Functional Analysis in Applied Mathematics and Engineering: Model Mid term exam

(1) (i) Let X be a set and $d : X \times X \to \mathbb{R}$. Say when d is called a *distance* on X. [2]

(ii) Let Y be a finite set, let $k \in \mathbb{N}$, and

$$X = Y^k = \{x = (x_1, \ldots, x_k) \ : \ x_i \in Y \ \text{ for all } i = 1, \ldots, k\}\,.$$

Set $d : X \times X \to \mathbb{N}$ as $d(x, y) = \#(\{j \in \{1, \ldots, k\} \ : \ x_j \neq y_j\})$. Prove that d is a distance on X. [4]

(iii) Let $(X, \|\cdot\|)$ be a Banach space and let $B \subset X$ be a closed subset. Prove that B is a complete metric space with the induced distance $d(x, y) = \|x - y\|$. [3]

(iv) Let C be a subset of a metric space.

(a) Say when C is called a sequentially compact set. [1]

(b) Say when C is called a totally bounded set. [1]

(c) Prove that every sequentially compact set is totally bounded. [3]

(2) (i) Let $X = C([0, 1])$ be the linear space of continuous functions $f : [0, 1] \to \mathbb{R}$ and let f_n be a sequence in X.

(a) Say when f_n is said to converge *pointwise* to f. [1]

(b) Say when f_n is said to converge *uniformly* to f. [1]

(c) Prove that if f_n converges uniformly to f then it converges pointwise to f. [1]

(ii) State (without proof) Arzelá-Ascoli's Theorem. [2]

(iii) Let $X = C([a, b])$ be the space of continuous functions $f : [a, b] \to \mathbb{R}$ equipped with the infinity norm $\|\cdot\|_\infty$. Let $B = \{f \in X \ : \ f \text{ is strictly increasing on } [a, b]\}$. Answer the following questions by proving all your statements.

(a) Prove that B is not closed. [2]

(b) Find the set $\overline{B}$. [2]

(c) Is B dense in X? [1]

(iv) Let $f_n : [-2, 3] \to \mathbb{R}$ be the sequence defined by

$$f_n(x) = \frac{nx^2}{5 + n^4 x^4}.$$

Answer the following questions by proving all your statements.

(a) Does f_n converge pointwise to some $f \in C([-2, 3])$? [2]

(b) Does f_n converge to f uniformly? [2]