Functional Analysis in Applied Mathematics and Engineering:

Mid term exam

Friday 4 December 2014

Solution

(1) (i) Provide the definition of Lebesgue measurable set $E \subset \mathbb{R}^d$. [2]

Solution.

A bounded set $E \subset \mathbb{R}^d$ is called Lebesgue measurable if the inner measure

$$m_*(E) = \sup\{m(K)\,,\ k \subset E\,,\ K \text{ compact}\}$$

and the outer measure

$$m^*(E) = \inf\{m(A)\,,\ A \supset E\,,\ A \text{ open}\}$$

coincide. An unbounded set $E \subset \mathbb{R}^d$ is called Lebesgue measurable if $E \cap B_R(0)$ is Lebesgue measurable according to the previous definition for all $R \geq 0$.

(ii) Say which of the following statements are true. Provide a short motivation for each answer (not more than two lines!).

(a) All sets $E \subset \mathbb{R}^d$ are Lebesgue measurable. [1]

Solution.

False. The Vitali set is an example of a non measurable set in $\mathbb{R}$. It can be constructed thanks to the axiom of choice.

(b) All countable sets are Lebesgue measurable with measure zero. [1]

Solution.

True. A countable set C can be written as the countable union of points, and points have zero measure. The countable additivity property of Lebesgue measure prove the assertion.

(c) All measurable sets with zero Lebesgue measure are at most countable. [1]

Solution.

False. The Cantor set is uncountable, and yet it has zero measure.

(d) If $m^*(E) = 0$ then E is Lebesgue measurable. [1]

Solution.

True. Since $0 \leq m_*(E) \leq m^*(E)$ for all sets $E \subset \mathbb{R}^d$, we have

$$0 \leq m_*(E) \leq m^*(E) = 0$$

which implies $m_*(E) = m^*(E) = 0$ and the set E is measurable.

(iii) Prove that a bounded set $E \subset \mathbb{R}^d$ is Lebesgue measurable if and only if for all $\varepsilon > 0$ there exists an open set $U \subset \mathbb{R}^d$ with $U \supset E$ and $m^*(U \setminus E) < \varepsilon$. [6]

Solution.

Assume first E Lebesgue measurable. The characterization of measurable sets with open and compact sets yields, for all $\varepsilon > 0$, the existence of a compact set K and an open set A such that $K \subset E \subset A$ and $m(A) - m(K) \leq \varepsilon$. In particular, a simple monotonicity property implies $m(A \setminus E) \leq \varepsilon$, and since $A \setminus E$ is measurable (A is open), we have $m^*(A \setminus E) \leq \varepsilon$.

Assume now that for all $\varepsilon > 0$ there exists an open set $A \subset \mathbb{R}^d$ with $A \supset E$ and $m^*(A \setminus E) < \varepsilon$. For all $n \in \mathbb{N}$, we choose $\varepsilon = 1/n$, and get an open set $A_n \supset E$ such that $m^*(A_n \setminus E) \leq 1/n$. Now let $A := \bigcap_{n \in \mathbb{N}} A_n$. As intersection of countably many measurable sets, A is measurable. Since $A \setminus E \subset A_n \setminus E$ for all n, we clearly have

$$m^*(A \setminus E) \leq m^*(A_n \setminus E) \leq 1/n.$$

By arbitrariness of n (no dependency on n occurs on the left hand side), we have $m^*(A \setminus E) = 0$, and therefore $A \setminus E$ is Lebesgue measurable. Hence, $E = A \setminus (A \setminus E)$ is measurable.

(2) (i) Let $f : \mathbb{R}^d \to [0, +\infty]$ be a measurable function.

(a) Provide the definition of Lebesgue integral $\int_{\mathbb{R}^d} f(x)dx$. [1,5]

Solution.

Lebesgue' integral of $f \geq 0$ is defined as

$$\int f dx = \sup \left\{ \int \phi dx \,,\ 0 \leq \phi \leq f \,,\ \phi \text{ simple function} \right\} .$$

(b) State (without proof) *Markov*'s inequality. [1,5]

Solution.

Given $\lambda > 0$ and $f : \mathbb{R}^d \to [0, +\infty]$ measurable, then

$$m(\{x \mid f(x) \geq \lambda\}) \leq \frac{1}{\lambda} \int f dx \,.$$

(c) Show that if $\int_{\mathbb{R}^d} f(x)dx < +\infty$ then f is finite almost everywhere. [3]

Solution.

Let

$$E_\infty = \{x \in \mathbb{R}^d \mid f(x) = +\infty\} \,,$$

and for all $M \in \mathbb{N}$

$$E_M = \{x \in \mathbb{R}^d \mid f(x) \geq M\}\,.$$

Markov's inequality implies

$$m(E_M) \leq \frac{1}{M} \int_{\mathbb{R}^d} f(x)dx$$

and the assumption on the integral of f implies $m(E_M) \to 0$ as $M \to +\infty$. Now, since $E_{M+1} \subset E_M$ for all $M \in \mathbb{N}$, and since $m(E_1) < +\infty$ (once again in view of Markov's inequality), the continuity of Lebesgue measure implies

$$m(E_\infty) = m(\bigcap_M E_M) = \lim_{M \to +\infty} m(E_M) = 0\,.$$

Hence, f is finite outside E_∞ which has measure zero, i. e. f is finite almost everywhere.

(ii) State (without proof) *Lebesgue's dominated convergence* theorem. [3]

Solution.

Let $f_n : \mathbb{R}^d \to \mathbb{R}$ be a sequence of measurable functions converging almost everywhere to some measurable function $f : \mathbb{R}^d \to \mathbb{R}$. Assume there exists a nonnegative measurable function $g : \mathbb{R}^d \to \mathbb{R}$ with $\int_{\mathbb{R}^d} g(x)dx < +\infty$ and such that $|f_n(x)| \leq g(x)$ for almost every $x \in \mathbb{R}^d$. Then,

$$\int f(x)dx = \lim_{n \to +\infty} \int f_n(x)dx.$$

(iii) Find an example of a sequence of measurable functions $f_n : \mathbb{R} \to \mathbb{R}$ such that

$$\int_{\mathbb{R}^d} \left(\lim_{n \to +\infty} f_n(x) \right) dx < \lim_{n \to +\infty} \int_{\mathbb{R}^d} f_n(x)dx\,.$$

[3]

Solution.

There are many examples. One of them is

$$f_n(x) = \mathbf{1}_{[n,n+1]}(x)$$

for which one can easily see that

$$\lim_{n \to +\infty} f_n(x) = 0\,, \qquad \text{for all } x \in \mathbb{R}$$

and

$$\int f_n(x)dx = 1\,, \qquad \text{for all } n \in \mathbb{N}\,.$$

Hence

$$0 = \int 0dx < \int f_n(x)dx = 1\,.$$

(3) Let $(E, \|\cdot\|)$ be a normed space.

(i) Provide the definition of the weak topology $\sigma(E, E^*)$. [2]

Solution.

Let E^* be the dual space of E, i. e the space of all continuous linear functionals on E. For all $f \in E^*$ we set

$$\varphi_f(x) = \langle f, x \rangle \, .$$

Then, $\sigma(E, E^*)$ is the inverse limit topology of the family of maps $\{\varphi_f\}_{f \in E^*}$, i. e. the coarsest topology on E that makes all functionals in E^* continuous.

(ii) Let $x_0 \in E$. Let O be an open neighborhood or x_0 in the weak topology $\sigma(E, E^*)$. Explain in *two lines* why there exist $\varepsilon > 0$, $N \in \mathbb{N}$, and $f_1, \dots, f_N \in E^*$ such that

$$O \supset \{x \in E \, : \, |\langle f_i, x - x_0 \rangle| < \varepsilon \, , \; i = 1, \dots, N\}.$$

[2]

Solution.

The family of sets of the form $\{x \in E \, : \, |\langle f_i, x - x_0 \rangle| < \varepsilon \, , \; i = 1, \dots, N\}$, with $\varepsilon > 0$, $N \in \mathbb{N}$, and $f_1, \dots, f_N \in E^*$, is a base of neighborhoods for x_0 in the weak topology. This is easily understood by recalling that the inverse limit topology can be constructed by taking arbitrary unions of finite intersections of inverse images via φ_f of an open interval in $\mathbb{R}$.

(iii) Let $S = \{x \in E \, : \, \|x\| = 1\}$. Prove that if E is infinite dimensional then S is not closed in $\sigma(E, E^*)$. [8]

Solution.

Let $x_0 \in E$ be such that $\|x_0\| < 1$. For some $\varepsilon > 0$, $N \in \mathbb{N}$, and $f_1, \dots, f_N \in E^*$, let

$$V = \{x \in E \, | \, |\langle f_i, x \rangle| < \varepsilon \, , \text{ for all } i = 1, \dots, N\}$$

be an arbitrary basic open neighborhood of x_0 in $\sigma(E, E^*)$. We prove that $V \cap S \neq \emptyset$, which will imply that $x_0 \notin S$ belongs to the weak closure of S, and hence S is not weakly closed. We claim that there exists $y_0 \in E$, $y_0 \neq 0$ such that $\langle f_i, y_0 \rangle = 0$ for all $i = 1, \dots, N$. Assuming that the claim is false, we define the map $\Phi : E \to \mathbb{R}^N$ as

$$\Phi(x) = (\langle f_1, x \rangle, \dots, \langle f_N, x \rangle) \, .$$

If the claim is false then Φ is a linear isomorphism between E and $\Phi(E)$, which implies E is finite dimensional, and that is a contradiction. Now, let $G : [0, +\infty) \to$

$[0,+\infty)$ defined by

$$g(t) = \|x_0 + ty_0\| .$$

g is clearly continuous, with $g(0) < 1$ and $g(t) \to +\infty$ as $t \to +\infty$. Therefore, there exists some t_0 with $g(t_0) = \|x_0 + t_0 y_0\| = 1$. We compute, for all $i = 1, \ldots, N$,

$$\langle f_i, x_0 + t_0 y_0 \rangle = \langle f_i, x_0 \rangle ,$$

which implies $x_0 + t_0 y_0 \in V$. On the other hand, $x_0 + t_0 y_0 \in S$, i. e. $V \cap S \neq \emptyset$, which proves the assertion.