ON THE STRUCTURE OF POSITIVE SCALAR CURVATURE TYPE GRAPHS

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KEY WORDS. Scalar Curvature, Curvature Estimates, Bernstein-type Theorem.

AMS Classification. 53A10, 35J15

1. INTRODUCTION

In this paper we study some aspects of the partial differential equation satisfied by a hypersurface of $\mathbb{R}^{n+1}$ of positive scalar curvature. Purely interior estimates for the principal curvatures of such hypersurfaces are far from being understood. On the contrary there is a wide litterature about such estimates for constant mean curvature hypersurfaces. A leading paper on the last subject is [SSY].

We find purely interior estimates for the principal curvatures of graphs of positive scalar curvature satisfying a uniformly ellipticity condition (cf. Definition 1). In fact our estimates hold for a wider class of fully nonlinear equations. Then we discuss the behaviour at infinity of the solutions over the whole plane of such fully nonlinear equations (Bernstein-type theorem).

We use techniques from [CNS2] and our results are strongly inspired by those of [CNS2]. Caffarelli, Nirenberg and Spruck find purely interior curvature estimates for a class of partial differential equations that include the mean curvature equation but not the scalar curvature equation. Then, they use curvature estimates to prove Bernstein-type theorems for minimal graphs.

* This research was partially supported by Humboldt Stiftung. I would like to thank the members of sfb288-TU Berlin for their hospitality during the preparation of this work.

2. Statement of the results

Let $D_R = \{|x| \leq R\}$ be the open disk of radius R in $\mathbb{R}^n$ and let u be a smooth function defined over D_R. Throughout this paper M will denote the graph of u. At any point $x \in D_R$, let $\kappa(x) = (\kappa_1(x), \ldots, \kappa_n(x))$ be the set of the principal curvatures of the graph M.

Assume that $\kappa(x)$ satisfies the relation

$$f(\kappa(x)) = \Psi(u(x)), \tag{1}$$

where

$$\Psi \text{ is a positive bounded function on } \mathbb{R},\ \Psi',\ \Psi'' \geq 0. \tag{2}$$

The function f satisfies the following conditions:

$$\begin{gathered} f \text{ is a smooth symmetric (under interchange of any two } \kappa_i) \\ \text{concave function defined in an open convex cone } \Gamma \subset \mathbb{R}^n \\ \text{with vertex at the origin and containing the positive cone,} \\ \Gamma \text{ is symmetric in the } \kappa_i, \end{gathered} \tag{3}$$

$$f_i = \frac{\partial f}{\partial \kappa_i} > 0, \text{ for all } i, \ \sum \kappa_i f_i > 0 \text{ in } \Gamma, \tag{4}$$

there exist constants $c_0 > 0$ and $0 < \alpha < 2$ such that

$$\sum f_i(\kappa) \leq c_0 \sum |\kappa_i|^\alpha \text{ in } \Gamma. \tag{5}$$

We remark that the first condition in (4) means ellipticity of f.

Remark 1. Let $1 \leq r \leq n$. Consider the r^{th} symmetric function of the principal curvatures of M, i.e.

$$S_r(\kappa(x)) = \sum_{1 \leq i_1 < \ldots < i_r \leq n} \kappa_{i_1}(x) \ldots \kappa_{i_r}(x).$$

S_1 and S_2 are the mean curvature and the scalar curvature of M respectively. Let c be a positive constant. The mean curvature equation $S_1(\kappa(x)) = c$ satisfies conditions (3) and (4); furthermore $\sum f_i(\kappa)$ is bounded. For $r \geq 3$ the equation $S_r^{\frac{1}{r}}(\kappa(x))$ does not always satisfy conditions (3), (4) and never satisfies condition (5). The scalar curvature equation $S_2^{\frac{1}{2}}(\kappa(x)) = c$ satisfies condition (3), (4) and (5) (with $\alpha = 1$ and $c_0 = 2S_2^{-\frac{1}{2}}$). Ellipticity follows by the following observation. $S_1^2 = \sum_{j=1}^n \kappa_j^2 + 2S_2 \geq k_j^2$ for every j. Hence $S_1 - \kappa_j = \frac{\partial S_2}{\partial \kappa_j} > 0$. Ellipticity allows us to use maximum principle to compare hypersurfaces of positive scalar curvature (cf. [NR] for similar arguments).

Let us state the maximum principle for scalar curvature in a formulation convenient for our purpose. In the following we will say surface and plane instead of hypersurface and hyperplane.

Maximum Principle. *Let M, N be two compact surfaces of $\mathbb{R}^{n+1}$ such that M has positive constant scalar curvature $S_2(M) = S$ and the scalar curvature of N satisfies inequality $0 < S_2(N) \leq S$ at any point (M and N oriented by their mean curvature vector). Let $p \in \operatorname{int}(M) \cap \operatorname{int}(N)$ be a point such that M and N are tangent at p and the mean curvature vectors at p point towards the same side (say upwards). Then, near p, N can not lie above M. The same holds if $p \in \partial M \cap \partial N$ and ∂M and ∂N are tangent at p.*

We remark that the maximum principle holds for mean curvature too (without positiveness condition).

Definition 1. Let $\delta > 0$. A function $u \in C^2(\overline{D}_R)$ is called δ-*admissible* with respect to f if at every point $x \in D_R$ it satisfies

$$\inf_{1 \leq i \leq n} f_i(\kappa(x)) \geq \delta. \tag{6}$$

Remark 2. When $f = S_2^{\frac{1}{2}}$, we have

$$C_2(x) := \inf_{1 \leq i \leq n} f_i(\kappa(x)) = \frac{1}{2} S_2^{-\frac{1}{2}} \big(\sum \kappa_i(x) - \max_{1 \leq i \leq n} \kappa_i(x) \big). \tag{7}$$

A positive constant scalar curvature surface has the property that $C_2(x) > 0$ at any point. If $C_2(x) \geq \delta$ at every point of a surface we say that the surface is δ-admissible. In [NS] we use the condition of admissibility in order to prove a compactness theorem.

Let $w = \sqrt{1 + |\nabla u|^2}$ and set

$$k = 2 \sup_{D_R} w(x). \tag{8}$$

Theorem A. *Let u be a smooth δ-admissible solution in D_R of*

$$f(\kappa(x)) = \Psi(x)$$

with f and Ψ satisfying (2)-(5). Then, there exists a constant A depending only on f and n such that at any principal curvature $\kappa(0)$ of the graph of u at $(0, u(0))$ satisfies

$$|\kappa(0)| \leq \frac{A}{w(0)} \left(\frac{k^{4-\alpha}}{R^2 \delta}\right)^{\frac{1}{2-\alpha}}. \tag{9}$$

Remark 3. When Ψ is specialized to be the scalar curvature, we can take $\alpha = 1$ so that (9) becomes

$$|\kappa(0)| \leq \frac{A}{w(0)} \frac{k^3}{\delta R^2}. \tag{9'}$$

Remark 4. Assume that $\Psi = \Psi(x)$ is a positive smooth function on D_R. In [CNS3] it is proved that, for such a Ψ, if the curvature of M is bounded at the boundary of M, then it is bounded in the interior too. It is easy to see that an analogous proof works for any positive $\Psi = \Psi(u)$ with Ψ' and Ψ'' bounded but not necessarily $\Psi', \ \Psi'' \geq 0$.

From the curvature estimates of Theorem A, we infert the following result.

Theorem B. *Let u be a smooth δ-admissible solution in $\mathbb{R}^n$ of*

$$f(\kappa(x)) = \Psi(u(x))$$

with f and Ψ satisfying (2)-(5). Then, near infinity we have

$$\nabla u(x) = O(|x|^{\frac{2}{4-\alpha}}). \tag{10}$$

Proof.

Assume $\nabla u(x) = o(|x|^{\frac{2}{4-\alpha}})$ at infinity. Letting $R \to \infty$ and using (9), we see that the curvatures of the graph vanish at the origin, which may be any point. Hence the graph is not δ-admissible. Contradiction.

□

Remark 5. When Φ is specialized to be the scalar curvature (10) becomes

$$\nabla u(x) = O(|x|^{\frac{2}{3}}). \tag{10'}$$

Remark 6. From Theorem B, it follows that it does not exist a $\delta > 0$ such that a positive scalar curvature graph on the plane is δ-admissible. In fact, a stronger result is true. A graph M over the whole plane, with scalar curvature satisfying

$$0 < c = \inf_{x \in M} S_2(x)$$

does not exists.

This can be seen by a classical argument using the maximum principle and it is well known to the experts on the subject, but we do not know a reference for the proof. Hence it is worth to prove it here.

By contradiction, assume that such graph M exists. At any point $x \in M$ the mean curvature of M satisfies

$$S_1(x) \geq \big(\frac{n-1}{2n} S_2(x)\big)^{\frac{1}{2}} \geq \big(\frac{n-1}{2n} c\big)^{\frac{1}{2}} > 0$$

where the first inequality follows by a classical inequality for elementary symmetric functions (cf. [HLP]). As the mean curvature has positive sign, we can assume that the mean curvature vector points upwards at every point of M. Then, consider a sphere S of scalar curvature equal to c and translate it vertically to be disjoint from M. Then, come down till the first contact point between M and S. At this point the two surfaces are tangent and their mean curvature vectors both point upwards. Furthermore S is above M and the scalar curvature of S is smaller or equal to the scalar curvature of M at the contact point. This is a contradiction by the maximum principle.

Remark 7. Let us now consider the equation $S_2 = 0$. If a zero scalar curvature surface is δ-admissible, then it can not have flat points (points where all sectional curvature are zero). In fact, at a flat point, at most one pricipal curvature is different from zero (and we can assume that it is ≥ 0), hence

$$\sum_{i=1}^{n} \kappa_i(x) - \max_{1\leq i\leq n} \kappa_i(x) = 0.$$

On a zero scalar curvature surface, the absence of flat points is equivalent to $S_3 \neq 0$, hence the equation of scalar curvature is elliptic on a δ-admissible zero scalar curvature surface (cf. [HL]).

Proposition 1. *Let $\delta > 0$. A zero scalar curvature δ-admissible graph over the whole plane does not exist.*

Proof.

By contradiction assume that such a graph M exists. δ-admissibility implies that the mean curvature at any point of M is bigger or equal to δ. Hence we can orient M by its mean curvature vector and assume it points upwards. Then we consider a sphere S of mean curvature δ and we proceed as in the proof of remark 6.

□

We have results analogous to Theorem A and B when f is a function of the eigenvalues of the Hessian of u. In this case we estimate the second derivatives of u. Such functions are also considered in [CNS1] and in [CNS3], where the authors prove some existence results.

Let f be a function of the eigenvalues $\lambda = (\lambda_1, \ldots, \lambda_n)$ of the Hessian $\{u_{ij}\}$ of the function u

$$f(\lambda(u_{ij}(x)) = \Psi(u(x)). \tag{11}$$

Assume that f, Ψ satisfy conditions (2)-(5).

Theorem A'. *Let u be a smooth δ-admissible solution of* (11) *in D_R. Then for any second derivative D^2u we have*

$$|D^2u(0)| \leq A\Big(\frac{\sup_{D_R} |Du|^2}{R^2\delta}\Big)^{\frac{1}{2-\alpha}} \tag{12}$$

where A is a constant depending only of f and n.

Theorem B'. *Let u be a smooth δ-admissible solution of* (11) *in* $\mathbb{R}^n$, *with f and Ψ satisfying (2)-(5). Then near infinity we have*

$$\nabla u(x) = O(|x|). \tag{13}$$

3. The proofs

Proof of theorem A.

Our computations till inequality (23) are analogous to those in [CNS2].

Let ζ be a smooth function on Ω with $0 \leq \zeta \leq 1$, $\zeta \equiv 1$ in $D_{\frac{R}{2}}$ and satisfying

$$|D\zeta|^2,\ |D^2\zeta| \leq \frac{C}{R^2}. \tag{14}$$

Let

$$\tau = \frac{1}{w},$$

$$a = \frac{1}{k} = \frac{1}{2\inf_{D_R}\tau}$$

and set

$$M := \max_{D_R,\ 1\leq i\leq n} \zeta \frac{1}{\tau - a}\kappa(x).$$

We can assume $M > 0$ and it is achieved at some point $x^0 \in D_R$.

It suffices to prove that

$$M \leq \frac{Ak^{\frac{4-\alpha}{2-\alpha}}}{R^{\frac{2}{2-\alpha}}\delta^{\frac{1}{2-\alpha}}}.$$

It is convenient to use tangential coordinates to the surface at the point $(x^0, u(x^0))$. Let $e_1, \ldots, e_{n+1}$ denote the unit vectors in the directions of the axes, and introduce new orthonormal vectors

$$\varepsilon_1, \ldots, \varepsilon_n, \varepsilon_{n+1},$$

where $\varepsilon_{n+1} = w^{-1}(-u_1, \dots, -u_n, 1)$ is the unit normal at x^0 and ε_1 correspond to the tangential direction at x^0 with largest principal curvature. We represent the surface near $(x^0, u(x^0))$ by tangential coordinates $y_1, \dots, y_n$ and $v(y)$ (summation is from 1 to n):

$$x_j e_j + u(x)e_{n+1} = x_j^0 e_j + u(x^0)e_{n+1} + y_j\varepsilon_j + v(y)\varepsilon_{n+1}$$

thus $\nabla v(0) = 0$. Set

$$\omega = (1 + |\nabla v|^2)^{\frac{1}{2}}$$

Then the normal curvature in the ε_1 direction is

$$k = \frac{v_{11}}{(1+v_1^2)\omega}.$$

In the y coordinates the unit normal is

$$N = -\frac{1}{\omega}v_j\varepsilon_j + \frac{1}{\omega}\varepsilon_{n+1},$$

and

$$\tau = \frac{1}{w} = N \cdot e_{n+1} = \frac{1}{\omega W} - \frac{1}{\omega}\sum a_j v_j, \tag{15}$$

where $W = w(x^0)$ and $a_j = \varepsilon_j \cdot e_{n+1}$, so $\sum a_j^2 \le 1$.

At the point $y = 0$, since the y_1 direction is a direction of principal curvature, we have $v_{1j}(0) = 0$ for $j > 1$. By rotating the $\varepsilon_2, \dots, \varepsilon_n$, we may achieve that the Hessian $\{v_{ij}(0)\}$ is diagonal. Note that in the y coordinates inequalities (14) still hold. At the point $y = 0$ the function

$$\log\big(\zeta\frac{1}{\tau - a}\frac{v_{11}}{(1+v_1^2)\omega}\big) \tag{16}$$

takes its maximum, hence

$$\frac{v_{11i}}{v_{11}} + \frac{\zeta_i}{\zeta} - \frac{\tau_i}{\tau - a} - \frac{2v_1 v_{1i}}{1+v_1^2} - \frac{\omega_i}{\omega} = 0 \quad \forall i \tag{17}$$

(18) $$\frac{v_{11ii}}{v_{11}} - \frac{v_{11i}^2}{v_{11}^2} + \left(\frac{\zeta_i}{\zeta}\right)_i - \left(\frac{\tau_i}{\tau - a}\right)_i - 2v_{1i}^2 - v_{ii}^2 \le 0 \quad \forall i$$

From (15), we find at $y = 0$, $i = 1, \ldots, n$

(19) $$\tau_i = -a_i v_{ii}, \quad \tau_{ii} = -a_j v_{jii} - \frac{v_{ii}^2}{W}$$

The principal curvatures of the surfaces (in y-coordinates) are the eigenvalues of the matrix

$$a_{il} = \frac{1}{\omega}\left\{v_{il} - \frac{v_i v_j v_{jl}}{\omega(1+\omega)} - \frac{v_l v_k v_{ki}}{\omega(1+\omega)} + \frac{v_i v_l v_j v_k v_{jk}}{\omega^2(1+\omega)^2}\right\}$$

Hence at the origin the matrix $a_{il} = v_{il}$ is diagonal and for every $j = 1, \ldots, n$

(20) $$\frac{\partial a_{il}}{\partial y_j} = v_{ilj}, \quad \frac{\partial^2 a_{il}}{\partial y_1^2} = v_{il11} - v_{11}^2(v_{il} + \delta_{i1}v_{1l} + \delta_{1l}v_{1i}).$$

The smooth concave function $f(\kappa)$ can be written as a smooth concave function F of the symmetric matrix $A = \{a_{il}\}$ and at $y = 0$

(21) $$\frac{\partial F}{\partial a_{il}} = \frac{\partial f}{\partial \kappa_i}\delta_{il} = f_i \delta_{il}$$

By differentiating the equation $F(a_{il}) = \Psi(u) = \tilde{\Psi}(y)$ with respect to y_1 we obtain

$$\frac{\partial F}{\partial a_{il}}\frac{\partial a_{il}}{\partial y_1} = \tilde{\Psi}_1.$$

Differentiating once more with respect to y_1 and using concavity of F, we have

$$\tilde{\Psi}_{11} \le \frac{\partial F}{\partial a_{il}}\frac{\partial^2 a_{il}}{\partial y_1^2}$$

Using (20) and (21) we infert at $y = 0$

(22) $$f_i v_{iij} = \tilde{\Psi}_j \quad \forall j$$

and

$$\tilde{\Psi}_{11} \leq f_i(v_{ii11} - v_{11}^2 v_{ii}) - 2f_1 v_{11}^3 \tag{23}$$

Replacing (17), (18) and (22) in (23) and using the fact that $\Psi', \Psi'' \geq 0$ we obtain

$$\frac{a\zeta^2}{\tau - a} \sum f_i v_{ii}^2 \leq \sum f_i \left(\frac{C}{R^2} + \frac{C\zeta}{R} \frac{|v_{ii}|}{\tau - a}\right) \tag{24}$$

By Young's inequality we have for every i

$$\frac{C\zeta}{R(\tau - a)} f_i |v_{ii}| \leq \frac{a\zeta^2}{2(\tau - a)} f_i v_{ii}^2 + \frac{C^2}{2R^2 a(\tau - a)} f_i$$

Replacing last inequality in (24) we obtain (with a different constant C)

$$\frac{\zeta^2}{(\tau - a)} \sum f_i v_{ii}^2 \leq \frac{1}{a^2} \frac{C}{R^2} \frac{1}{\tau - a} \sum f_i.$$

Using conditions (5) and (6), and recalling that $v_{11}(0)$ is the maximal principal curvature at $(0, u(0))$, we find

$$\frac{\zeta^2}{\tau - a} \delta v_{11}^2 \leq \frac{1}{a^2} \frac{Cc_0}{R^2} \frac{n}{\tau - a} v_{11}^{\alpha}$$

Multiplying both sides by $\zeta^{-\alpha}(\tau - a)^{\alpha - 1} v_{11}^{-\alpha} \delta^{-1}$ we have

$$\frac{\zeta^{2-\alpha}}{(\tau - a)^{2-\alpha}} v_{11}^{2-\alpha} \leq \frac{1}{a^2} \frac{Cc_0}{\delta R^2} \frac{n}{(\tau - a)^{2-\alpha}}$$

Hence

$$M^{2-\alpha} := \left(\frac{\zeta}{\tau - a} v_{11}\right)^{2-\alpha} \leq \frac{Cc_0 n}{\delta R^2} k^{4-\alpha.}$$

Then, there exists $A = A(C, c_0, \alpha, n)$ such that

$$M \leq \frac{A}{\delta^{\frac{1}{2-\alpha}} R^{\frac{2}{2-\alpha}}} k^{\frac{4-\alpha}{2-\alpha}}$$

□

Proof of Theorem A'.

Here also, computation till inequality (31) are analogous to those in [CNS2].

Since $\lambda(u_{ij})$ is assumed to be in Γ (hence in the halfspace $\sum \lambda_i > 0$) it suffices to established the following estimate for any directional derivative ∂_ξ :

$$\partial_\xi^2 u(0) \leq \frac{A}{R^{\frac{2}{2-\alpha}} \delta^{\frac{1}{2-\alpha}}} \sup_{D_R} |Du|^{\frac{2}{2-\alpha}}. \tag{25}$$

Set

$$a = \frac{1}{4} (\sup_{D_R} |\nabla u|)^{-2}.$$

With ζ as in the previous proof, we establish the estimate

$$M := \max_{\xi \in S^{n-1}, x \in D_R} \zeta(x) \exp\big(\frac{a}{2} |\nabla u|^2\big) \partial_\xi^2 u(x) \leq \frac{A}{R^{\frac{2}{2-\alpha}} \delta^{\frac{1}{2-\alpha}}} a^{\frac{1}{\alpha-2}} \tag{26}$$

from which (25) follows. We may suppose M is achieved at some point $x^0 \in D_R$, for the direction $\xi = (1, 0, \ldots, 0)$ and $M > 0$. Furthemore we may assume that the Hessian matrix $\{u_{ij}(x^0)\}$ is diagonal. Then the function

$$\zeta \exp(\frac{a}{2} |\nabla u|^2) u_{11}$$

achieves its maximum M at x^0. So, we have there

$$\frac{u_{11i}}{u_{11}} + \frac{\zeta_i}{\zeta} + a u_j u_{ji} = 0 \ \forall i, \tag{27}$$

$$0 \geq \frac{u_{11ii}}{u_{11}} - \frac{u_{11i}^2}{u_{11}^2} + \big(\frac{\zeta_i}{\zeta}\big)_i + a \sum_k (u_{ki}^2 + u_k u_{kii}) \ \forall i. \tag{28}$$

Differentiating equation (11) we obtain (recall that f is concave)

$$f_i u_{iij} = \Psi' u_j \ \forall j, \tag{29}$$

$$(30) \qquad f_i u_{ii11} \geq \Psi' u_{11} + \Psi'' u_1^2.$$

Replacing (27), (28) and (29) in (30) and using the fact that $\Psi', \Psi'' \geq 0$, we obtain

$$(31) \qquad 0 \geq \sum f_i \big[-\big(\frac{\zeta_i}{\zeta} + a \sum u_k u_{ki}\big)^2 + \big(\frac{\zeta_i}{\zeta}\big)_i + a \sum u_{ki}^2 \big]$$

$$= \sum f_i \big[-\big(\frac{\zeta_i}{\zeta}\big)^2 - a^2 (u_i u_{ii})^2 - 2a \frac{\zeta_i}{\zeta} u_i u_{ii} + \big(\frac{\zeta_i}{\zeta}\big)_i + a u_{ii}^2 \big]$$

where last equality depends on the fact that the matrix $\{u_{ij}\}$ is diagonal at x^0.

By Young's inequality, for every i :

$$-2a \frac{\zeta_i}{\zeta} u_i u_{ii} \geq -\big(\frac{\zeta_i}{\zeta}\big)^2 - a^2 u_i^2 u_{ii}^2$$

Substituting in (31) we have

$$0 \geq \sum f_i \big[-2\big(\frac{\zeta_i}{\zeta}\big)^2 + \big(\frac{\zeta_i}{\zeta}\big)_i - 2a^2 u_i^2 u_{ii}^2 + a u_{ii}^2 \big]$$

$$\geq \sum f_i \big[-2\big(\frac{\zeta_i}{\zeta}\big)^2 + \big(\frac{\zeta_i}{\zeta}\big)_i + \frac{a}{2} u_{ii}^2 \big]$$

by definition of a. Then

$$\frac{a}{2} \zeta^2 \sum f_i u_{ii}^2 \leq \frac{C}{R^2} \sum f_i.$$

Using conditions (5) and (6), and recalling that $u_{11}(0)$ is the maximal second derivative at $(0, u(0))$, we find

$$\frac{a}{2} \zeta^2 \delta u_{11}^2 \leq \frac{C c_0 n}{R^2} u_{11}^{\alpha}$$

Multiplying by $2\zeta^{-\alpha} a^{-1} \delta^{-1} \exp\big(\frac{a|\nabla u|^2}{2}(2-\alpha)\big) u_{11}^{-\alpha}$
we obtain

$$\zeta^{2-\alpha} \exp\big(\frac{a|\nabla u|^2}{2}(2-\alpha)\big) u_{11}^{2-\alpha} \leq \frac{2 C c_0 n}{\delta R^2} \frac{\zeta^{-\alpha}}{a} \exp\big(\frac{a|\nabla u|^2}{2}(2-\alpha)\big)$$

and then there exists a constant $A = A(C, c_0, \alpha, n)$ such that

$$M^{2-\alpha} \leq \frac{A^{2-\alpha}}{\delta R^2} a^{-1}$$

that gives the desired estimate on M :

$$M \leq \frac{A}{\delta^{\frac{1}{2-\alpha}} R^{\frac{2}{2-\alpha}}} a^{\frac{1}{\alpha-2}}$$

□

Theorem B' follows from theorem A' in the same way as theorem B follows from Theorem A.

Remark 8. Let us give a slightly different definition of admissibility.

Definition 2'. Let $\delta > 0$, $\alpha \geq 0$. A function $u \in C^2(\overline{D}_R)$ is called *δ-admissible* if at every point $x \in D_R$, it satisfies

$$\inf_{1 \leq i \leq n} f_i(\mathrm{k}(x)) \geq \delta \max_{1 \leq i \leq n} |\mathrm{k}_i|^{\alpha-1}. \tag{6'}$$

This definition of admissibility allows us to extend all our results (with the same proof) to functions f that satisfy conditions (3), (4) and (5) without the restriction $\alpha < 2$ of last section.

Remark 9. It would be interesting to study equation (1) when $\Psi \equiv 0$ - zero scalar curvature graphs, for example. As we have previously observed (see Remark 7) we know that the equation of a δ-admissible zero scalar curvature graph is always elliptic. But when f is the square root of the scalar curvature (and any $S_r^{\frac{1}{r}}$ in fact), f it is not smooth at zero, then we can not apply the same techniques as above.

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