Functional Analysis in Applied Mathematics and Engineering:

Second Mid term exam

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Full name:____

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(1) (a) State (without proof) Hölder's inequality in L^p spaces. [2]

Solution.

Let $p, q \in [1, +\infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$ (with the convention that $1/+\infty = 0$). Let $f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$. Then,

$$\|fg\|_{L^1(\mathbb{R}^d)} \leq \|f\|_{L^p(\mathbb{R}^d)} \|g\|_{L^q(\mathbb{R}^d)} \,.$$

(b) Let $p, q, r \in [1, +\infty]$ be such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ and let $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$. Prove that

$$\|fg\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q} \,.$$

[4]

Solution.

From the assumptions we have

$$1 = \frac{r}{p} + \frac{r}{q} \,,$$

hence p/r and q/r are conjugate exponents. Therefore we can apply Hölder's inequality to $|f|^r$ and $|g|^r$:

$$\|fg\|^r_{L^r(\mathbb{R}^d)} = \||f|^r |g|^r\|_{L^1} \leq \||f|^r\|_{L^{p/r}} \||g|^r\|_{L^{q/r}} = \|f\|^r_{L^p} \|g\|^r_{L^q} \,,$$

and that implies the assertion by taking the $1/r$ power above.

(c) Say which of the following statements are true and justify your answer shortly (two lines for each statement):

(i) Every uniformly bounded sequence in L^2 admits a converging subsequence in the weak L^2 topology. [1]

Solution.

True. L^2 is a reflexive space, so the unit ball is compact in the weak topology by Kakutani's theorem, and the weak compactness is sequential due to the separability of the subspace generated by the sequence itself.

(ii) The sequence $f_n(x) = n\mathbf{1}_{[0,\frac{1}{n}]}$ is uniformly bounded in $L^2(\mathbb{R})$. [1]

Solution.

False. Compute

$$\|f_n\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} f_n(x)^2 dx = \int_0^{1/n} n^2 dx = n \xrightarrow[n\to+\infty]{} +\infty\,,$$

so the sequence is not uniformly bounded in L^2.

(iii) The space $L^\infty(\mathbb{R})$ is separable. [1]

Solution.

False. It is possible to construct an uncountable family of disjoint open balls with unit radius. Hence there can be no dense countable subset.

(d) Prove that f_n in point (ii) does not converge to zero weakly in $L^1(\mathbb{R})$. [3]

Solution.

Weak convergence to zero in L^1 is equivalent to

$$\int_{\mathbb{R}} f_n(x)g(x)dx \xrightarrow[n\to+\infty]{} 0 \quad \text{for all } g \in L^\infty(\mathbb{R}).$$

Take $g \equiv 1 \in L^\infty(\mathbb{R})$. We get

$$\int_{\mathbb{R}} f_n(x)g(x)dx = \int_0^{1/n} n dx = 1 \nrightarrow 0 \qquad \text{as } n \to +\infty\,.$$

(2) Let H be a Hilbert space.

(a) For $x, y \in H$, prove that

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2\,.$$

[3]

Solution.

Compute

$$\begin{aligned}
\|x+y\|^2 + \|x-y\|^2 &= \|x\|^2 + \|y\|^2 + 2\mathrm{Re}(x,y) + \|x\|^2 + \|y\|^2 + 2\mathrm{Re}(x,-y) \\
&= \|x\|^2 + \|y\|^2 + 2\mathrm{Re}(x,y) + \|x\|^2 + \|y\|^2 - 2\mathrm{Re}(x,y) \\
&= 2\|x\|^2 + 2\|y\|^2\,.
\end{aligned}$$

(b) Let $M \subset H$ be non-empty, closed, and convex. Let $x_0 \in H$. Prove that there exists a unique $y_0 \in M$ such that

$$\|x_0 - y_0\| = \inf_{y\in M} \|x_0 - y\|\,.$$

[4]

Solution.

The statement is trivial is $x_0 \in M$. Let $x_0 \notin M$. Set

$$d = \operatorname{dist}(x_0, M) = \inf_{y \in M} \|x_0 - y\|,$$

where $d > 0$ is a consequence of M being closed. From the properties of inf, there exists a sequence $\{y_n\}_n \subset M$ such that $\|y_n - x_0\| \to d$. Use (a) with the vectors $x = x_0 - y_n$ and $y = x_0 - y_m$, for $n, m \in \mathbb{N}$. We get

$$\|2x_0 - y_n - y_m\|^2 + \|y_n - y_m\|^2 = 2\|x_0 - y_n\|^2 + 2\|x_0 - y_m\|^2.$$

Now, we have

$$\|2x_0 - y_n - y_m\|^2 = 4 \left\| x_0 - \frac{(y_m + y_m)}{2} \right\|^2,$$

and since M is convex, the mid-point of the segment connecting y_n and y_m, i. e. $\frac{(y_m+y_m)}{2}$, belongs to M. Therefore,

$$\|2x_0 - y_n - y_m\|^2 \geq 4d^2.$$

Hence, we get

$$4d^2 + \|y_n - y_m\|^2 \leq 2\|x_0 - y_n\|^2 + 2\|x_0 - y_m\|^2 \xrightarrow[n,m \to +\infty]{} 4d^2,$$

and this implies that $\|y_n - y_m\| \xrightarrow[n,m\to+\infty]{} 0$, i. e. $\{y_n\}_n$ is a Cauchy sequence. Since H is complete, let $y_0 = \lim_{n\to+\infty} y_n$. By continuity of the norm, we get $\|x_0 - y_0\| = d$, and y_0 is the desired point in M. Uniqueness: assume y_0, y_0' are two points in M with the same property. We can use (a) to get

$$4d^2 + \|y_0 - y_0'\|^2 \leq 2\|x_0 - y_0\|^2 + 2\|x_0 - y_0'\|^2,$$

and this implies $\|y_0 - y_0'\| = 0$, i. e. $y_0 = y_0'$ as desired.

(c) State (without proof) Riesz' representation Theorem. [2]

Solution.

Let $f \in H^*$ be a linear and continuous functional on H. Then, there exists a unique $z \in H$ such that

(1) $$\langle f, x \rangle = (x, z), \qquad \text{for all } x \in H.$$

The map $\sigma : H^* \ni f \mapsto z \in H$ is a bijection of H^* onto H, it is an isometry, i. e. $\|\sigma(f)\|_H = \|f\|_{H^*}$, and it is anti-linear, i. e. $\sigma(f + \lambda g) = \sigma(f) + \overline{\lambda}\sigma(g)$ for all $f, g \in H^*$ and all $\lambda \in \mathbb{C}$.

(d) Provide an example of an infinite dimensional separable Hilbert space. Provide an example of a countable orthonormal base for such a space. [3]

Solution.

For instance one can take the sequence space ℓ^2 with the inner product

$$(x, y)_{\ell^2} = \sum_{i=1}^{+\infty} x_i y_i \,.$$

Such a space is a separable Hilbert space. A countable orthonormal base is $\{e_i\}_i$ with

$$(e_i)_n = \begin{cases} 1 & \text{if } n = i \\ 0 & \text{otherwise}\,. \end{cases}$$

(3) (a) Let E, F be Banach spaces. Provide the definition of *compact* linear operator $T : E \to F$. [1]

Solution.

A linear operator $T : E \to F$ is called *compact* if $T(B_E)$ has compact closure in the strong topology of F. Here B_E is the closed unit ball in E.

(b) Explain the meaning of the expression $\lambda \in \sigma(T)$ (λ is an element of the spectrum of T). [1]

Solution.

$\lambda \in \sigma(T)$ means that $\lambda\mathbb{I} - T$ does not have a continuous inverse.

(c) Let $T : E \to F$ be linear and compact and let $\lambda_n \in \sigma(T) \setminus \{0\}$ with $\lambda_n \to \lambda \in \mathbb{R}$ as $n \to +\infty$. Prove that $\lambda = 0$. [6]

Solution.

We know from a previous result that the λ_n's are eigenvalues. Let $e_n \neq 0$ such that $(A - \lambda_n\mathbb{I})e_n = 0$. Let $E_n = \text{span}[e_1, \ldots, e_n]$. Clearly $E_n \subset E_{n+1}$. We claim that $E_n \neq E_{n+1}$ for all n. To see that, it suffices to show that the vectors e_n are all linearly independent. We prove that by induction on n. Assume this holds up to n and suppose $e_{n+1} = \sum_{i=1}^{n} \alpha_i e_i$. Then

$$A(e_{n+1}) = \sum_{i=1}^{n} \alpha_i A(e_i) = \sum_{i=1}^{n} \alpha_i \lambda_i e_i.$$

On the other hand,

$$A(e_{n+1}) = \lambda_{n+1} e_{n+1} = \sum_{i=1}^{n} \lambda_{n+1} \alpha_i e_i.$$

The two above identities imply $\alpha_i(\lambda_i - \lambda_{n+1}) = 0$ for all $i = 1, \ldots, n$ since $e_1, \ldots, e_n$ are linearly independent. Since the eigenvalues are all distinct, we have $\alpha_i = 0$ for all $i = 1, \ldots, n$, a contradiction with $\lambda_{n+1} \neq 0$. By Riesz's lemma, we construct a sequence $u_n \in E$ such that $u_n \in E_n$ for all $n \in \mathbb{N}$, $\|u_n\| = 1$, and $\text{dist}(u_n, E_{n-1}) \geq 1/2$ for all $n \geq 2$. For $2 \leq m < n$ we have

$$E_{m-1} \subset E_m \subset E_{n-1} \subset E_n.$$

On the other hand, $(A - \lambda_n\mathbb{I})E_n \subset E_{n-1}$. Indeed, let $y \in (A - \lambda_n\mathbb{I})E_n$, i. e. $y = (A - \lambda_n\mathbb{I})(g_{n-1} + \alpha e_n)$ for some $g_{n-1} \in E_{n-1}$. We have

$$y = (A - \lambda_n\mathbb{I})(g_{n-1}) + \alpha(A - \lambda_n\mathbb{I})(e_n) = (A - \lambda_n\mathbb{I})(g_{n-1}),$$

and the last term above is an element of E_{n-1} (because $A(g_{n-1})$ is a linear combination of vectors in E_{n-1}). Therefore, we can write

$$\begin{aligned}&\left\|\frac{A(u_n)}{\lambda_n}-\frac{A(u_m)}{\lambda_m}\right\|\\&=\left\|\frac{(A(u_n)-\lambda_n u_n)}{\lambda_n}-\frac{(A(u_m)-\lambda_m u_m)}{\lambda_m}+u_n-u_m\right\|\geq \operatorname{dist}(u_n,E_{n-1})\geq 1/2.\end{aligned}$$

Now, assume by contradiction that $\lambda_n\to\lambda\neq 0$. Then, $\{A(u_n)\}_n$ has a convergent subsequence because A is a compact operator. Therefore, $\frac{A(u_n)}{\lambda_n}$ has a convergent subsequence too, but that contradicts the above inequality.

(d) Consider the Volterra integral operator $T: L^2([0,1])\to L^2([0,1])$

$$(Tf)(x)=\int_0^x f(y)dy\,.$$

Prove that $\lambda=1$ is not an eigenvalue of T. [4]

Solution.

Assume $\lambda=1$ is an eigenvalues. Then, there exists $f\in L^2([0,1])$ such that

$$\int_0^x f(z)dz=f(x)\,,\qquad \text{for all } x\in[0,1]\,.$$

Hence, for $x,y\in[0,1]$ with $x\leq y$ one has

$$|f(x)-f(y)|=\left|\int_0^x f(z)dz-\int_0^y f(z)dz\right|\leq\int_x^y|f(z)|dz$$

and Hölder's inequality implies

$$\int_x^y|f(z)|dz\leq\left(\int_x^y|f(z)|^2dz\right)^{1/2}\left(\int_x^y dz\right)^{1/2}\leq\|f\|_{L^2}|x-y|^{1/2}\,.$$

Hence, f is continuous. But then, the assumption $\int_0^x f(z)dz=f(x)$ and the fundamental theorem of calculus imply that f has a continuous derivative. By differentiating the latter expression on $[0,1]$ we get

$$f'(x)=f(x)\,,$$

and clearly

$$f(0)=\int_0^0 f(x)dx=0\,.$$

Hence, the solution of the differential equation above gives

$$f(x)=f(0)e^x=0\,,$$

and f cannot be an eigenvector.