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An example of an immersed complete genus one minimal surface in $\mathbb{R}^3$ with two convex ends

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Abstract. We prove the existence of a compact genus one immersed minimal surface M, whose boundary is the union of two immersed locally convex curves lying in parallel planes. M is a part of a complete minimal surface with two finite total curvature ends.

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1. Introduction

In 1978 Meeks conjectured that a connected minimal surface bounded by two convex curves in two parallel planes is topologically an annulus; hence it has genus zero. The conjecture has never been proved and the most general result, due to Schoen, is the following.

Let $\Gamma = \Gamma_1 \cup \Gamma_2$ be any boundary consisting of two Jordan curves in parallel planes; assume that Γ is invariant by reflection through two planes P_1, P_2 orthogonal to the planes of the Γ_i and that both P_1 and P_2 divide Γ into pieces which are graphs with locally bounded slope over the dividing plane. Then any minimal surface spanning Γ is topologically an annulus and is an embedded surface meeting each parallel plane between the planes of the Γ_i in smooth Jordan curves.

In particular, if Γ_1 and Γ_2 are circles such that the line joining their centers is perpendicular to the planes in which they lie, then M is a catenoid (cf. [Sc]).

In 1991, Meeks and White studied the space of minimal annuli bounded by convex curves in parallel planes (cf. [MW]).

In this paper we prove the existence of a compact genus one immersed minimal surface M, whose boundary is the union of two immersed locally convex curves lying in parallel planes. In fact M is a part of a complete minimal surface with two finite total curvature ends.

The method we use to construct our surface is the following.

It is well known that a minimal surface of genus g and k ends can be described by its Weierstrass representation, that is a triple $\{\overline{R}\setminus[p_1,\dots,p_k],\omega=fdz,g\}$, where $\overline{R}$ is a compact Riemann surface of genus g, $p_1,\dots,p_k$ are points in $\overline{R}$, ω is a holomorphic differential on R and g is a meromorphic function on R.

In our setting $\overline{R}$ is a torus, so we can choose f and g to be elliptic functions. For references about the use of elliptic functions in the Weierstrass representation, see [A], [A1], [C], [C1], [R]).

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2. Statement of results

Consider the lattice $L(1,i)$ on $\mathbb{C}$ generated by 1 and i and let T^2 be the torus $\mathbb{C}/L(1,i)$. Let $\pi:\mathbb{C}\longrightarrow T^2$ be the standard projection to the quotient and set $p_o=\pi(0), p_1=\pi(\frac{1}{2}), p_2=\pi(\frac{1+i}{2})$ and $p_3=\pi(\frac{i}{2})$. Finally, let $\wp$ be the Weierstrass function associated to the lattice $L(1,i)$ and $\wp'$ its derivative.

Theorem 2.1. *Let $f,g:T^2\setminus\{p_o,p_2\}\longrightarrow\mathbb{C}$ be the two meromorphic functions defined by*

$$f=\wp^2 \quad g=\frac{\alpha\wp'}{\wp^3}$$

where α is a real constant depending only on $L(1,i)$ and $\wp$.

Then $\{T^2\setminus[p_o,p_2],fdz,g\}$ is the Weierstrass representation of a complete genus one immersed minimal surface M with finite total curvature.

Remark 2.2. The ends of M cannot be embedded. In fact, if a complete finite total curvature minimal surface has two embedded ends, it is a catenoid (cf. [Sc]).

The functions f and g extend meromorphically to T^2 and we have $g(p_o)=0$ and $g(p_2)=\infty$. Hence the limit normal vector at both ends of M is vertical. Then we have the following result.

Theorem 2.3. *There exists a positive constant $c\in\mathbb{R}$ such that $M\cap\{|x_3|\le c\}$ is a compact genus one immersed minimal surface having the property that each of the boundary curves $M\cap\{x_3=\pm c\}$ is a compact locally convex immersed curve.*

3. Proof of the theorems

We list some useful classical properties of the function $\wp$ (cf. [B], [WW]).

By abuse of notation, we often identify points of $\mathbb{C}$ with points of T^2. Let $'$ be the differentiation with respect to the variable $z\in\mathbb{C}$.

(i) $\wp$ is even and $\wp'$ is odd. We have $\wp(z),\ \wp'(z) \in \mathbb{R}$ when $z \in \mathbb{R}$, $\wp(p_1) = e_1 \in \mathbb{R}^*_+$, $\wp(p_2) = 0$ and $\wp(p_3) = -e_1$.

The following identities hold:

(ii) $(\wp')^2 = 4\wp(\wp^2 - e_1^2)$, $\wp'' = 2(3\wp^2 - e_1^2)$.

(iii) $\wp(z+p_1) = \dfrac{e_1(\wp(z)+e_1)}{\wp(z)-e_1}$, $\wp(z+p_3) = \dfrac{e_1(\wp(z)-e_1)}{\wp(z)+e_1}$, $\wp(z+p_2) = -\dfrac{e_1^2}{\wp(z)}$.

(iv) $\wp'(z+p_2) = e_1^2 \dfrac{\wp'(z)}{\wp(z)^2}$.

(v) $\wp(iz) = -\wp(z)$, $\wp'(iz) = i\wp'(z)$.

(vi) The local expansion of $\wp$ and $\wp'$ around p_o is

$$\wp(z) = \frac{1}{z^2} + \frac{e_1^2}{5} z^2 + O(z^6),$$

$$\wp'(z) = -\frac{2}{z^3} + \frac{2e_1^2}{5} z + O(z^5).$$

Proof of Theorem 2.1. It is sufficient to prove that the following conditions are satisfied.

(A) z is a pole of order m of $g \Longleftrightarrow z$ is a zero of order $2m$ of f.

(B) $\int_\gamma (1+|g|^2)|f| = \infty$ for every divergent path γ in M.

(C) $\mathrm{Re} \int_\gamma fg = 0$ and $\int_\gamma fg^2 = \overline{\int_\gamma f}$ for every closed path in M.

Zeros and poles of f, g, fg, fg^2 in a fundamental region are as in figure 1.

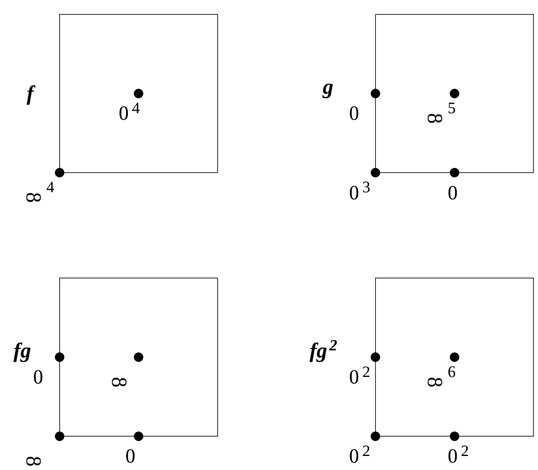

Figure 1.

The function g does not have poles in $T^2 \setminus \{p_o, p_2\}$, hence condition (A) is satisfied.

The expression of the metric on M in terms of $\wp$ is

$$ds = \left(1 + \alpha^2 \frac{|\wp'|^2}{|\wp|^6}\right) |\wp|^2$$

hence the metric is complete at the ends and condition (B) is satisfied.

We must verify (C) on paths that are not homologous to 0 in $T^2 \setminus \{p_o, p_2\}$, i.e. paths around p_o and p_2 and paths that generate the homology of T^2. Denote by $\alpha(p_o)$ and $\alpha(p_2)$ any closed path around p_o and p_2 respectively, and by γ_1 and γ_2 the following paths generating the homology of T^2 :

$$\gamma_1(t) = \frac{i}{4} + t \quad t \in [0,1]$$

$$\gamma_2(t) = \frac{1}{4} + it \quad t \in [0,1]$$

The functions f and fg^2 are even, so they have no residue at p_o, i.e.

$$\int_{\alpha(p_o)} fg^2 = \int_{\alpha(p_o)} f = 0$$

Furthermore

$$\text{Re} \int_{\alpha(p_o)} fg = \text{Re} \int_{\alpha(p_o)} \frac{\alpha\wp'}{\wp} = \text{Re} \left[\text{Res}_{p_o}(2\pi i\alpha \frac{\wp'}{\wp})\right]$$

By the local expansion of $\wp$ and $\wp'$ around 0 we have that $\text{Res}_{p_o}(2\pi i\alpha\frac{\wp'}{\wp}) = -4\pi i\alpha$, hence for $\alpha \in \mathbb{R}$ we have

$$\text{Re} \int_{\alpha(p_o)} fg = 0$$

By (iii) and (iv) we have

$$f(z + p_2) = \frac{e_1^4}{\wp^2(z)},$$

$$fg^2(z + p_2) = \frac{\alpha^2}{e_1^4}(\wp'(z))^2.$$

Hence $f(z + p_2)$ and $fg^2(z + p_2)$ are even functions of z and this gives

$$\int_{\alpha(p_2)} fg^2 = \int_{\alpha(p_2)} f = 0.$$

By (iii) and (iv) we have

$$fg(z+p_2) = -\alpha\frac{\wp'(z)}{\wp(z)}.$$

Hence, by the computation above, for $\alpha \in \mathbb{R}$ we have

$$\operatorname{Re}\int_{\alpha(p_2)} fg = 0.$$

Now we verify (C) over γ_1 and γ_2. We have

$$\operatorname{Re}\int_{\gamma_i} fg = \operatorname{Re}\int_{\gamma_i} \alpha\frac{\wp'}{\wp} = \alpha[\ln|\wp|]_{\gamma_i(0)}^{\gamma_i(1)} = 0$$

by periodicity of $\wp$, as α is real.

Integral of f over γ_1 : by Cauchy theorem and periodicity we can move γ_1 up to the segment from p_3 to p_3+1, hence

$$\int_{\gamma_1} f = \int_0^1 f(p_3+t)dt = \int_0^1 e_1^2\frac{(\wp(t)-e_1)^2}{(\wp(t)+e_1)^2}dt$$

where the last equality is given by (iii).

Integral of f over γ_2 : we can move γ_2 to the vertical segment from p_1 to p_1+i, hence by (iii) and (iv)

$$\int_{\gamma_2} f = \int_0^1 f(p_1+t)idt = i\int_0^1 e_1^2\frac{(\wp(t)-e_1)^2}{(\wp(t)+e_1)^2}dt.$$

Integral of fg^2 over γ_1 : we can move γ_1 down to the real segment from p_o to p_o+1, hence

$$\int_{\gamma_1} fg^2 = \int_0^1 f(t)g^2(t)dt = \int_0^1 \alpha^2\frac{\wp'(t)^2}{\wp(t)^4}dt.$$

Integral of fg^2 over γ_2 : we can move γ_2 to the vertical segment from p_o to p_o+i, hence

$$\int_{\gamma_2} fg^2 = \int_0^1 f(it)g^2(it)idt = -i\int_0^1 \alpha^2\frac{\wp'(t)^2}{\wp(t)^4}dt.$$

Then α must satisfy

$$\alpha^2\int_0^1 \frac{\wp'(t)^2}{\wp(t)^4}dt = \int_0^1 e_1^2\frac{(\wp(t)-e_1)^2}{(\wp(t)+e_1)^2}dt.$$

If $t \in \mathbb{R}$ we have $\wp(t)$, $\wp'(t) \in \mathbb{R}$, hence the two integrals involved in the definition of α are positive real numbers. Furthermore they are convergent, so $\alpha \in \mathbb{R}$.

Since g and f extend meromorphically to T^2, M has finite total curvature. □

Before proving Theorem 2.3 we need the following lemma.

Lemma 3.1. *Consider a minimal surface M with Weierstrass representation given by $\{f dz, g\}$ such that the vector corresponding to $g(0)$ is parallel to the x_3-axis. Then the planar curvature of the intersection curves of M with the horizontal planes is*

$$k = \frac{1}{|f^2 g|(1+|g|^2)} \mathrm{Re}\left(\overline{fg}\frac{g'}{g}\right).$$

Proof. Let $\theta = \arg g$ and s be the arc length of the curve $M \cap \{x_3 = c\}$; then $k(s) = \dfrac{d\theta}{ds}$. As $\arg g = \mathrm{Im}(\ln g)$, we have

$$k(s) = \frac{d\mathrm{Im}\ln g}{ds} = \mathrm{Im}(\frac{d\ln g}{dz}\frac{dz}{ds}) = \mathrm{Im}(\frac{g'}{g}\frac{dz}{ds}).$$

By the Weierstrass representation we have

$$x_3 = \mathrm{Re}\int fg.$$

Hence, on the curve $M \cap \{x_3 = c\}$, $\dfrac{dz}{ds}$ must satisfy

$$0 = \frac{d}{ds}\mathrm{Re}\int fg = \frac{1}{2}\mathrm{Re}(fg\frac{dz}{ds}).$$

By a straightforward computation we obtain

$$\frac{dz}{ds} = \frac{i}{(1+|g|^2)|f|}\frac{\overline{fg}}{|fg|}.$$

Then

$$k = \mathrm{Im}(\frac{i}{(1+|g|^2)|f|}\frac{\overline{fg}}{|fg|}\frac{g'}{g}) = \frac{1}{|f^2 g|(1+|g|^2)}\mathrm{Re}\left(\overline{fg}\frac{g'}{g}\right).$$

□

Proof of Theorem 2.3. The third coordinate of M is given by

$$x_3 = \mathrm{Re}\int fg = \mathrm{Re}\int \alpha\frac{\wp'}{\wp} = \alpha \ln|\wp|,$$

since α is real. Then, any level curve is given by $|\wp| = c$ and next to the ends this is a compact immersed curve with only one component.

By a straightforward computation, we obtain

$$g'(z) = 2\alpha\left[\frac{5e_1^2 - 3\wp(z)^2}{\wp(z)^3}\right],$$

$$\frac{g'(z)}{g(z)} = \frac{2(5e_1^2 - 3\wp(z)^2)}{\wp'(z)},$$

$$\overline{f(z)g(z)} = \overline{\alpha}\frac{\overline{\wp'(z)}}{\overline{\wp(z)}}.$$

By using the expansion of $\wp$ and $\wp'$ at p_o we have

$$\overline{f(z)g(z)} \sim -2\frac{\overline{\alpha}}{\overline{z}},$$

$$\frac{g'(z)}{g(z)} \sim \frac{3}{z},$$

where $\sim$ denotes equality between the principal parts of the functions in a neighborhood of zero. Hence the sign of the curvature of the level curve next to the end p_o is the same as the sign of

$$\mathrm{Re}(\frac{-6\overline{\alpha}}{\overline{z}z}) = -\frac{6\alpha}{|z|^2},$$

α being real.

We use the equality

$$\overline{f(z+p_2)g(z+p_2)} = -\overline{f(z)g(z)}$$

and the fact that in a neighborhood of zero we have

$$\frac{g'(z+p_2)}{g(z+p_2)} = \frac{2(5\wp(z)^2 - 3e_1^2)}{\wp'(z)} \sim -\frac{5}{z},$$

to conclude that the sign of the curvature of the level curve next to the end p_2 is the same as the sign of

$$\mathrm{Re}(\frac{-10\overline{\alpha}}{\overline{z}z}) = -\frac{10\alpha}{|z|^2}$$

since α is real.

Thus, if we choose a negative α, the level curves are locally convex next to the two ends of M. $\square$

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