Functional Analysis in Applied Mathematics and Engineering:

Model Mid term exam

(1) (i) Let K be a compact metric space. Prove that the space $C(K)$ of continuous functions on K equipped with the sup norm is complete. [4]

(ii) State (without proof) the contraction mapping theorem. [2]

(iii) Let $f_n \in C([0,1])$ be such that $\|f_n\|_\infty \leq 1$. Set

$$F_n(x) = \int_0^x f_n(y)dy.$$

Show that the sequence F_n has a subsequence that converges uniformly on $[0,1]$. [3]

(iv) Let $f_n : [0,+\infty) \to \mathbb{R}$ defined by $f_n(x) = x^n$. Find the maximal subset of $\mathbb{R}$ on which f_n converges pointwise. Find the subsets of $\mathbb{R}$ on which f_n converges uniformly. [5]

(2) (i) Given a measurable function $f : \mathbb{R}^d \to [0,+\infty]$, provide the definition of the Lebesgue integral of f. [2]

(ii) Prove that if $f \geq 0$ almost everywhere on $\mathbb{R}^d$ then

$$m(\{x \in \mathbb{R}^d \ : \ f(x) \geq \lambda\}) \leq \frac{1}{\lambda} \int f(x)dx.$$

[4]

(iii) Let $f, g \in L^p(\mathbb{R}^d)$ with $p \in [1,+\infty)$. Prove that

$$\|f+g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}.$$

[5]

(iv) Let $p \in [1,+\infty)$. Determine for which $\alpha \in (0,+\infty)$ the function $f : [0,1] \to \mathbb{R}$ with $f(x) = x^{-\alpha}$ belongs to $L^p([0,1])$. [3]