar{X}_i-\bar{X}_k)\delta X_k + \sum_{h=1}^{M} m_Y^h K''(\bar{X}_i-\bar{Y}_h)\delta Y_h \\
\delta\dot{Y}_j = -d_Y^j\delta Y_j + \displaystyle\sum_{h=1,\,h\neq j}^{M} m_Y^h S_2''(\bar{Y}_j-\bar{Y}_h)\delta Y_h - \alpha\sum_{k=1}^{N} m_X^k K''(\bar{Y}_j-\bar{X}_k)\delta X_k
\end{cases}. \tag{28}
$$

We can easily rewrite the above system (28) in matrix form. We consider the matrices:

$$
\bar{S}_1 = \left(m_X^k S_1''(\bar{X}_i-\bar{X}_k)(1-\delta_{ik})\right)_{i,k}, \quad i=1,...,N, k=1,...,N \tag{29}
$$

$$
\bar{S}_2 = \left(m_Y^h S_2''(\bar{Y}_j-\bar{Y}_h)(1-\delta_{j,h})\right)_{j,h}, \quad j=1,...,M, h=1,...,M \tag{30}
$$

$$
\bar{K}_X = \left(m_Y^h K''(\bar{X}_i-\bar{Y}_h)\right)_{i,h}, \quad i=1,...,N, h=1,...,M \tag{31}
$$

$$
\bar{K}_Y = \left(m_X^k K''(\bar{Y}_j-\bar{X}_k)\right)_{j,k}, \quad j=1,...,M, k=1,...,N, \tag{32}
$$

which collect all the second derivatives of the kernels in the steady state. Then, the perturbed system (28) can be written as

$$
\frac{d}{dt}\delta\Omega = (D+H)\,\delta\Omega,
$$

with

$$
H = \begin{pmatrix} 0 & \bar{K}_X \\ -\alpha\bar{K}_Y & 0 \end{pmatrix}, \qquad D = \begin{pmatrix} \bar{S}_1 + \mathrm{diag}\big(-d_X^i\big)_{i=1}^N & 0 \\ 0 & \bar{S}_2 + \mathrm{diag}\big(-d_Y^j\big)_{j=1}^M \end{pmatrix}.
$$

Therefore, the system (26) is stable if and only if the matrix $D+H$, restricted to the subspace (27), is stable, i. e. if all the eigenvalues corresponding to eigenvectors in (27) have non-positive real part.

On the other hand, system (28) features a more refined structure that makes the computation of the spectrum of $D+H$ much easier. For fixed masses $m_X^1,\dots,m_X^N$, $m_Y^1,\dots,m_Y^M$ with

$$
\sum_{i=1}^N m_X^i = \sum_{j=1}^M m_Y^j = 1,
$$

we consider the weighted inner product on $\mathbb{R}^{N+M}$

$$
\langle\Omega_1,\Omega_2\rangle_{2,w} = \alpha\sum_{i=1}^N m_X^i X_{1,i}X_{2,i} + \sum_{j=1}^M m_Y^j Y_{1,i}Y_{2,i},
$$

where we are using the block notation

$$
\Omega_1 = (X_1,Y_1), \qquad \Omega_2 = (X_2,Y_2)
$$

$$
X_i = (X_{i,1},\dots,X_{i,N}), \quad Y_1 = (Y_{i,1},\dots,Y_{i,M}), \quad \text{for } i=1,2.
$$

Clearly, the inner product induces the weighted norm

$$
\|\Omega\|_{2,w}^2 = \alpha\sum_{i=1}^N m_X^i X_i^2 + \sum_{j=1}^M m_Y^j Y_j^2, \qquad \Omega=(X,Y)\in\mathbb{R}^{N+M}.
$$

One can easily see that the bilinear form

$$
\langle\Omega_1, D\Omega_2\rangle_{2,w}
$$

is symmetric, whereas the bilinear form

$$
\langle\Omega_1, H\Omega_2\rangle_{2,w}
$$

is anti-symmetric. Indeed, due to the symmetry of S'', we have

$$\begin{aligned}
&\langle \Omega_1, D\Omega_2\rangle_{2,w}\\
&= -\alpha \sum_{i=1}^{N} m_X^i X_{1,i}X_{2,i}d_X^i + \alpha \sum_{i=1}^{N}\sum_{k\neq i} m_X^i m_X^k S_1''(\bar{X}_i - \bar{X}_k)X_{1,i}X_{2,k}\\
&- \sum_{j=1}^{M} m_Y^j Y_{1,j}Y_{2,j}d_Y^j + \sum_{j=1}^{M}\sum_{h\neq j} m_Y^j m_Y^h S_2''(\bar{Y}_j - \bar{Y}_h)Y_{1,j}Y_{2,h}\\
&= \langle D\Omega_1, \Omega_2\rangle_{2,w},
\end{aligned}$$

and, due to the symmetry of K'',

$$\begin{aligned}
&\langle \Omega_1, H\Omega_2\rangle_{2,w} =\\
&= \alpha \sum_{i=1}^{N}\sum_{h=1}^{M} m_X^i m_Y^h K''(\bar{X}_i - \bar{Y}_h)X_{1,i}Y_{2,h}\\
&- \alpha \sum_{j=1}^{M}\sum_{k=1}^{N} m_Y^j m_X^k K''(\bar{Y}_j - \bar{X}_k)Y_{1,j}X_{2,k} = -\langle H\Omega_1, \Omega_2\rangle_{2,w}.
\end{aligned}$$

Consequently, the perturbed system (28) satisfies the energy estimate

$$\frac{d}{dt}\|\delta\Omega\|_{2,w}^2 = \langle \delta\Omega, D\delta\Omega\rangle_{2,w} + \langle \delta\Omega, H\delta\Omega\rangle_{2,w} = \langle \delta\Omega, D\delta\Omega\rangle_{2,w}, \tag{33}$$

and only the spectrum of D plays a role in the stability of $\bar{\Omega}$. Notice that, in the weighed space $(\mathbb{R}^{d(N+M)}, \langle\cdot,\cdot\rangle_{2,w})$, D is the symmetric part of the bilinear form $D + H$.

3.3. **Examples.** We conclude this section by presenting some simple illustrative examples of how the above stability condition results into a competition between the predators' chasing ability and the prey ability to organize (e.g. via the attractive potential S_2) and to escape from the predators. A key role is played by the parameter α, which somehow rules the (normalised) velocity of the prey to escape from the predators.

3.3.1. *One predator and one prey.* Following the first example in subsection 3.1.1, let us consider the case of one particle per species, both with unit masses, i. e.

$$\begin{cases} \dot{X}(t) = K'(Y(t) - X(t)) \\ \dot{Y}(t) = -\alpha K'(X(t) - Y(t)) \end{cases}, \tag{34}$$

and let $\bar{\Omega} = \left(\bar{X}, \bar{Y}\right) \in \mathbb{R}^2$ be a steady state. The linearised equations (28) are

$$\begin{cases} \delta\dot{X} = -K''(0)\delta X + K''(0)\delta Y \\ \delta\dot{Y} = -\alpha K''(0)\delta X + \alpha K''(0)\delta Y \end{cases}. \tag{35}$$

Clearly, the perturbations δX and δY must satisfy $\alpha\delta X = \delta Y$ in order to preserve the joint center of mass.

This linear system (35) possesses the eigenvalues $\lambda_1 = 0$ e $\lambda_2 = \alpha - 1$, therefore the system is stable (but not asymptotically) if and only if $\alpha < 1$. On the other hand, by imposing the constraint $\alpha\delta X = \delta Y$, system (35) becomes

$$\delta\dot{Y} = (\alpha - 1)\, K''(0)\delta Y,$$

which is asymptotically stable if and only if $\alpha < 1$, see Figure 1. Basically, in this case the constraint on the joint center of mass forces the dynamical system (35) to evolve only along the eigenspace corresponding to the unique non-zero eigenvalue.

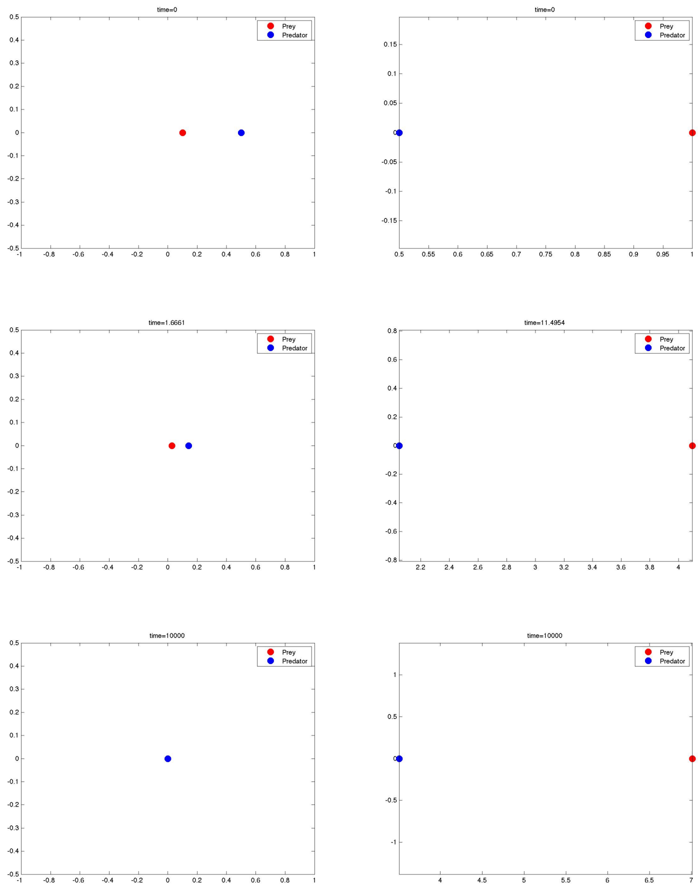

FIGURE 1. Different behavior for a one particle per species system, governed by a normalized Gaussian potential $K(x) = \frac{1}{2}(1 - e^{-x^2})$. In the first column with $\alpha = 0.2$ we have convergence to a single point. In the second column instability for $\alpha > 1$.

Notice that, in case $\alpha = 1$ the linearised system is degenerate, with both δX and δY constant in time, i. e. we have a continuum of admissible steady states, none of which is asymptotically stable.

On the other hand, apart from any linearisation, we note that in this simple case of one predator vs. one prey the nonlinear system (34) is explicitly solvable. Indeed, assume

$$M_\alpha = \alpha X - Y = c \in \mathbb{R}.$$

By substituting $\alpha X = Y - c$ into the first equation, we get

$$\dot{X}(t) = K'((\alpha - 1)X(t) - c),$$

which has the implicit solution

$$\int_{X(0)}^{X(t)} \frac{d\xi}{K'((\alpha - 1)\xi - c)} = t.$$

Now, if $\alpha = 1$ and $c \neq 0$ one has the explicit formulas

$$X(t) = X(0) + K'(-c)t, \qquad Y(t) = X(0) + K'(-c)t + c,$$

which shows the predator never catches the prey. If $\alpha = 1$ and $c = 0$, then $X(t) = Y(t) = X(0) = Y(0)$ for all $t \geq 0$. If $\alpha < 1$, we easily see that the particle $X(t)$ approaches the steady configuration $(\alpha - 1)/c$ in an infinite time, due to the fact that $K'(x)$ gets to zero linearly as $x \to 0$. If $\alpha > 1$, one easily sees that the predator $X(t)$ gets far away from the steady state. This is due to the fact that the prey does the same thing, but with a larger speed compared to the predator.

3.3.2. *One predator and two prey.* We focus now on a system with one predator with mass $m_X = 1$ and two prey with masses $m_Y^1 = m_Y^2 = \frac{1}{2}$:

$$\begin{cases} \dot{X}(t) = \frac{1}{2}\left(K'(Y_1(t) - X(t)) + K'(Y_2(t) - X(t))\right) \\ \dot{Y}_1(t) = \frac{1}{2}S_2'(Y_2(t) - Y_1(t)) - \alpha K'(X(t) - Y_1(t)) \\ \dot{Y}_2(t) = \frac{1}{2}S_2'(Y_1(t) - Y_2(t)) - \alpha K'(X(t) - Y_2(t)) \end{cases}. \tag{36}$$

Here, the necessary condition (18) is

$$K'(Y_1 - X) + K'(Y_2 - X) = 0,$$

i. e. it is just equivalent to the steady condition for the first equation in (36). As can be seen in Subsection 3.1.2, a steady state $(\bar{X}, \bar{Y}_1, \bar{Y}_2)$ in this case must satisfy

$$\bar{Y}_1 - \bar{X} = \bar{X} - \bar{Y}_2,$$

where $\bar{Y}_1 \leq \bar{Y}_2$ without restriction. This implies

$$\bar{X} = \frac{\bar{Y}_1 + \bar{Y}_2}{2}.$$

The usual constraint on the joint center of mass implies then

$$c = \alpha\bar{X} - \frac{\bar{Y}_1 + \bar{Y}_2}{2} = \frac{\alpha - 1}{2}(\bar{Y}_1 + \bar{Y}_2) = (\alpha - 1)\bar{X},$$

and that identifies $\bar{X}$. The remaining condition is

$$\frac{S_2'(\bar{Y}_2 - \bar{Y}_1)}{2} = \alpha K'\left(\frac{\bar{Y}_2 - \bar{Y}_1}{2}\right).$$

It is clear that $\bar{X} = \bar{Y}_1 = \bar{Y}_2 = \frac{c}{\alpha - 1}$ is a solution. Whether or not another solution with $\bar{Y}_2 - \bar{Y}_2 > 0$ exists, it depends on the form of the kernels S_2 and K.

The linearised equations are:

$$\delta\dot{X}(t) = -\frac{1}{2}\left(K''(\bar{Y}_1 - \bar{X}) + K''(\bar{Y}_2 - \bar{X})\right)\delta X + \frac{1}{2}K''(\bar{Y}_1 - \bar{X})\delta Y_1 + \frac{1}{2}K''(\bar{Y}_2 - \bar{X})\delta Y_2,$$

$$\delta\dot{Y}_1(t) = \left(\alpha K''(\bar{X} - \bar{Y}_1) - \frac{1}{2}S_2''(\bar{Y}_2 - \bar{Y}_1)\right)\delta Y_1 + \frac{1}{2}S_2''(\bar{Y}_2 - \bar{Y}_1)\delta Y_2 - \alpha K''(\bar{X} - \bar{Y}_1)\delta X,$$

$$\delta\dot{Y}_2(t) = \left(\alpha K''(\bar{X} - \bar{Y}_2) - \frac{1}{2}S_2''(\bar{Y}_1 - \bar{Y}_2)\right)\delta Y_2 + \frac{1}{2}S_2''(\bar{Y}_1 - \bar{Y}_2)\delta Y_1 - \alpha K''(\bar{X} - \bar{Y}_2)\delta X.$$

Let us introduce $A_i = \frac{1}{2}K''(\bar{X} - \bar{Y}_i)$, for $i = 1, 2$ and $B_2 = \frac{1}{2}S_2''(\bar{Y}_2 - \bar{Y}_1)$. As in the previous example, the conserved joint momentum M_α decouples the above 3×3 system into a 2×2 one. Indeed, the momentum constraint reads in this case

$$\alpha\delta X = \frac{1}{2}\left(\delta Y_1 + \delta Y_2\right),$$

and that gives the reduced system

$$\delta\dot{Y}_1(t) = \left((2\alpha - 1)A_1 - B_2\right)\delta Y_1 + \left(B_2 - A_1\right)\delta Y_2,$$
$$\delta\dot{Y}_2(t) = \left((2\alpha - 1)A_2 - B_2\right)\delta Y_2 + \left(B_2 - A_2\right)\delta Y_1.$$

Consider first the case $\bar{X} = \bar{Y}_1 = \bar{Y}_2$. We have in this case $A_1 = A_2 = A := \frac{1}{2}K''(0) > 0$ and $B_2 = \frac{1}{2}S_2''(0)$. Hence, we reduce to

$$\delta\dot{Y}_1(t) = ((2\alpha-1)A - B_2)\,\delta Y_1 + (B_2 - A)\,\delta Y_2,$$
$$\delta\dot{Y}_2(t) = ((2\alpha-1)A - B_2)\,\delta Y_2 + (B_2 - A)\,\delta Y_1,$$

with eigenvalues $\lambda_1 = 2A(\alpha-1)$ and $\lambda_2 = 2\alpha A - 2B_2$, hence we obtain the following sufficient stability conditions on α:

$$\alpha < 1 \qquad \alpha < \frac{B_2}{A}. \tag{37}$$

Let us now consider the possible nontrivial configuration $\bar{Y}_2 > \bar{Y}_1$. Due to the symmetry of K, we must have $\bar{X} - \bar{Y}_1 = \bar{Y}_2 - \bar{X}$, so $A_1 = A_2 = A := \frac{1}{2}K''(\bar{X} - \bar{Y}_1)$, with $B_2 = S_2''(\bar{Y}_2 - \bar{Y}_1)$ as usual. Note that steady state is determined in unique way by the symmetry and the conserved quantity

$$M_\alpha = \alpha\bar{X} - \frac{1}{2}(\bar{Y}_1 + \bar{Y}_2) = c$$

if $\alpha \neq 1$. Indeed due to the symmetry of K', we have $2\bar{X} = \bar{Y}_1 + \bar{Y}_2$, so the conserved quantity M_α is equal to $(\alpha-1)\,\bar{X}$. Hence, if $\alpha \neq 1$, we have $\bar{X} = \frac{c}{\alpha-1}$, and $\bar{Y}_1 + \bar{Y}_2 = \frac{2c}{\alpha-1}$. If $\alpha = 1$, as before there are no steady states if $c \neq 0$. In the case $\alpha \neq 1$, the remaining condition to determine uniquely the steady state is

$$S_2'(\bar{Y}_2 - \bar{Y}_1) = 2\alpha K'\left(\frac{\bar{Y}_2 - \bar{Y}_1}{2}\right).$$

This configuration yields the same expression for the eigenvalues as in the configuration $\bar{X} = \bar{Y}_1 = \bar{Y}_2$, except that $\bar{Y}_1$, A and B_2 are now depending on α, so the conditions (37) are somewhat implicit in α. We do not tackle this issue in detail at this stage. In the next example we shall prove the existence of nontrivial steady states in the special case of Gaussian interactions.

In Figure 2 we consider normalized Gaussian cross-interaction potential and self-interaction potentials is given by $S_2(x) = \frac{\beta_2}{2}(1 - e^{-x^2})$ respectively. Note that $B_2 = \frac{1}{2}\beta_2$. For such a potential only the fully aggregated steady state can be stable, whereas the separated states $\bar{Y}_2 > \bar{Y}_1$ are always unstable.

3.3.3. *Two predator and two prey.* As a last example, we consider the case of two particle per species, all the agents having mass equal to $1/2$, i. e.

$$\begin{cases} \dot{X}_1(t) = \frac{1}{2}S_1'(X_2(t) - X_1(t)) + \frac{1}{2}\left(K'(Y_1(t) - X_1(t)) + K'(Y_2(t) - X_1(t))\right) \\ \dot{X}_2(t) = \frac{1}{2}S_1'(X_1(t) - X_2(t)) + \frac{1}{2}\left(K'(Y_1(t) - X_2(t)) + K'(Y_2(t) - X_2(t))\right) \\ \dot{Y}_1(t) = \frac{1}{2}S_2'(Y_2(t) - Y_1(t)) - \frac{\alpha}{2}\left(K'(X_1(t) - Y_1(t)) + K'(X_2(t) - Y_1(t))\right) \\ \dot{Y}_2(t) = \frac{1}{2}S_2'(Y_1(t) - Y_2(t)) - \frac{\alpha}{2}\left(K'(X_1(t) - Y_2(t)) + K'(X_2(t) - Y_2(t))\right) \end{cases}. \tag{38}$$

A first trivial steady state is given by $\bar{X}_1 = \bar{X}_2 = \bar{Y}_1 = \bar{Y}_2 = \frac{c}{2(\alpha-1)}$, with c given by (25) and $\alpha \neq 1$. We shall detect an example of nontrivial steady state as follows. Assume

$$\bar{X}_1 < \bar{Y}_1 < \bar{Y}_2 < \bar{X}_2,$$

and

$$\bar{X}_2 - \bar{Y}_2 = \bar{Y}_2 - \bar{Y}_1 = \bar{Y}_1 - \bar{X}_1 = d > 0.$$

We consider for simplicity the following Gaussian cross interaction kernel

$$K(x) = -e^{-\frac{x^2}{2}}.$$

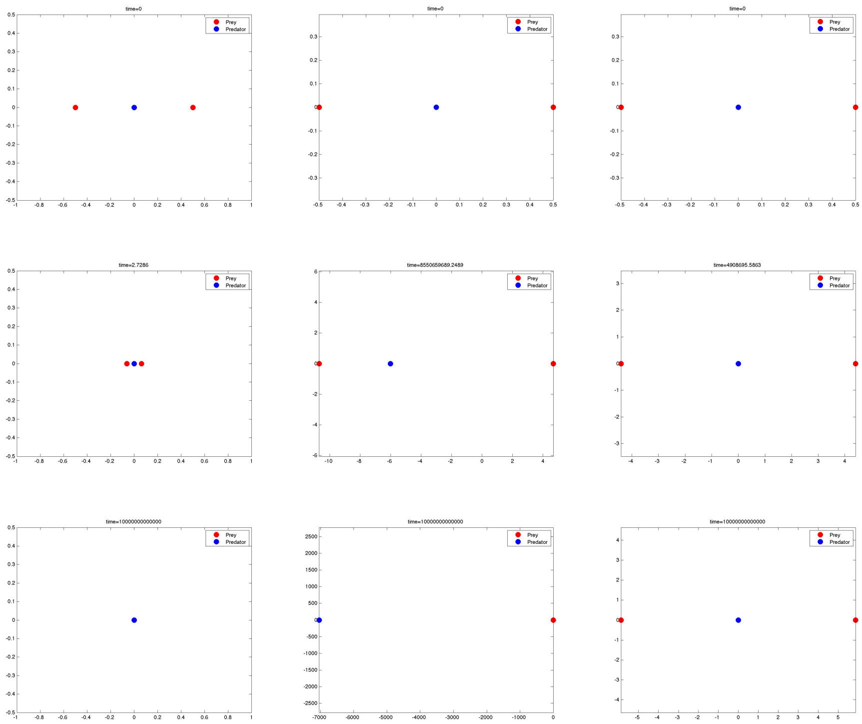

FIGURE 2. Different behavior for the 1×2 system. In the first column we can see convergence to the single steady state at the origin with $\beta_2 = 1.4$ and $\alpha = 0.3$. In second column $\beta_2 = 1$ and $\alpha = 0.5$ so the second stability condition in (37) is violated. In the last column completely unstable configuration with $\beta_2 = 1.4$ and $\alpha = 1.5$.

Let us set for simplicity

$$b_1 = \frac{1}{2}S_1'(\bar{X}_2 - \bar{X}_1), \quad b_2 = \frac{1}{2}S_2'(\bar{Y}_2 - \bar{Y}_1).$$

By requiring that both S_1 and S_2 are attractive kernels, one has $b_1, b_2 > 0$. We set

$$a_{ij} = \frac{1}{2}K'(\bar{X}_1 - \bar{Y}_j), \qquad i, j \in \{1, 2\}.$$

The stationary version of system (38) is equivalent to

$$\begin{aligned}
&a_{11} + a_{12} + a_{21} + a_{22} = 0,\\
&b_1 = \frac{1}{2}(a_{21} + a_{22} - a_{11} -_{12}),\\
&b_2 = \frac{\alpha}{2}(a_{12} + a_{22} - a_{11} - a_{21}).
\end{aligned}$$

With the above choice of K, the first equation above is automatically satisfied. The constants b_1 and b_2 are given by

$$b_1 = de^{-2d^2} + \frac{d}{2}e^{-\frac{d^2}{2}}, \qquad b_2 = \alpha\left(\frac{d}{2}e^{-\frac{d^2}{2}} - de^{-2d^2}\right).$$

Since $b_2 > 0$, one needs the following condition

$$d > \sqrt{\frac{2\log 2}{3}},$$

which allows to detect b_1 and b_2 as a function of d. We leave the computation to the reader, and we just note that by requiring e.g.

$$S_1(x) = -A_1 e^{-\frac{\sigma x^2}{2}}, \qquad S_2(x) = -A_2 e^{-\frac{\sigma x^2}{2}}, \qquad A_1, A_2, \sigma > 0,$$

a non trivial solution exists if

$$2A_1\sigma > 1, \quad \text{and} \quad \sigma > 1.$$

Note that the constraint (25) allows to fix the center of mass

$$\frac{\bar{X}_1 + \bar{X}_2}{2} = \frac{\bar{Y}_1 + \bar{Y}_2}{2} = \frac{c}{\alpha - 1}.$$

Assume now that a steady state for (38) is given, and let us linearise around it. We impose $\alpha\,(\delta X_1 + \delta X_2) = \delta Y_1 + \delta Y_2$, and we reduce to the following 3×3 system

$$\begin{aligned}
\delta\dot{X}_2 &= -\left(2B_1 + A_{21} + A_{22}\right)\delta X_2 + \left(A_{21} + \frac{1}{\alpha}B_1\right)\delta Y_1 + \left(A_{22} + \frac{1}{\alpha}B_1\right)\delta Y_2,\\
\delta\dot{Y}_1 &= \alpha\left(A_{11} - A_{21}\right)\delta X_2 + \left((\alpha-1)A_{11} + \alpha A_{21} - B_2\right)\delta Y_1 + \left(B_2 - A_{11}\right)\delta Y_2,\\
\delta\dot{Y}_1 &= \alpha\left(A_{12} - A_{22}\right)\delta X_2 + \left(B_2 - A_{12}\right)\delta Y_1 + \left((\alpha-1)A_{12} + \alpha A_{22} - B_2\right)\delta Y_2,
\end{aligned}$$

where we set $A_{ij} = \frac{1}{2}K''(\bar{X}_i - \bar{Y}_j)$, $B_1 = \frac{1}{2}S_1''(\bar{X}_1 - \bar{X}_2)$ and $B_2 = \frac{1}{2}S_2''(\bar{Y}_1 - \bar{Y}_2)$.

Let us consider only the trivial stay state, where $A_{ij} = \frac{1}{2}K''(0) = A > 0$ (we assume for simplicity that $K''(0) > 0$), $B_1 = \frac{1}{2}S_1''(0)$ and $B_2 = \frac{1}{2}S_2''(0)$. We have

$$\begin{aligned}
\delta\dot{X}_2 &= -\left(2B_1 + 2A\right)\delta X_2 + \left(A + \frac{1}{\alpha}B_1\right)\delta Y_1 + \left(A + \frac{1}{\alpha}B_1\right)\delta Y_2,\\
\delta\dot{Y}_1 &= \left((2\alpha-1)A - B_2\right)\delta Y_1 + \left(B_2 - A\right)\delta Y_2,\\
\delta\dot{Y}_1 &= \left(B_2 - A\right)\delta Y_1 + \left((2\alpha-1)A - B_2\right)\delta Y_2.
\end{aligned}$$

Since $A > 0$ the stability conditions are:

$$A + B_1 > 0, \quad \alpha < \frac{B_2}{A}, \quad \alpha < 1.$$

In Figure 3 we show some simulation for a normalized Gaussian cross-interaction potential. The self-interaction potentials are given by $S_1(x) = \frac{\beta_1}{2}(1 - e^{-x^2})$ and $S_2(x) = \frac{\beta_2}{2}(1 - e^{-x^2})$ respectively.

4. Local nonlinear stability

In this section we provide a reasonable sufficient condition for the nonlinear stability of the particle steady states considered in the previous section, when they are seen as steady states to the one-dimensional continuum system

$$\begin{cases}\partial_t\mu_1 = \partial_x(\mu_1(S_1' * \mu_1 + K' * \mu_2))\\ \partial_t\mu_2 = \partial_x(\mu_2(S_2' * \mu_2 - \alpha K' * \mu_1))\end{cases}. \tag{39}$$

More precisely, we shall perturb such steady states in a space of measure, namely the ∞-Wasserstein space of probability measures. Our analysis is inspired by the result for one species contained in [19, 20]. Consider the particle steady state

$$\bar{\boldsymbol{\mu}} = (\mu_1, \mu_2) = \left(\sum_{i=1}^{N} m_X^i \delta_{\bar{X}_i}, \sum_{j=1}^{M} m_Y^j \delta_{\bar{Y}_j}\right), \tag{40}$$

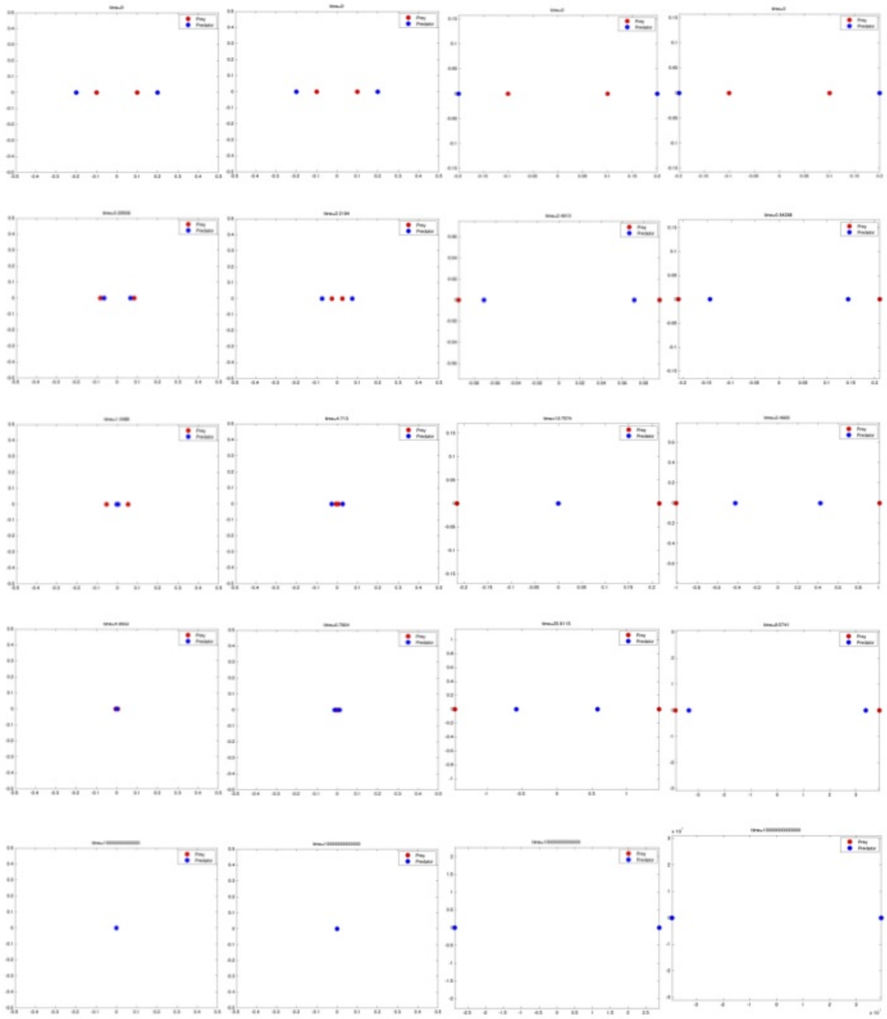

FIGURE 3. In first column convergence into the single steady state for $\alpha = 0.2$, attractive S_1, i.e. $\beta_1 > 0$ and $\beta_2 < 1$. In second column convergence with repulsive S_1, i.e. $0 > \beta_1 > -1$, $\alpha = 0.2$ and $\beta_2 < 1$. Unstable configuration for $\alpha = \beta_2 = 0.5$ and $\alpha = \beta_2 = 1$ with $0 > \beta_1 > -1$ in third and fourth columns.

Following the notation in subsection 2.3, a particle steady state for the pseudo inverse system

$$\begin{cases} \partial_t u_1(z,t) = \displaystyle\int_0^1 S_1'(u_1(\zeta,t) - u_1(z,t))d\zeta + \int_0^1 K'(u_2(\zeta,t) - u_1(z,t))d\zeta \\ \partial_t u_2(z,t) = \displaystyle\int_0^1 S_2'(u_2(\zeta,t) - u_2(z,t))d\zeta - \alpha\int_0^1 K'(u_1(\zeta,t) - u_2(z,t))d\zeta \end{cases}. \tag{41}$$

can be written in the pseudo-inverse formalism as the non-decreasing step function $\bar{u}(z)$

$$(\bar{u}_1(z), \bar{u}_2(z)) = \left(\sum_{i=1}^{N} \bar{X}_i \mathbb{1}_{I_X^i}(z), \sum_{h=1}^{M} \bar{Y}_h \mathbb{1}_{I_Y^h}(z) \right), \tag{42}$$

where the intervals above are given by

$$I_X^h = \left[\sum_{j<h} m_X^j, \sum_{j\leq m} m_X^j\right) \quad \text{with} \quad |I_X^h| = m_X^h,$$

$$I_Y^h = \left[\sum_{j<h} m_Y^j, \sum_{j\leq m} m_Y^j\right) \quad \text{with} \quad |I_Y^h| = m_Y^h.$$

We show the local non-linear stability for the stationary states (42) of the pseudo-inverse equation (41) under small L^∞-perturbations $v_i(t,z) = u_i(t,z) - \bar{u}_i(z)$ for $i = 1, 2$. Let us recall the definitions of the following quantities

$$d_X^h = \sum_{i=1}^{N} S_1''(\bar{X}_i - \bar{X}_h) m_X^i + \sum_{k=1}^{M} K''(\bar{Y}_k - \bar{X}_h) m_Y^k \quad h = 1, ..., N, \tag{43}$$

$$d_Y^h = \sum_{k=1}^{M} S_2''(\bar{Y}_k - \bar{Y}_h) m_Y^k - \alpha \sum_{i=1}^{N} K''(\bar{X}_i - \bar{Y}_h) m_X^i \quad h = 1, ..., M. \tag{44}$$

and define, for $k = 1, ..., M-1$, the following matrix

$$\mathcal{H} = \begin{pmatrix} \mathrm{diag}(d_X^h) & 0 \\ 0 & \mathrm{diag}(d_Y^k) \end{pmatrix} - \begin{pmatrix} \bar{S}_1 & \bar{K} \\ -\alpha\bar{K} & \bar{S}_2 \end{pmatrix} = -(D+H), \tag{45}$$

with D and H defied in Section 3.2. We also set

$$\Lambda = (\alpha m_X^1, \dots, \alpha m_X^N, -m_Y^1, \dots, -m_Y^M) \in \mathbb{R}^{N+M},$$

and the hyperplane

$$\mathcal{V}_\Lambda = \{\Omega \in \mathbb{R}^{M+N} \ : \ \Lambda \cdot \Omega = 0\}.$$

In order to prove the local non-linear stability, we need the following lemma of linear algebra, a proof of which can be found e.g. in [19, 20].

Lemma 4.1. *If, for all $t > 0$, a matrix $A(t) \in L^\infty(M_n(\mathbb{C}))$ satisfies $\sigma(A) \subset \{z \in \mathbb{C} | \Re(z) > \eta > 0\}$, then for any induced matrix norm $||\cdot||$, there exists a constant $C > 0$ such that for $t > 0$*

$$||e^{\int_0^t A(s)ds}|| \leq C\left(1 + t^{n-1}\right) e^{-\eta t}.$$

In particular, for $A \in M_n(\mathbb{C})$,

$$||e^{-At}|| \leq C\left(1 + t^{n-1}\right) e^{-\eta t}.$$

We are now ready to state our main result.

Theorem 4.1. *Let S_1, S_2 and K potentials under (A1), (A2) and (A3) assumptions. Consider a particle steady state (40) for system (39), and assume*

- (NS1) *d_X^h, d_Y^k in (43) and (44) are strictly positive for all $h = 1, ..., N$ and $k = 1, ..., M$;*
- (NS2) *on the hyperplane $\mathcal{V}_\Lambda$, the matrix $\mathcal{H}$ defined in (45) has strictly positive spectrum, i.e. for some $\nu > 0$, $\sigma(\mathcal{H}|_{\mathcal{V}_\Lambda}) \subset \{z \in \mathbb{C} | \Re(z) > \nu > 0\}$.*

Then there exist $C, \eta, \epsilon > 0$ such that, for all initial data $\boldsymbol{\mu}_0 = (\mu_{1,0}, \mu_{2,0})$ with compact support and with

$$W_\infty(\mu_{1,0}, \bar{\mu}_1) + W_\infty(\mu_{2,0}, \bar{\mu}_2) < \epsilon,$$

for all $t > 0$ we have

$$W_\infty(\mu_1(t), \bar{\mu}_1) + W_\infty(\mu_2(t), \bar{\mu}_2) < C\left(1 + t^{n-1}\right) e^{-\eta t}.$$

Proof. We shall work with the pseudo-inverse variables $u_1(t)$ and $u_2(t)$. We recall that, for two compactly supported probability measures μ, ν, one has

$$W_\infty(\mu,\nu) = \|u - v\|_\infty,$$

where u and v are the pseuso-inverse variables corresponding to μ and ν respectively.

The perturbations around the stationary states $u_i(t,z) = \bar{u}_i(z) + v_i(t,z)$ for $i = 1, 2$, satisfy the equations

$$\begin{aligned}
\partial_t v_1(t,z) &= \int_0^1 S_1'(\bar{u}_1(\zeta) - \bar{u}_1(z) + v_1(t,\zeta) - v_1(t,z))d\zeta \\
&+ \int_0^1 K'(\bar{u}_2(\zeta) - \bar{u}_1(z) + v_2(t,\zeta) - v_1(t,z))d\zeta, \\
\partial_t v_2(t,z) &= \int_0^1 S_2'(\bar{u}_2(\zeta) - \bar{u}_2(z) + v_2(t,\zeta) - v_2(t,z))d\zeta \\
&+ \int_0^1 K'(\bar{u}_1(\zeta) - \bar{u}_2(z) + v_1(t,\zeta) - v_2(t,z))d\zeta.
\end{aligned}$$

Thanks to a first order expansion around the stationary states and invoking the definition of the step functions $\bar{u}_i$ in (42), the equation for v_1 above reduces to

$$\begin{aligned}
&\partial_t v_1(t,z) = \sum_{i=1}^N \int_{I_X^i} S_1''(\bar{X}_i - \bar{u}_1(z)) \left(v_1^i(t,\zeta) - v_1(t,z)\right) d\zeta \\
&+ \sum_{k=1}^M \int_{I_Y^k} K''(\bar{Y}_k - \bar{u}_1(z)) \left(v_2^i(t,\zeta) - v_1(t,z)\right) d\zeta + O(\|v_1\|_\infty^{1+\gamma}) + O(\|v_2\|_\infty^{1+\gamma}) \\
&= -v_1(t,z) \left(\sum_{i=1}^N S_1''(\bar{X}_i - \bar{u}_1(z)) m_X^i + \sum_{k=1}^M K''(\bar{Y}_k - \bar{u}_1(z)) m_Y^k \right) \\
&+ \sum_{i=1}^N S_1''(\bar{X}_i - \bar{u}_1(z)) \int_{I_X^i} v_1^i(t,\zeta) d\zeta \\
&+ \sum_{k=1}^M K''(\bar{Y}_k - \bar{u}_1(z)) \int_{I_Y^k} v_2^k(t,\zeta) d\zeta + O(\|v_1\|_\infty^{1+\gamma}) + O(\|v_2\|_\infty^{1+\gamma}),
\end{aligned}$$

where $v_1^i(t,\zeta)$ denotes the restriction of $v_1(t,\zeta)$ on I_X^i and $v_2^i(t,\zeta)$ denotes the restriction of $v_2(t,\zeta)$ on I_Y^i. Taking d_X^h as defined in (43) we have

$$\begin{aligned}
\partial_t v_1^h(t,z) &= -d_X^h v_1^h(t,z) + \sum_{i=1}^N S_1''(\bar{X}_i - \bar{X}_h) \int_{I_X^i} v_1^i(t,\zeta) d\zeta \\
&+ \sum_{k=1}^M K''(\bar{Y}_k - \bar{X}_h) \int_{I_Y^k} v_2^k(t,\zeta) d\zeta + O(\|v_1\|_\infty^{1+\gamma}) + O(\|v_2\|_\infty^{1+\gamma}). \qquad (46)
\end{aligned}$$

By integrating the above equation over the intervals I_X^h, for $h = 1, ..., N$, we have

$$\begin{aligned}
\partial_t V_1^h(t) &= -V_1^h(t) d_X^h + \sum_{i=1}^N S_1''(\bar{X}_i - \bar{X}_h) m_X^h V_1^i(t) \\
&+ \sum_{k=1}^M K''(\bar{Y}_k - \bar{X}_h) m_X^h V_2^i(t) + O(\|v_1\|_\infty^{1+\gamma}) + O(\|v_2\|_\infty^{1+\gamma}),
\end{aligned}$$

with, for $h = 1, ..., N$ and $k = 1, ..., M$

$$V_1^h(t) = \int_{I_X^h} v_1^h(t,z) dz, \quad V_2^k(t) = \int_{I_Y^k} v_2^h(t,z) dz. \qquad (47)$$

Performing the same expansion for $v_2(t,z)$, for $z \in I_Y^h$, we obtain that

$$\begin{aligned}\partial_t v_2^h(t,z) &= -d_Y^h v_2^h(t,z) + \sum_{k=1}^{M} S_2''(\bar{Y}_k - \bar{Y}_h) V_2^k(t) \\ &- \alpha \sum_{i=1}^{N} K''(\bar{X}_i - \bar{Y}_h) V_1^i(t) + O(\|v_1\|_\infty^{1+\gamma}) + O(\|v_2\|_\infty^{1+\gamma}) \end{aligned} \tag{48}$$

and

$$\begin{aligned}\partial_t V_2^h(t) &= -d_Y^h V_2^h(t) + \sum_{k=1}^{M} S_2''(\bar{Y}_k - \bar{Y}_h) m_Y^h V_2^i(t) \\ &- \alpha \sum_{i=1}^{N} K''(\bar{X}_i - \bar{Y}_h) m_X^h V_1^i(t) + O(\|v_1\|_\infty^{1+\gamma}) + O(\|v_2\|_\infty^{1+\gamma}).\end{aligned}$$

with d_Y^h defined in (44). Remember that the system posses a conserved quantity $M_\alpha(t)$, that reduce to

$$\alpha \sum_{i=1}^{N} V_1^i(t) = \sum_{j=1}^{M} V_2^j(t),$$

for perturbations. This allows to eliminate one component, for example we can express $V_2^M(t)$ as a linear combination of the other masses V_1^i and V_2^j. Multiplying (46) and (48) for $\mathrm{sign}(v_1^h(t,z))$ and $\mathrm{sign}(v_2^h(t,z))$ respectively, we have

$$\begin{aligned}\partial_t |v_1^h| &= -m_1^h |v_1^h| + \mathrm{sign}(v_1^h) \sum_{i=1}^{N} S_1''(\bar{X}_i - \bar{X}_h) V_1^i \\ &+ \mathrm{sign}(v_1^h) \sum_{k=1}^{M} K''(\bar{Y}_k - \bar{X}_h) V_2^k + O(\|v_1\|_\infty^{1+\gamma}) + O(\|v_2\|_\infty^{1+\gamma})\end{aligned}$$

and

$$\begin{aligned}\partial_t |v_2^h| &= -m_2^h |v_2^h| + \mathrm{sign}(v_2^h) \sum_{k=1}^{M} S_2''(\bar{Y}_k - \bar{Y}_h) V_2^k(t) \\ &- \alpha \mathrm{sign}(v_2^h) \sum_{i=1}^{N} K''(\bar{X}_i - \bar{Y}_h) V_1^i(t) + O(\|v_1\|_\infty^{1+\gamma}) + O(\|v_2\|_\infty^{1+\gamma}).\end{aligned}$$

Introducing

$$\Omega(t) = \left(|v_1^1|, ..., |v_1^N|, |v_2^1|, ..., |v_2^M|, V_1^1, ..., V_1^N, V_2^1, .., V_2^{M-1}\right) \in \mathbb{R}^{2(N+M)-1},$$

we have,

$$\frac{d}{dt}\Omega(t) = \mathcal{A}(t)\Omega(t) + O\left(||\Omega||^{1+\gamma}\right) \quad \text{with} \quad \mathcal{A}(t) = \begin{pmatrix} -\mathcal{D} & \mathcal{O}(t) \\ 0 & -\mathcal{H} \end{pmatrix}.$$

with $\mathcal{D} = \mathrm{diag}(d_X^1, ..., d_X^N, d_Y^1, ..., d_Y^M)$, for some time dependent matrix $\mathcal{O}(t)$ which is uniformly bounded in time. Thanks to $(NS1)$, $(NS2)$, reducing the matrix $\mathcal{H}$ into an upper triangular form that we call $\mathcal{H}$ again

$$\max\{\mathcal{R}(a_{ii})\} = \max\{-d_i, -\mathcal{R}(h_{ii})\} \leq \max\{-d_i, -\nu\} < 0.$$

$\Omega(t)$ is given by

$$\Omega(t) = e^{\int_0^t \mathcal{A}(s)ds}\Omega(0) + \int_0^t e^{\int_s^t \mathcal{A}(\tau)d\tau} O\left(||\Omega||^{1+\gamma}\right)(\tau) d\tau.$$

For Lemma 4.1, calling $\eta = \max\{d_i, \nu\}$

$$\begin{aligned}||\Omega(t)||_\infty &\le C\left(1+t^{n-1}\right)e^{-\eta t}||\Omega(0)||_\infty \\ &+ C\int_0^t ||\Omega(s)||_\infty^{1+\gamma}\left(1+(t-s)^{n-1}\right)e^{-\eta(t-s)}ds,\end{aligned}$$

and a Gronwall type estimate closes the argument.

Remark 4.1. Similarly to what we have observed in particle stability analysis, only the matrix

$$D = \begin{pmatrix} \bar{S}_1 + \mathrm{diag}\big(-d_X^i\big)_{i=1}^N & 0 \\ 0 & \bar{S}_2 + \mathrm{diag}\big(-d_Y^j\big)_{j=1}^M \end{pmatrix}$$

contributes to the spectrum of the matrix $\mathcal{H}$

4.1. **Stability for a single Dirac Delta.** In this subsection we consider, as an illustrative example, the case in which the stationary states are constituted by a single delta, namely

$$(\mu_1, \mu_2) = (\delta_{\bar{x}}, \delta_{\bar{x}}) \quad (u_1, u_2) = (\bar{x}, \bar{x})\,.$$

We are going to find conditions on the kernels and on the coefficient α that ensure linear stability of the stationary states. For S_1, S_2 and K smooth potentials, the linearized equations for $u_i = \bar{x} + v_i$, $i = 1, 2$ read as

$$\begin{cases} \partial_t v_1(z,t) = -(S_1''(0) + K''(0))v_1(z,t) + S_1''(0)V_1(t) + K''(0)V_2(t) \\ \partial_t v_2(z,t) = -(S_2''(0) - \alpha K''(0))v_2(z,t) + S_2''(0)V_2(t) - \alpha K''(0)V_1(t) \end{cases}. \tag{49}$$

where V_1 and V_2 are the partial masses, defined by

$$V_i(t) = \int_0^1 v_i(z,t)dz \quad i = 1, 2.$$

We recall that the system has a conserved joint center of mass, that is equivalent to the conservation of the joint mass M_α. Therefore, the conservation of M_α yields

$$\alpha V_1 = V_2.$$

In order to study the linear stability of (49), we need a control on the terms involving V_i. An integration over $[0,1]$ provides equations for V_1 and V_2:

$$\begin{aligned}\frac{d}{dt}V_1(t) &= -(S_1''(0) + K''(0))V_1(t) + S_1''(0)V_1(t) + \alpha K''(0)V_1(t) \\ &= (\alpha - 1)K''(0)V_1(t).\end{aligned}$$

Now, with an explicit equation for V_1 (and V_2), we rewrite (49), that turns out to be a completely decoupled system of first order linear integro-differential equations:

$$\begin{cases} \partial_t v_1(z,t) = -(S_1''(0) + K''(0))v_1(z,t) + (S_1''(0) + \alpha K''(0))V_1(t) \\ \partial_t v_2(z,t) = -(S_2''(0) - \alpha K''(0))v_2(z,t) + (S_2''(0) - K''(0))\alpha V_1(t) \\ \partial_t V_1(t) = (\alpha - 1)K''(0)V_1(t) \end{cases}. \tag{50}$$

Note that, assuming $K''(0) > 0$, the equation for V_1 provides the same stability condition found in the particle case, namely $\alpha < 1$. The equations for v_1 and v_2, on the other hand, provide a non trivial condition on the combination between the self-interaction kernels and the cross-interaction one. The matrix $\mathcal{A}$ reads as

$$\mathcal{A} = \begin{pmatrix} -(S_1''(0) + K''(0)) & 0 & (S_1''(0) + \alpha K''(0)) \\ 0 & -(S_2''(0) - \alpha K''(0)) & (S_2''(0) - K''(0))\alpha \\ 0 & 0 & (\alpha - 1)K''(0) \end{pmatrix},$$

and the stability conditions are

$$\alpha < 1, \quad S_1''(0) + K''(0) > 0, \text{ and } S_2''(0) > \alpha K''(0). \tag{51}$$

We emphasize that the matrix

$$\mathcal{O} = \begin{pmatrix} S_1''(0) + \alpha K''(0) \\ (S_2''(0) - K''(0))\alpha \end{pmatrix},$$

does not contribute to the stability analysis.

4.2. **Stability of a non trivial singular state.** We conclude this section considering the following steady state

$$(\bar{u}_1, \bar{u}_2) = \left(\bar{X}, \sum_{i=1}^{2} \bar{Y}_i \mathbb{1}_{I_Y^i}(z) \right),$$

with interval's lengths given by m_Y^1 and m_Y^2 respectively. We have seen in 3.1.2 that the stationary condition for $(\bar{u}_1, \bar{u}_2)$ is

$$K'(\bar{X} - \bar{Y}_1) = \frac{m_Y^2}{m_Y^1} K'(\bar{Y}_2 - \bar{X}).$$

and choosing $m_Y^1 = m_Y^2 = \frac{1}{2}$, we have that the steady state

$$\bar{Y}_2 - \bar{X} = \bar{X} - \bar{Y}_1$$

is uniquely determined. A straightforward computation, using the conservation condition $\alpha V_1 = V_2^1 + V_2^2$, yields the following linearized equations around $(\bar{u}_1, \bar{u}_2)$:

$$\begin{aligned}
\partial_t v_1 &= -\left(S_1''(0) + \frac{1}{2} \sum_{k=1}^{2} K''(\bar{Y}_k - \bar{X}) \right) v_1 + S_1''(0) V_1 + \sum_{k=1}^{2} K''(\bar{Y}_k - \bar{X}) V_2^k, \\
\partial_t v_2^1 &= \left(\alpha K''(\bar{X} - \bar{Y}_1) - \frac{1}{2} \sum_{k=1}^{2} S_2''(\bar{Y}_k - \bar{Y}_1) \right) v_2^1 + \sum_{k=1}^{2} S_2''(\bar{Y}_k - \bar{Y}_1) V_2^k \\
&\quad - \alpha K''(\bar{X} - \bar{Y}_1) V_1, \\
\partial_t v_2^2 &= \left(\alpha K''(\bar{X} - \bar{Y}_2) - \frac{1}{2} \sum_{k=1}^{2} S_2''(\bar{Y}_k - \bar{Y}_2) \right) v_2^2 + \sum_{k=1}^{2} S_2''(\bar{Y}_k - \bar{Y}_2) V_2^k \\
&\quad - \alpha K''(\bar{X} - \bar{Y}_2) V_1, \\
\partial_t V_2^1 &= \left(\alpha K''(\bar{X} - \bar{Y}_1) - \frac{1}{2} S_2''(\bar{Y}_2 - \bar{Y}_1) \right) V_2^1 + \frac{1}{2} S_2''(\bar{Y}_2 - \bar{Y}_1) V_2^2 - \alpha K''(\bar{X} - \bar{Y}_1) V_1, \\
\partial_t V_2^2 &= \left(\alpha K''(\bar{X} - \bar{Y}_2) - \frac{1}{2} S_2''(\bar{Y}_1 - \bar{Y}_2) \right) V_2^2 + \frac{1}{2} S_2''(\bar{Y}_1 - \bar{Y}_2) V_2^1 - \alpha K''(\bar{X} - \bar{Y}_2) V_1,
\end{aligned}$$

We set $A = K''(\bar{X} - \bar{Y}_2) = K''(\bar{X} - \bar{Y}_2)$, $B_1 = S_1''(0)$ and

$$B_2 = \frac{1}{2} S_2''(\bar{Y}_2 - \bar{Y}_1), \quad \hat{B}_2 = \frac{1}{2} \left(S_2''(0) + S_2''(\bar{Y}_1 - \bar{Y}_2) \right),$$

we have

$$\begin{aligned}
\partial_t v_1 &= -(A + B_1) v_1 + \left(A + \frac{1}{\alpha} B_1 \right) V_2^1 + \left(A + \frac{1}{\alpha} B_1 \right) V_2^2, \\
\partial_t v_2^1 &= \left(\alpha A - \hat{B}_2 \right) v_2^1 + (S_2''(0) - A) V_2^1 + \left(S_2''(\bar{Y}_2 - \bar{Y}_1) - A \right) V_2^2, \\
\partial_t v_2^2 &= \left(\alpha A - \hat{B}_2 \right) v_2^2 + \left(S_2''(\bar{Y}_1 - \bar{Y}_2) - A \right) V_2^1 + (S_2''(0) - A) V_2^2, \\
\partial_t V_2^1 &= ((\alpha - 1)A - B_2) V_2^1 + (B_2 - A) V_2^2, \\
\partial_t V_2^2 &= (B_2 - A) V_2^1 + ((\alpha - 1)A - B_2) V_2^2.
\end{aligned}$$

with associated matrix

$$\mathcal{A} = \begin{pmatrix} A + B_1 & 0 & 0 & A + \frac{1}{\alpha} B_1 & A + \frac{1}{\alpha} B_1 \\ 0 & \alpha A - \hat{B}_2 & 0 & S_2''(0) - A & S_2''(\bar{Y}_2 - \bar{Y}_1) - A \\ 0 & 0 & \alpha A - \hat{B}_2 & S_2''(\bar{Y}_1 - \bar{Y}_2) - A & S_2''(0) - A \\ 0 & 0 & 0 & (\alpha - 1)A - B_2 & B_2 - A \\ 0 & 0 & 0 & B_2 - A & (\alpha - 1)A - B_2 \end{pmatrix}.$$

A simple stability analysis can be performed here as in the previous example, and one can get similar conditions to (51). We omit the details.

5. Numerical simulations in two dimensions

In the following we show some numerical simulations for system (12), in dimension $d = 2$. We start considering two species with the same number of individuals, $N = M = 200$, and the dynamic driven by normalized Gaussian potentials. In Figure 4 the self-interaction is attractive for both predators and prey. Predators, represented by blue dots, are subordinate to two attractive potentials, so they collapse very fast to the center of mass of the whole system, while prey run far away from the center, thus creating a circular pattern and, after a while, they start aggregating. Note that, after the aggregation, predators remain in the position of the center of mass.

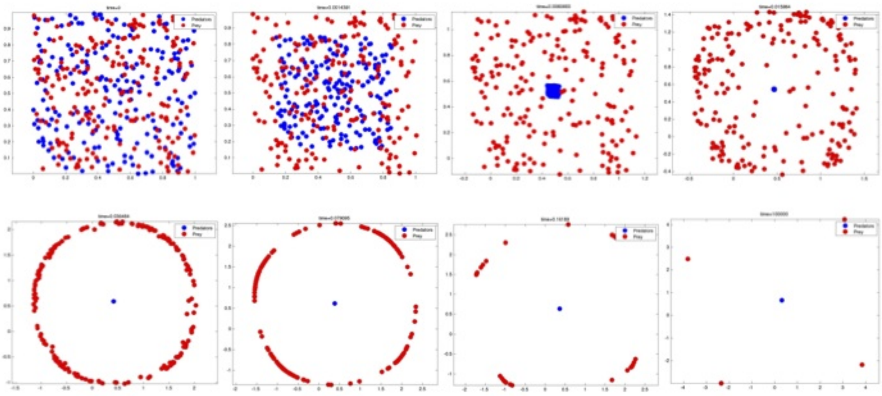

Figure 4. Normalized Gaussian self-attractive potentials for both predators and prey.

A similar situation is shown in Figure 5, where we put attractive self-interaction for the predators and repulsive self-interaction for prey. Predators behave as in the previous simulation, prey start running far away from the center but the repulsive self-interaction organize the particles in a uniform distribution.

Let us consider now $N = 10$, $M = 200$, so we have less predators than prey. Still considering in Figure 6 normalized Gaussian potentials with self-attraction for predators and self-repulsion for prey, we can see that predators start to rotating around the center of mass trying to catch the prey.

A more realistic situation can be reproduced considering different potentials. This is the case of figure Figure 7 where we consider cross-interaction potential

$$K(x) = 1 - (|x| + 1)e^{-|x|} \tag{52}$$

and the usual normalized Gaussian for the self interaction, attracting for predators and repulsive for prey.

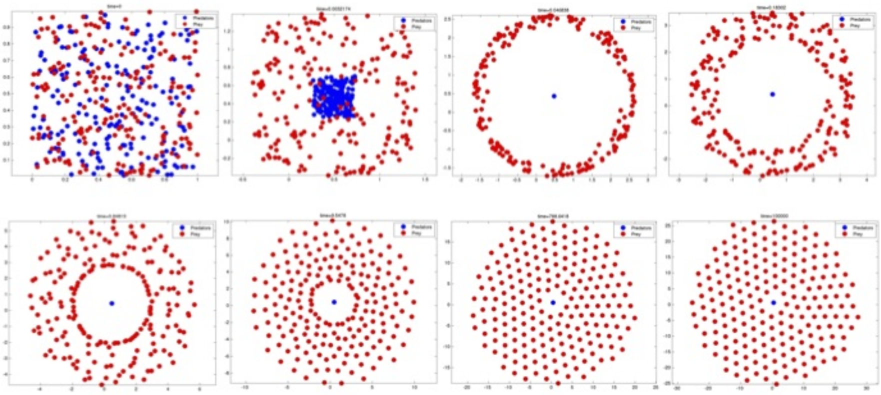

FIGURE 5. Normalized Gaussian self-attractive potentials predators and repulsive for prey.

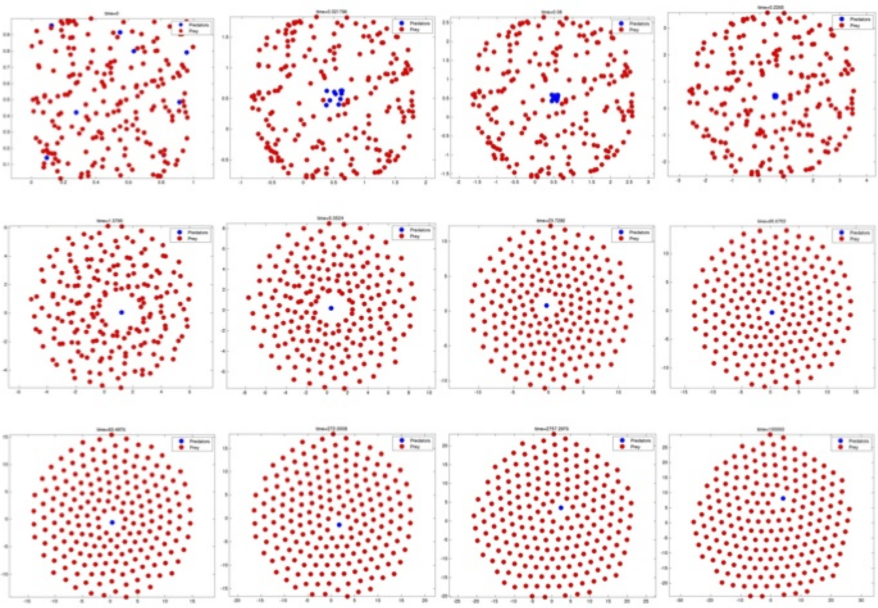

FIGURE 6. Normalized Gaussian self-attractive potentials predators and repulsive for prey with less predators then prey.

Another realistic situation is shown in Figure 8, where we consider repulsive self-interaction for prey and no self interaction for the predators. In Figure 9, we present the opposite situation with no self-interaction for the prey.

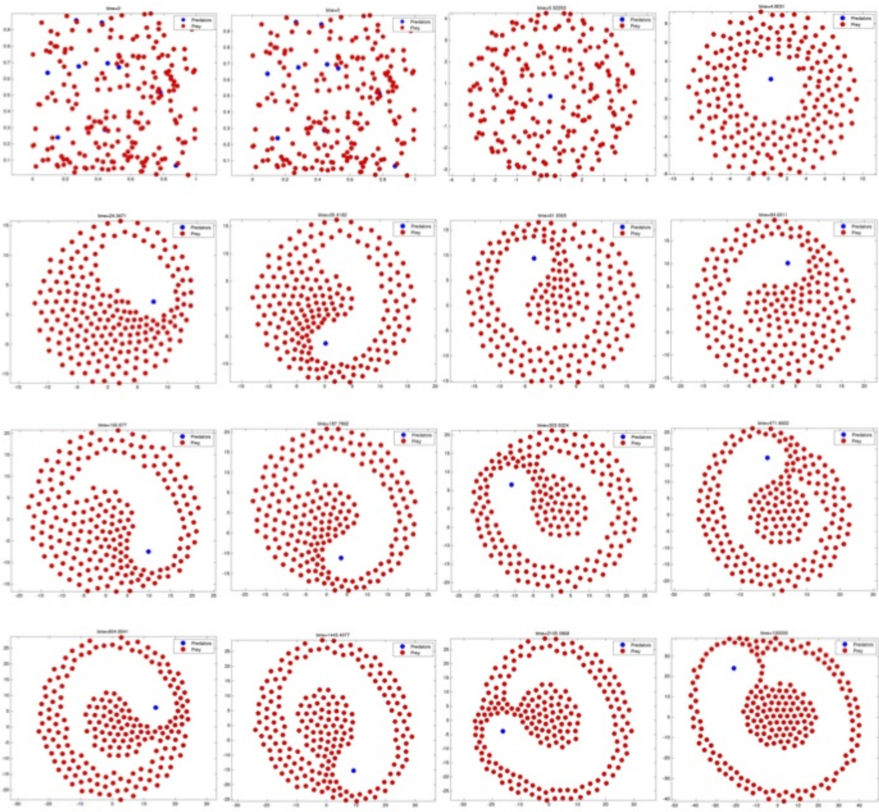

FIGURE 7. A realistic catching with cross-interaction given by (52).

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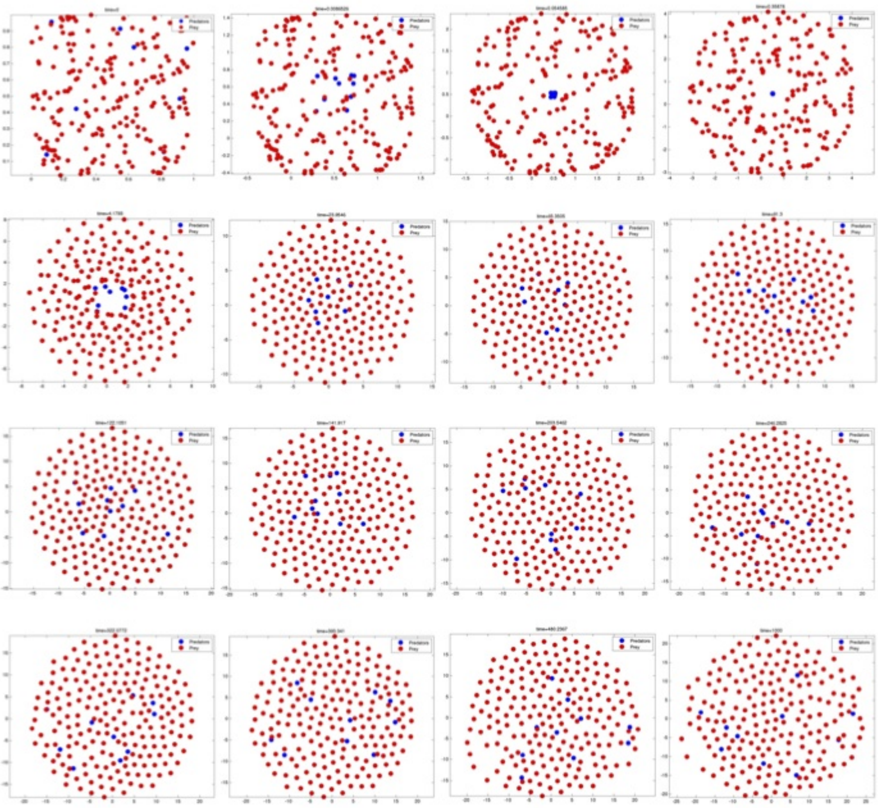

FIGURE 8. Repulsive predators and prey.

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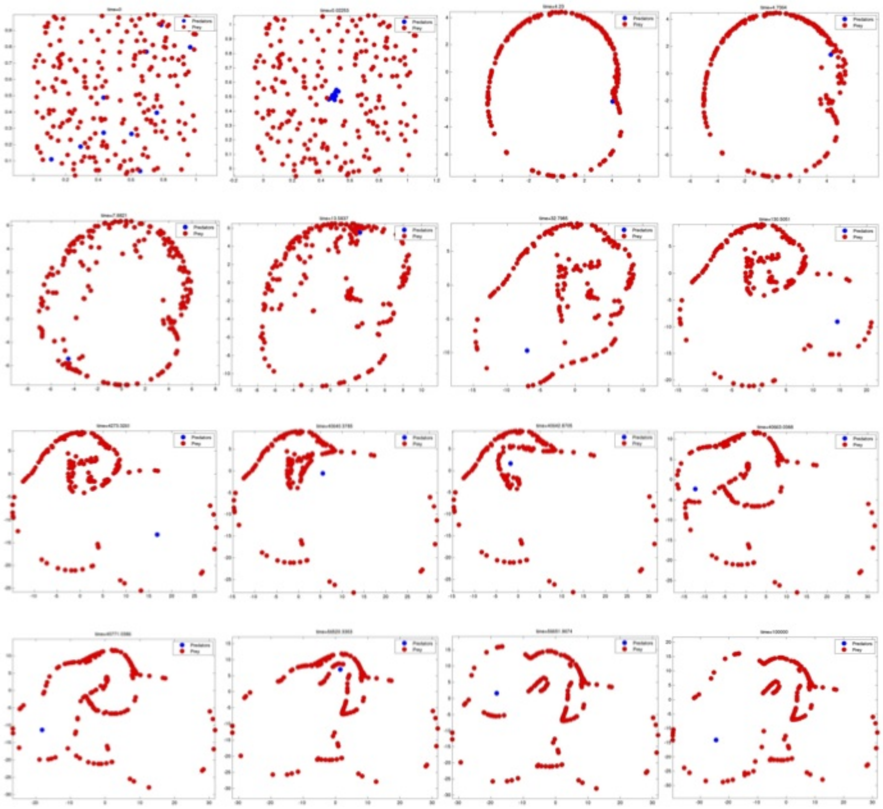

FIGURE 9. .

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