Functional Analysis in Applied Mathematics and Engineering: First Mid term exam - 09/11/2018

(1) (i) Let X be a metric space and $C(X)$ be the space of continuous functions $f : X \to \mathbb{R}$.

(a) Define the uniform norm $\| \cdot \|_\infty$ on $C(X)$. [1]

Solution: $\|f\|_\infty = \sup_{x \in X} |f(x)|$.

(b) Let $f_n \in C(X)$ and $f_n \to f$ uniformly. Prove that f is continuous. [4]

Solution: Let $x \in X$ and let $\varepsilon > 0$. We need to prove that there exists $\delta > 0$ such that $d(x, y) < \delta$ implies $|f(x) - f(y)| < \varepsilon$. Because of the uniform convergence of f_n to f, there exists $N \in \mathbb{N}$ such that $\|f_n - f\|_\infty < \varepsilon/3$ for all $n \geq N$. In particular, $\|f_N - f\|_\infty < \varepsilon/3$. Since f_N is continuous, there is a $\delta > 0$ such that $d(x, y) < \delta$ implies $|f_N(x) - f_N(y)| < \varepsilon/3$. Now, assuming $d(x, y) < \delta$ we get

$$|f(x) - f(y)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)|$$
$$\leq 2\|f_N - f\|_\infty + |f_N(x) - f_N(y)| < 2\varepsilon/3 + \varepsilon/3 = \varepsilon,$$

and the assertion is proven.

(ii) (a) State (without proof) Arzelá-Ascoli Theorem. [2]

Solution: Let (K, d) be a compact metric space and let $C(K)$ be the space of continuous functions $f : K \to \mathbb{R}$ equipped with the $\| \cdot \|_\infty$ norm. A subset $\mathcal{F} \subset C(K)$ is precompact if and only if it is bounded and equicontinuous.

(b) Let $M > 0$. Prove that the set

$\mathcal{B}_M = \{f : [0, 1] \to \mathbb{R} : f \text{ is Lipschitz, } \mathrm{Lip}(f) \leq M, \text{ and } f(0) = 0\}$

is relatively compact in $C([0, 1])$. (Hint: Use Arzelá-Ascoli Theorem) [3]

Solution: We use Arzelá-Ascoli Theorem. We need to show that $\mathcal{B}_M$ is bounded and equicontinuous. The Lipschitz condition $|f(x) - f(y)| \leq M|x - y|$ with $y = 0$ implies

$$|f(x)| \leq |f(x) - f(0)| + |f(0)| \leq M|x| \leq M$$

because $x \in [0, 1]$. Hence, $\|f\|_\infty = \sup_{x \in [0,1]} |f(x)| \leq M$ for all $f \in \mathcal{B}_M$, which means that $\mathcal{B}_M$ is bounded. Moreover, the same Lipschitz condition above implies that, for a given $\varepsilon > 0$, choosing $\delta = \varepsilon/M$ and assuming $|x - y| < \delta$ one gets

$$|f(x) - f(y)| \leq M|x - y| < M\delta = M\frac{\varepsilon}{M} = \varepsilon,$$

and therefore $\mathcal{B}_M$ is equicontinuous.

(iii) Consider the sequence of continuous functions $f_n : [0,1] \to \mathbb{R}$

$$f_n(x) = \frac{n^2x^2}{1+n^3x^3}.$$

(a) Prove that $f_n \to 0$ pointwise on $[0,1]$ as $n \to +\infty$. [2]

Solution: for $x = 0$ we get $f_n(0) = 0$, and therefore $f_n(0)$ trivially tends to zero for $n \to +\infty$. If $x \neq 0$ we have

$$0 \le \frac{n^2x^2}{1+n^3x^3} \le \frac{n^2x^2}{n^3x^3} = \frac{1}{nx} \to 0$$

as $n \to +\infty$. Therefore, by comparison we get $f_n(x) \to 0$ for all $x \in [0,1]$ as $n \to +\infty$.

(b) Does $f_n \to 0$ uniformly on $[0,1]$? Motivate your answer suitably. [2]

Solution: We need to compute $\|f_n\|_\infty = \max_{x\in[0,1]} |f_n(x)|$. Since $f_n \ge 0$, we need to find the maximum of f_n, which we may do by computing

$$f_n'(x) = \frac{2n^2x(1+n^3x^3) - 3n^3x^2n^2x^2}{(1+n^3x^3)^2} = \frac{2n^2x - n^5x^4}{(1+n^3x^3)^2} = \frac{n^2x(2-n^3x^3)}{(1+n^3x^3)^2},$$

which gives a stationary point for f_n at $x_n = 2^{1/3}/n$. Computing $f_n(x_n) = \frac{2^{2/3}}{3} > 0$, which means that $\|f_n\|_\infty$ is positive and constant with respect to n, so it cannot converge to zero. Hence, f_n does not converge to zero uniformly.

(2) (i) (a) Provide the definition of *Lebesgue integral* $\int \phi(x)\,dx$ for a *simple function* $\phi : \mathbb{R}^d \to \mathbb{R}$. [1]

Solution: Given a simple function $\varphi(x) = \sum_{i=1}^n \alpha_i \mathbf{1}_{E_i}$ for some $n \in \mathbb{N}$, some $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ without repetitions, and for some Lebesgue measurable sets $E_1, \dots, E_n$ such that $E_i \cap E_j = \emptyset$ if $i \neq j$, we get

$$\int \varphi(x)dx = \sum_{i=1}^n \alpha_i m(E_i).$$

(b) Provide the definition of *Lebesgue integral* $\int f(x)\,dx$ for a *measurable function* $f : \mathbb{R}^d \to [0, +\infty]$. [1]

Solution:

$$\int f(x)dx = \sup\left\{\int \varphi(x)dx \;:\; \varphi(x) \le f(x) \text{ for a. e. } x \in \mathbb{R}^d \text{ and } \varphi \text{ is simple}\right\}.$$

(c) Define the L^p norm for a measurable function $f : \mathbb{R}^d \to \mathbb{R}$ with $p < +\infty$. [1]

Solution: $\|f\|_{L^p} = \left(\int |f(x)|^p dx\right)^{1/p}$.

(ii) (a) State (without proof) *Fatou's Lemma.* [1]

Solution: Let $f_n : \mathbb{R}^d \to [0, +\infty]$ be a sequence of measurable functions. Then

$$\int \left(\liminf_{n \to +\infty} f_n(x) \right) dx \leq \liminf_{n \to +\infty} \int f_n(x) dx.$$

(b) Provide an example of sequence $f_n : \mathbb{R} \to \mathbb{R}$ of measurable functions such that

$$\lim_{n \to +\infty} \int_{\mathbb{R}} f_n(x)\, dx \neq \int_{\mathbb{R}} \left(\lim_{n \to +\infty} f_n(x) \right) dx.$$

[2]

Solution: Set $f_n : \mathbb{R} \to [0, +\infty)$ with $f_n(x) = \mathbf{1}_{n,n+1}(x)$. For a given $x \in \mathbb{R}$, let n be the a positive integer such that $n - 1 \geq x$. Then $f_k(x) = 0$ for all $k \geq n$, therefore $f_n(x) \to 0$ for all $x \in \mathbb{R}$. So,

$$\int \left(\liminf_{n \to +\infty} f_n(x) \right) dx = 0$$

in this case. On the other hand, for all $n \in \mathbb{N}$ we have

$$\int f_n(x) dx = \int_n^{n+1} 1 dx = 1,$$

which implies that $\liminf_{n \to +\infty} \int f_n(x) dx = 1$ in this case.

(iii) State *and* prove Hoelder's inequality. [4]

Solution: The inequality states

$$\|fg\|_{L^1} \leq \|f\|_{L^p} \|g\|_{L^q}$$

if p and q are conjugate numbers in $[1, +\infty]$. The proof in the case $p = 1$ and $q = +\infty$ follows from the estimate

$$\|fg\|_{L^1} = \int |f(x)g(x)| dx \leq \int |f(x)| \|g\|_{L^\infty} dx = \|g\|_{L^\infty} \|f\|_{L^1}.$$

In the case $p > 1$, for all $\alpha > 0$ Young's inequality implies

$$|f(x)g(x)| \leq \alpha |f(x)| \frac{|g(x)|}{\alpha} \leq \frac{\alpha^p |f(x)|^p}{p} + \frac{|g(x)|^p}{q \alpha^q}.$$

Integrating we get

$$\|fg\|_{L^1} \leq \frac{\alpha^p}{q} \|f\|_{L^p}^p + \frac{1}{q \alpha^q} \|g\|_{L^q}^q.$$

Optimizing in $\alpha > 0$ the above right hand side we obtain that the minimum value is achieved at

$$\alpha = \frac{\|g\|_{L^q}^{1/p}}{\|f\|_{L^p}^{1/q}}.$$

We then get

$$\|fg\|_{L^1} \leq \frac{1}{q}\|f\|_{L^p}^{p-p/q}\|g\|_{L^q} + \frac{1}{q}\|g\|_{L^q}^{q-q/p}\|f\|_{L^p}.$$

Since p and q are conjugate, we get $p - p/q = p(1-1/q) = p/p = 1$ and $q - q/p = q(1-1/p) = q/q = 1$, which implies the assertion.

(iv) Let $f : (0,+\infty) \to \mathbb{R}$ be defined by

$$f(x) = \frac{\arctan x}{\sqrt{x}}.$$

Find all $p \in [1,+\infty]$ such that $f \in L^p((0,+\infty))$. Motivate your answer. [4]

Solution: We have to find all $p \in [1,+\infty]$ such that

$$\int_0^{+\infty} \frac{(\arctan x)^p}{x^{p/2}} dx < +\infty.$$

We immediately notice that the above integrand is bounded near $x = 0$. Indeed,

$$\frac{(\arctan x)^p}{x^{p/2}} \simeq x^{p/2} \qquad \text{as } n \to 0,$$

which follows by a simple first order Taylor expansion of the arctan function. Therefore, the problem is only affected by the behavior of the integrand at $+\infty$. Now, if $p > 2$, since $\arctan x \leq \pi/2$, we have

$$\begin{aligned}
&\int_1^{+\infty} \frac{(\arctan x)^p}{x^{p/2}} dx \leq \int_1^{+\infty} \frac{\pi^p}{2^p x^{p/2}} dx \\
&= \frac{\pi^p}{2^p} \int_1^{+\infty} x^{-p/2} dx = \frac{\pi^p}{2^p} \lim_{R\to+\infty} \frac{1}{p/2-1}\left(1 - R^{1-p/2}\right) \\
&= \frac{\pi^p}{2^p} \frac{1}{p/2-1} < +\infty.
\end{aligned}$$

On the other hand, if $p \in [1,2]$, we know that $\arctan x \geq \pi/4$ for all $x \geq 1$, which implies

$$\int_1^{+\infty} \frac{(\arctan x)^p}{x^{p/2}} dx \geq \frac{\pi^p}{4^p} \int_1^{+\infty} x^{-p/2} dx,$$

which is infinite because $-p/2 \geq\leq -1$. Therefore, the answer is $p > 2$.