Introductory material for the Functional Analysis course

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1 Introduction

The student is supposed to be familiar with concepts of basic real analysis and linear algebra typically covered in first and second year of a BSc course. Here we shall recall some of them, and send the reader to textbooks for some others. We suggest the student less familiar with theoretical mathematical analysis to study the reading material in detail before Semester 1 starts, and to engage in some of the proposed exercises in the references recommended below.

2 Set theory

We shall assume that the reader is familiar with the basics of set theory. For a comprehensive introduction to its main concepts, we refer to [2, Section 1.1]. More in detail, the reader should be familiar with the notions of finite set, infinite set, countable set, uncountable set, which are recalled in [2, Section 1.3]. The concept of mathematical induction is also pretty crucial in the world of mathematical analysis in general. We refer to [2, Section 1.2].

3 Basic concepts of real analysis

3.1 Real numbers

We assume the reader knows the basics axioms and properties of the real line $\mathbb{R}$. They are recalled in [2, Chapter 2]. Sections 1 and 2 in [2, Chapter 2] deal with the elementary algebraic properties on the real line, which should be well known since secondary school. The material in Section 3 deals with the completeness property of the real line, and in particular with the fundamental notions of supremum (sup), infimum (inf), maximum, minimum, lower bounds, upper bounds, bounded and unbounded sets, etc. Sections 4 and 5 are also crucial, as they introduce two main properties of the real line: *density* and *uncountability*.

The notion of supremum (and infimum) of a set of real numbers is a fundamental requirement to engage in a functional analysis course. Most of the course rely on such concept. If you are not familiar with it, please study Section 2.3 of [2] in detail, and solve the exercises to that Section.

3.2 Sequences and series of real numbers

Although this may not be noticed at a first glance, sequences of real numbers will take part in our life everyday during our course. We refer to [2, Chapter 3]. The limit of a sequence is the fundamental concept of the whole mathematical analysis. The reader should be as familiar with it as with his/her own birth date. Section 1 and Section 2 recall the notion of limit of a sequence and elementary properties of limits. Section 3 deals with a fundamental subfamily of real sequences, namely monotone sequences.

An important issue is the following. Very often, in Calculus or Analysis basic courses, three topics are not covered. They are indeed crucial in Functional Analysis. They are:

- The concept of *subsequence* of a sequence.
- The Bolzano-Weierstrass Theorem.
- The notion of *compact set*.

They first two are covered in [2, Chapter 3, Section 3.4]. The notion of compactness for one dimensional sets is covered in [2, Section 11.2]. In particular, a theorem of paramount importance in Calculus, namely Heine-Borel theorem, is stated in two forms in Theorem 11.2.5 and Theorem 11.2.6 of [2].

A property which distinguishes the real line e. g. from the set of rational numbers is the *Cauchy property of sequences*. Such a property is of paramount important in Functional Analysis. It will be indeed be re-defined in a more generalised setting. In the meantime, it would help the student if such concept would be easily manageable, and therefore we recommend a careful study of [2, Section 3.5]. The notion of properly divergent sequence is fundamental. It is recalled in [2, Section 3.6].

Infinite series will be crucial in our course, e. g. in the study of Hilbert spaces. The student is required to handle the notion of converging sequence and related concepts very easily. The material in [2, Setion 3.7] contains pretty much what should be known from the theoretical point of view, but it would be better for the student to have a good practical experience with detecting the behaviour of a series (convergent, divergent, oscillating, etc.). Some important notions about infinite series are not covered in [2]. Among them we mention

- the definition of the real number e (Napier's constant),
- the product of two series,
- rearrangements.

For them we refer to [6, Chapter 3]. The reader should, of course, be extremely familiar with the definition of the number e. The notions of product of two series and rearrangements can be skipped at a first read.

3.3 Functions of one real variable

3.3.1 Continuity

The core of a basic course in Calculus or Mathematical Analysis is the study of functions of one real variable. We assume the student is very familiar with them. In particular with the concepts of

- monotonic (increasing, decreasing) functions,
- sup, inf, max, min of a function,
- limit of a function,
- continuous function.

Our recommended reference for this subject is, once again, [2], in particular Chapter 5. The reader will, in all likelihood, read this chapter quickly just to check that he/she knows everything about it. Perhaps he/she will encounter some keywords which may look not totally familiar. An example is Section 5.4 on uniform continuity. The reader should take this opportunity to get familiar with the notion of uniform continuity and with the notion of Lipschitz continuous function. The material in this section after Theorem 5.4.8 can be skipped. Section 5.5 can be skipped. Section 5.6 is important, as it introduces the notion of monotone (increasing, decreasing) functions and their basic properties.

3.3.2 Differentiation

An applicant to a Master degree in Applied Mathematics (or in Mathematical Engineering) *must be able to compute derivatives*. Period. Therefore, no reference will be provided for that. However, some theoretical properties of differentiation are sometimes skipped in basic Calculus courses. Hence, in order to avoid misunderstandings, we will assume the student is totally familiar with [6, Chapter 5]. Moreover, we strongly suggest the student to pick at random some of the exercises at the end of the chapter, and try to tackle them.

3.4 Complex numbers

Complex numbers are an essential basic tool to learn functional analysis. They are well covered in a dedicated section in [6, Chapter 1].

4 Riemann theory of integral calculus

We assume the student is familiar with the concept of Riemann integral calculus. For functions of one variable, this topic is usually covered in the first year of Bachelor studies. We refer e. g. to [5, Chapters 11 and 12], or to the lecture notes available at the link `http://www.math.klte.hu/~maksa/Riemann%20integral.pdf`. The Riemann integral in higher dimension is usually covered in the first or second year of Bachelor studies. We refer to the book [7, Chapter 11].

5 Linear algebra

Linear algebra is typically covered in the first year of a BSc course in Mathematics, Physics, or Engineering (among others). In our journey towards Functional Analysis, this is the first unit in which the student faces a *multi-dimensional* world. Linear algebra provides a sound theory in which the geometrical properties of a *finite dimensional* universe can be explored.

Linear algebra is a fundamental corner stone in learning functional analysis. **Functional analysis does not even make sense without linear algebra, as the main purpose of functional analysis is the study of vector spaces with *infinite dimensions*.**

We suggest the student to review all the study material in linear algebra. As an example, we refer here to [4] and to [3]. The list of topic below contains the *must know* material.

- Solving linear systems (Gauss elimination, linear geometry): [4, Chapter 1]
- Vector spaces, linear independence, basis and dimension: [4, Chapter 2]
- Linear transformations: [3, Chapter 6]
- Matrices: [3, Chapter 7]
- Determinants: [3, Chapter 8]
- Diagonalization, eigenfunctions, eigenvectors: [3, Chapter 12, Chapter 13, Chapter 15]
- Orthogonal bases: [3, Chapter 14]
- Kernel, range, nullity, rank: [3, Chapter 16].

The student is welcome to consider alternative references to those suggested here, as long as all the above topics are well understood and the student can handle them easily.

6 Analysis of multi-dimensional spaces

During the second year of BSc study (typically in units still labelled by Calculus, or Analysis), BSc students learn how to generalise concepts in the one-dimensional analysis (namely the one recalled in Section 3) to a multi-dimensional setting. Functions will no longer have one-dimensional intervals or half-lines as domains, but rather spherical or cylindrical domains in two or three dimensions. Here we shall recall some of them. Some notions will still be mentioned via references to suitable textbooks.

6.1 Basic topology in Euclidean spaces

In this section we collect a fundamental set of concepts needed in order to develop mathematical analysis on the linear space $\mathbb{R}^n$. These concepts are often skipped in first and second year calculus. They are typically proposed in first or second year analysis courses. They are very important, and we therefore recall them in detail.

We recall the definition of Euclidean norm, for $x = (x_1, \dots, x_d) \in \mathbb{R}^d$:

$$\|x\| = \left(\sum_{i=1}^{d} x_i^2 \right)^{1/2}.$$

Let us also recall that the Euclidean norm can be used to measure the *distance* between two points $x, y \in \mathbb{R}^d$:

$$d(x, y) = \|x - y\|.$$

The Euclidean norm is generated by the *standard scalar product*

$$(x, y) = \sum_{i=1}^{d} x_i y_i,$$

for $x = (x_1, \dots, x_d)$, $y = (y_1, \dots, y_d)$, i. e.

$$\|x\|^2 = (x, x).$$

The standard scalar product is a bilinear, symmetric form, i. e. for $x, y, z \in \mathbb{R}^d$ and $a \in \mathbb{R}$,

$$(x, y) = (y, x), \tag{1}$$

$$(ax, y) = a(x, y), \tag{2}$$

$$(x + y, z) = (x, z) + (y, z). \tag{3}$$

Exercise 6.1. Prove the properties (1), (2), and (3).

Exercise 6.2. Prove the following *Schwartz inequality.* Let $x = (x_1, \dots, x_d)$, $y = (y_1, \dots, y_d)$ be vectors in $\mathbb{R}^d$. Prove

$$|(x, y)| \leq \|x\| \|y\|.$$

Solution.
Set $c = x + \lambda y$ for a real number $\lambda \in \mathbb{R}$. By the linearity properties (1)-(2)-(3) we have

$$0 \leq \|c\|^2 = (c, c) = (x + \lambda y, x + \lambda y) = \|y\|^2 \lambda^2 + 2(x, y)\lambda + \|x\|^2.$$

Now, since the above quadratic polynomial in λ is always nonnegative, its discriminant should be non positive, i. e.

$$(x, y)^2 - \|x\|^2 \|y\|^2 \leq 0,$$

and this proves the assertion.

Exercise 6.3. Prove the following *triangular inequality*:

$$\|x + y\| \leq \|x\| + \|y\|, \tag{4}$$

for all $x, y \in \mathbb{R}^d$.
Solution.
We partially leave the solution to the student. Hint: start computing the quantity $\|x + y\|^2$, use linearity, and finally use Schwartz inequality to estimate the mixed term.

The notion of distance in $\mathbb{R}^d$ clearly allows to introduce the notion of boundedness in Euclidean spaces.

Definition 6.1. A subset $B \subset \mathbb{R}^d$ is called *bounded* if there exists $R > 0$ such that $K \subset B_R(0)$. A subset which is not bounded is called *unbounded.*

We can consider sequences $(x_n)_{n \in \mathbb{N}}$ in $\mathbb{R}^d$, i. e. maps $\mathbb{N} \ni n \mapsto x_n \in \mathbb{R}^d$. The notion of converging sequence in $\mathbb{R}^d$ can be easily stated as follows using the notion of Euclidean distance.

Definition 6.2. A sequence $(x_n)_{n \in \mathbb{N}} \subset \mathbb{R}^d$ is said to *converge* to $x \in \mathbb{R}^d$ as $n \to +\infty$ if, for all $\epsilon > 0$ there exists $M = M(\epsilon) > 0$ such that $\|x_n - x\| < \epsilon$ for all $n > M$.

Theorem 6.1 (Cauchy criterion in $\mathbb{R}^d$)**.** *A sequence $(x_n)_n \subset \mathbb{R}^d$ is convergent if and only if it has the following* Cauchy property*: for all ϵ there exists $M > 0$ such that, for all $n, m \geq M$ one has $\|x_n - x_m\| < \epsilon$.*

Proof. It is a simple consequence of the one dimensional Cauchy criterion (see [2, Section 3.5]), and is left as en exercise. □

Definition 6.3 (Open ball in $\mathbb{R}^d$)**.** Let $x_0 \in \mathbb{R}^d$ be a point, and let $R > 0$. The *open ball of radius R centered on x_0* is the set

$$B_R(x_0) = \{x \in \mathbb{R}^d\colon\ \|x - x_0\| < R\}.$$

Definition 6.4 (Open set in $\mathbb{R}^d$)**.** Let $A \subset \mathbb{R}^d$.

- A point $x_0 \in A$ is said to be an *interior point of A* if there exists an open ball B centered on x_0 with $B \subset A$.
- The set of all interior points of A is called the interior part of A, and is denoted by $\mathring{A}$.
- A is said to be an *open set* if all its points are interior points, i. e. if $A = \mathring{A}$.

Exercise 6.4 (Examples of open sets).

- On the real line $\mathbb{R}$, examples of open sets are the open intervals $(a, b) = \{x \in \mathbb{R}\colon\ a < x < b\}$, the half-lines $(-\infty, a) = \{x \in \mathbb{R}\colon\ x < a\}$, $(b, +\infty) = \{x \in \mathbb{R}\colon\ bx > b\}$.
- Open balls in $\mathbb{R}^d$ are open sets.
- Rectangles in $\mathbb{R}^d$, i. e. sets of the form

  $$(a_1, b_1) \times \ldots \times (a_d, b_d) = \{x = (x_1, \dots, x_d)\colon\ a_i < x_i < b_i \text{ for all } i = 1, \dots, d\}$$

  are open sets.
- The empty set $\emptyset \subset \mathbb{R}^d$ is trivially an open set (there is no condition to test).

- The whole space $\mathbb{R}^d$ is trivially an open set, as it contains all the open balls.

Definition 6.5 (Diameter of a set)**.** Let $B \subset \mathbb{R}^d$. We set the *diameter* of B as

$$\mathrm{diam}(B) = \sup\{|x - y| : x, y \in B\}.$$

Definition 6.6 (Closed sets in $\mathbb{R}^d$)**.** Let $C \in \mathbb{R}^d$.

- A point $x_0 \in \mathbb{R}^d$ is a called an *adherent point for* C if every open ball centered on x_0 has a non empty intersection with C.
- The set of all adherent points for C is called the *closure of* C, and is denoted by $\overline{C}$.
- C is said to be a *closed set* if all adherent points of C belongs to C, i. e. $C = \overline{C}$.

Note that all points $x \in C$ are adherent points for C. Also, note that the closure of an arbitrary set is a closed set. The *boundary of* C is the set $\partial C = \overline{C} \setminus \mathring{A}$.

Example 6.1 (Examples of closed sets).

- On the real line, examples of closed sets are closed intervals $[a, b] = \{x \in \mathbb{R} \colon a \leq x \leq b\}$, and the closed half lines $(-\infty, b] = \{x \in \mathbb{R} \colon x \leq b\}$ and $[a, +\infty) = \{x \in \mathbb{R} \colon x \geq a\}$.
- In $\mathbb{R}^d$, the set
$$\overline{B_R(x_0)} = \{x \in \mathbb{R}^d \colon \|x - x_0\| \leq R\}$$
is called the closed ball centered on x_0 with radius R. $\overline{B_R(x_0)}$ is the closure of $B_R(x_0)$, and is therefore a closed set.
- The closed rectangles $[a_1, b_1] \times \ldots \times [a_d, b_d]$ are also closed sets. They are the closure of the open rectangles considered in Example 6.1.

Open sets and closed sets share the following property.

Proposition 6.2.

- *Let $A \subset \mathbb{R}^d$ be an open set, and let $A^c = \{x \in \mathbb{R}^d,\, x \notin A\}$ be its complement. Then A^c is a closed set.*
- *Let $B \subset \mathbb{R}^d$ be a closed set, and let $B^c = \{x \in \mathbb{R}^d,\, x \notin B\}$ be its complement. Then B^c is an open set.*

Proof. Let us prove the first statement. For a given adherent point for A^c, we have to prove that $x \in A^c$. By contradiction, if $x \notin A^c$, then $x \in A$. Hence, since A is open, there exists an open ball $B_r(x)$ contained in A. This contradicts the fact that x is an adherent point for A^c.

For the second statement, let $x \in B^c$. We have to prove that there exists a ball $B_r(x)$ entirely contained in B^c. By contradiction, if such a ball does not exist, then no open ball centered on x is entirely contained in B^c. Therefore, every ball centered on x intersects B. But then x is an adherent point for B, and x is not an element of B. This contradicts that B is closed. □

A crucial property of closed sets is the following.

Proposition 6.3. *Let $C \subset \mathbb{R}^d$ be a closed subset. Let $(x_n)_{n \in \mathbb{N}} \subset C$ be a sequence contained in C, i. e. such that $x_n \in C$ for all $n \in \mathbb{N}$, such that x_n converges to some point $x \in \mathbb{R}^d$. Then $x \in C$.*

Proof. Assume by contradiction that $x \notin C$. Then, since C^c is open, there exists an open ball $B_r(x) \subset C^c$, and hence no points in $B_r(x)$ belong to C. Since all x_n belong to C, this means $x_n \notin B_r(x)$, which reads

$$\|x_n - x\| \geq r,$$

which implies that x_n cannot converge to x. This proves the desired assertion. □

Definition 6.7. Let $A \subset \mathbb{R}^d$.

- A point $x_0 \in \mathbb{R}^d$ (not necessarily in A) is called a *limit point* (or *accumulation point*) for A if every ball centered on x_0 has a non empty intersection with $A \setminus \{x_0\}$.
- A point $x_0 \in A$ is called *isolated point* in A if there exists a ball centered on x_0 having empty intersection with $A \setminus \{x_0\}$.
- A set $B \subset A$ is called a *dense subset in* A if $A \subseteq \overline{B}$. $B \subset \mathbb{R}^d$ is called *dense* if $\overline{B} = \mathbb{R}^d$.

Example 6.2. The set $\mathbb{Q}$ of rational numbers in $\mathbb{R}$ is dense. This property is easily proven as follows. Let $r \in \mathbb{R}$ be a real number. If $r \in \mathbb{Q}$, then obviously $r \in \overline{\mathbb{Q}}$. If $r \in \mathbb{R} \setminus \mathbb{Q}$, we have to prove that $B_\epsilon(r) \cap \mathbb{Q} \neq \emptyset$ for all $\epsilon > 0$. To perform this task, assume by contradiction that there exists a ball $B_\epsilon(r)$ which contains no rational numbers. This easily implies that there is a real number $s > r$ such that no rational numbers exists in the interval $[r, s]$. Now, let $n \in \mathbb{N}$ be such that $s - r > \frac{1}{n}$, and let $q \in \mathbb{N}$ be the smallest integer such that $\frac{q}{2n} > r$. By definition of q, $\frac{q-1}{2n} < r$, which gives $\frac{q}{2n} - r < \frac{1}{2n}$. Therefore, we have

$$s - \frac{q}{2n} > r + \frac{1}{n} - \frac{q}{2n} > \frac{1}{n} - \frac{1}{2n} = \frac{1}{n} > 0,$$

and this implies that $\frac{q}{2n} \in (r, s)$, which yields a contradiction as $\frac{q}{2n} \in \mathbb{Q}$.

We now recall the notion of *compactness* for a subset of $\mathbb{R}^d$.

Definition 6.8. A subset $A \subset \mathbb{R}^d$ is called a *pre-compact* subset of $\mathbb{R}^d$ if every sequence $(x_n)_{n\in\mathbb{N}} \subset A$ contained in A has a converging subsequence. A is called *compact* if every sequence $(x_n)_{n\in\mathbb{N}} \subset A$ contained in A has a converging subsequence to a limit $x \in A$.

The notion of compactness for a set can be formulated in other ways. More general definitions will be provided during the course in environments more general than the Euclidean space $\mathbb{R}^d$. In the special case of $\mathbb{R}^d$, compact sets can be characterised in the following theorem.

Theorem 6.4 (Heine-Borel)**.** *A set $K \subset \mathbb{R}^d$ is compact if and only if K is closed and bounded.*

Proof. Assume first K is closed and bounded. As K is contained in a ball, it is clearly contained in a cube $Q = [-M, M]^d$ for large enough M. Now, let $x_n = (x_n^1, \ldots, x_n^d)$ be a sequence contained in K. Clearly, the one-dimensional sequence x_n^1 is contained in $[-M, M]$. Hence, due to the one-dimensional Heine-Borel Theorem (see [2, Theorem 11.2.5]), x_n^1 has a subsequence $x_{n_k}^1$ converging to a real number x^1. Hence, the subsequence $x_{n_k} = (x_{n_k}^1, \ldots, x_{n_k}^d)$ of x_n has the first component converging to x^1. Now, the sequence $(x_{n_k}^2)_{k\in\mathbb{N}}$ has the second component $x_{n_k}^2$ contained in the interval $[-M, M]$. Therefore, we can re-apply the one-dimensional Heine-Borel Theorem to get a further subsequence of $x_{n_k}^2$ converging to some $x^2 \in \mathbb{R}$, and in the same way as above we get a subsequence of $(x_n)_{n\in\mathbb{N}}$ with the first two components converging to a limit. We repeat the same procedure for all the d components of the sequence. At the end we obtain a subsequence of $(x_n)_{n\in\mathbb{N}}$ with all components converging to a limit. $\square$

6.2 Continuity and differentiation of functions on $\mathbb{R}^d$

Multi dimensional calculus allows to generalise the notion of continuity and differentiation. These concepts are fundamental and should be well known by the student. The definition of continuous function is recalled here.

Definition 6.9 (Continuous function)**.** Let $A \subseteq \mathbb{R}^d$. A function f is said to be continuous at $x \in A$ if, for all $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|f(y) - f(x)| < \epsilon,$$

for all $y \in B_\delta(x)$.

Theorem 6.5 (Equivalent statement for continuity)**.** *A function $f : A \to \mathbb{R}$ is continuous at $x \in A$ if and only if, for all sequences $x_n \in A$ with $\lim_{n\to+\infty} x_n = x$ one has*

$$\lim_{n\to+\infty} f(x_n) = f(x).$$

Proof. Assume first that f is continuous at $x \in A$. Let $\epsilon > 0$, and let $\delta > 0$ such that $|f(y) - f(x)| < \epsilon$ for all y such that $|x - y| < \delta$. Let the sequence x_n converge to x. This means that there exists $M > 0$ such that $|x_n - x| < \delta$ for all $n > M$. Therefore, for all $n > M$ we have $|f(x_n) - f(x)| < \epsilon$, which means $f(x_n)$ converges to $f(x)$.

Conversely, assume $f(x_n)$ converges to $f(x)$ for all sequences $x_n \to x$. Assume by contradiction that f is not continuous at x. This means that there exists $\bar{\epsilon} > 0$ such that, for all $\delta > 0$ one can find y such that $|x - y| < \delta$ and $|f(y) - f(x)| \geq \bar{\epsilon}$. Let $\delta = \delta_n = 1/n$, and let $y = y_n$ such that $|x - y_n| < 1/n$ and $|f(y_n) - f(x)| \geq \bar{\epsilon}$. This means that $y_n \to x$ in the limit, but clearly $f(y_n)$ cannot converge to $f(x)$, which contradicts the assumption. $\square$

Another equivalent statement of continuity, which involves the topological properties of $\mathbb{R}^d$, is the following.

Theorem 6.6. *A function $f : \mathbb{R}^d \subseteq A \to \mathbb{R}$ is continuous if and only if, for all open sets $U \subset \mathbb{R}$ the set $f^{-1}(U)$ is open in $\mathbb{R}^d$.*

Proof. Let $U \subset \mathbb{R}$ be open. Let $x \in f^{-1}(U)$. This means that $f(x) \in U$. Since f is continuous, for all $\epsilon > 0$ there exists $\delta > 0$ such that $\|f(x) - f(y)\| \leq \epsilon$ for all $y \in A$ with $\|y - x\| \leq \delta$. Now, since U is open, let $\epsilon_0 > 0$ such that $(f(x) - \epsilon_0, f(x) + \epsilon_0) \subset U$. The continuity implies that there exists a $\delta_0 > 0$ such that $f((x - \delta_0, x + \delta_0)) \subset (f(x) - \epsilon_0, f(x) + \epsilon_0) \subset U$. This means that $(x - \delta_0, x + \delta_0) \subset f^{-1}(U)$, which means that $f^{-1}(U)$ is open. □

A definition which will be crucial in the course is the following.

Definition 6.10 (Uniformly continuous function)**.** Let $A \subseteq \mathbb{R}^d$. A function f is said to be uniformly continuous in A if, for all $\epsilon > 0$ there exists a $\delta > 0$ independent of $x, y \in A$ such that

$$|f(y) - f(x)| < \epsilon,$$

for all $x, y \in A$ with $\|x - y\| < \delta$.

Theorem 6.7 (Heine–Cantor)**.** *Let $B \subset \mathbb{R}^d$ be a compact set, and let $f : B \to \mathbb{R}$ be a continuous function. Then f is uniformly continuous.*

Proof. Assume by contradiction that f is not uniformly continuous. Hence, there exists $\epsilon_0 > 0$ such that, for all $\delta > 0$ there exist $x_\delta, y_\delta \in B$ with $\|x_\delta - y_\delta\| < \delta$ and $|f(x_\delta) - f(y_\delta)| \geq \epsilon_0$. Now, for all $n \in \mathbb{N}$, let us take $\delta = \delta_n = 1/n$, and then construct two sequences $x_{\delta_n} := x_n$ and $y_{\delta_n} := y_n$ with $\|x_n - y_n\| < \delta_n$ and $|f(x_n) - f(y_n)| \geq \epsilon_0$. Since B is compact, both sequences x_n and y_n admit converging subsequences $x_{n_k} \to x$ and $y_{n_k} \to y$ as $k \to +\infty$. Since $\|x_n - y_n\| < 1/n$, clearly $x = y$. Since f is continuous at x, from Theorem 6.5 we have $f(y_n) \to f(x)$, which contradicts with $|f(x_n) - f(y_n)| \geq \epsilon_0$.

□

Differential calculus in more than one dimension is covered in [1, Chapter V, Sections 20 and 21]. Section 20 contains basic concepts that the student must possess easily. Section 21 contains concepts that, although they might be not relevant to our course, are indeed part of a fundamental background in multi-dimensional analysis.

References

[1] Robert G. Bartle. *The elements of real analysis.* John Wiley & Sons, New York-London-Sydney, second edition, 1976. URL: https://hayanihamudi.files.wordpress.com/2014/01/the-elements-of-real-analysis-by-robert-g-bartle.pdf.

[2] Robert G. Bartle and Donald R. Sherbert. *Introduction to real analysis.* John Wiley & Sons, Inc., New York, second edition, 1992. URL: http://iuuk.mff.cuni.cz/~andrew/bartle_introduction-to-real-analysis-new-edition.pdf.

[3] David Cherney, Tom Denton, and Andrew Waldron. *Linear algebra.* URL: https://www.math.ucdavis.edu/~linear/linear-guest.pdf.

[4] Jim Hefferon. *Linear algebra.* URL: http://joshua.smcvt.edu/linearalgebra/book_ls.pdf.

[5] Jerrold Marsden and Alan Weinstein. *Calculus unlimited.* Benjamin/Cummings, 1981. URL: http://authors.library.caltech.edu/25054/1/CalcUnlimited.pdf.

[6] Walter Rudin. *Principles of mathematical analysis.* McGraw-Hill Book Co., New York-Auckland-Düsseldorf, third edition, 1976. International Series in Pure and Applied Mathematics, URL: http://dangtuanhiep.files.wordpress.com/2008/09/principles_of_mathematical_analysis_walter_rudin.pdf.

[7] H. Sagan. *Advanced calculus: of real-valued functions of a real variable and vector-valued functions of a vector variable.* Houghton Mifflin Co., 1974. URL: http://books.google.co.uk/books?id=ozLvAAAAMAAJ.