On Properties of Constant Mean Curvature Surfaces in $\mathbb{H}^2 \times \mathbb{R}^1$

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Abstract. We discuss global properties of constant mean curvature surfaces (H-surfaces) in $\mathbb{H}^2 \times \mathbb{R}$: maximum principle at infinity, halfspace type theorem, non existence of simply connected surfaces with one end.

1 Introduction

In last years, there has been work on H-surfaces in homogeneous 3-manifolds, in particular in product spaces $M^2 \times \mathbb{R}$, where M^2 is a Riemannian surface: new examples were produced and many theoretical results as well (see the bibliography for an overview about the subject).

In this survey, we describe some of our contributions to the theory of constant mean curvature surfaces in $\mathbb{H}^2 \times \mathbb{R}$ where $\mathbb{H}^2$ is the hyperbolic plane ([27], [28], [30]).

In $\mathbb{H}^2 \times \mathbb{R}$, the mean curvature $H = \frac{1}{2}$ plays the same role as the mean curvature zero in $\mathbb{R}^3$ and one in $\mathbb{H}^3$ ([7]). It will be clear in the following that the behavior of H-surfaces is quite different depending on H greater or smaller than $\frac{1}{2}$. For example, there is no entire graph with constant mean curvature $H > \frac{1}{2}$, while, for any $H \in (0, \frac{1}{2})$ there exists an entire rotational graph with constant mean curvature H ([39], [31], [30]). Furthermore, there is no compact embedded surface with constant mean curvature $H < \frac{1}{2}$, while for any $H > \frac{1}{2}$, there exists a compact sphere with constant mean curvature H ([16], [5]).

2 The Maximum Principle at Infinity

Theorem 2.1. *([27]) Let M_1, M_2 be two disjoint H-surfaces in $\mathbb{H}^2 \times \mathbb{R}$ complete, properly embedded, without boundary, with $H > \frac{1}{\sqrt{2}}$. Then M_2 cannot lie in the*

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mean convex side of M_1.

By mean convex side of M_1, we mean the following. As M_1 is properly embedded, it separates $\mathbb{H}^2 \times \mathbb{R}$ into two connected components. The mean convex side of M_1 is the component of $\mathbb{H}^2 \times \mathbb{R} \setminus M_1$ towards which the mean curvature vector of M_1 points.

For $H \leq \frac{1}{2}$, Theorem 2.1 is false, in fact rotational, entire vertical graphs are counterexamples to the result of the Theorem ([31], [30], [36]). We believe that Theorem 2.1 holds for $H > \frac{1}{2}$.

The analogue of the previous Theorem in the Euclidean case is proved in [35]. An analogous result in $\mathbb{H}^3$ is proved in [12], provided the curvature of M_1 and M_2 is bounded.

A key tool for the proof of Theorem 2.1 is the distance estimate established in the following Lemma.

Lemma 2.2. (Distance Lemma) *([27]) Let M be a stable H-surface in $\mathbb{H}^2 \times \mathbb{R}$ with $H > \frac{1}{\sqrt{3}}$. Then, for any $p \in M$*

$$dist_M(p, \partial M) < \frac{2\pi}{\sqrt{3(3H^2-1)}} \tag{1}$$

The hypothesis $H > \frac{1}{\sqrt{3}}$ seems to be due only to technical reasons. Actually, we believe that a similar estimate can be proven for $H > \frac{1}{2}$. The proof of Lemma 2.2 is a modification of Fisher-Colbrie's method ([11]).

In [34], Lemma 2.2 was extended to any homogeneously regular three manifold, provided the curvature H is great enough with respect to the sectional curvature of the ambient manifold.

Let us give an idea of the proof of Theorem 2.1.

The proof is by contradiction: assume that M_2 lies in the mean convex side of M_1.

We first prove that neither M_1 nor M_2 can be compact. If M_1 were compact, then the mean convex side of M_1 would be compact too and M_2 would be properly embedded in a compact set. Hence M_2 would be compact. Moving M_1 towards M_2, by an isometry of the ambient space, yields a first contact point where the mean curvature vectors of M_1 and M_2 are equal. This gives a contradiction by the standard maximum principle. Hence M_1 cannot be compact. If M_2 were compact, then, by moving M_2 towards M_1 as before, one obtains a contradiction by the standard maximum principle. So, M_1 and M_2 are both non compact and moving M_1 towards M_2, by an isometry of the ambient space, the first contact point cannot be a finite one, by the standard maximum principle. So we are left with the case in which the first contact point is at infinity. This explains the name "maximum principle at infinity".

In this case, we define W to be the closure of the component of $\mathbb{H}^2 \times \mathbb{R} \setminus (M_1 \cup M_2)$ satisfying $\partial W = M_1 \cup M_2$. The boundary of W is not connected and the mean curvature vector of M_1 points towards W.

Let S be a relatively compact domain in M_1 such that $\partial S = \Gamma$ is a smooth curve. One can prove that there exists a stable H-surface Σ in W with boundary Γ and homologous to S (for the definition of stability see [6], [28]). The proof of this fact is quite delicate and uses results by geometric measure theory. Then, by taking the domain S in M_1 larger and larger, one finds points of Σ very far from its boundary Γ. This gives a contradiction by the fact that the distance between a point of a stable H-surface and its boundary is bounded by Lemma 2.2.

As a Corollary of Lemma 2.2, we have the following result.

Theorem 2.3. *([27]) In $\mathbb{H}^2 \times \mathbb{R}$ there is no non compact complete stable H-surface with $H > \frac{1}{\sqrt{3}}$ either with compact boundary or without boundary.*

On the other hand, we obtain a bound on the topology of a stable compact H-surface, provided $H > \frac{1}{\sqrt{2}}$.

Theorem 2.4. *([27]) Let M be a compact weakly stable H-surface in $\mathbb{H}^2 \times \mathbb{R}$ with $H > \frac{1}{\sqrt{2}}$. Then the genus g of M satisfies $g \leq 3$.*

3 Simply Connected Surfaces

The distance Lemma is also a key point in the proof of the following result.

Theorem 3.1. *([28]) For $H > \frac{1}{\sqrt{3}}$, there is no properly embedded H-surface in $\mathbb{H}^2 \times \mathbb{R}$ with finite topology and one end.*

In [23], Meeks proved that if M is a properly embedded simply connected surface of constant mean curvature $H \neq 0$ in $\mathbb{R}^3$, then M is a round sphere. In particular, M can not be topologically $\mathbb{R}^2$. More generally, he proved there is no properly embedded H-surface of finite topology in $\mathbb{R}^3$, with exactly one end. Afterwards, in [22], a different proof of Meeks' Theorem was found and, in [21], it was extended to the hyperbolic space $\mathbb{H}^3$.

Theorem 3.1 answers to this problem in $\mathbb{H}^2 \times \mathbb{R}$. There are properly embedded H-surfaces in $\mathbb{H}^2 \times \mathbb{R}$ that are topologically $\mathbb{R}^2$; there are entire graphs (vertical graphs over $\mathbb{H}^2$) for each H, $0 \leq H \leq \frac{1}{2}$ ([27], [30], [31], [36], [39]). We prove that such a surface can not exist for $H > \frac{1}{\sqrt{3}}$. In [9] our result is extended to the case $H > \frac{1}{2}$.

It is interesting to consider to what extent Theorem 3.1 holds in other homogeneous 3-manifolds (for some other constant than $\frac{1}{\sqrt{3}}$). In $\mathbb{S}^2 \times \mathbb{R}$, there is no properly embedded H-surface with one end. To see this, notice that an end of such a surface M would have to go up, or down (but not both), since M is proper. So one can assume M is bounded below say by height zero. Then, do Alexandrov reflection with respect to the "planes" $\mathbb{S}^2 \times \{t\}$ coming up from $t = 0$, to conclude that the part of M below any $M \times \{t\}$, is a vertical graph. This contradicts the height estimates for such graphs (see [17]). So, no such M exists in $\mathbb{S}^2 \times \mathbb{R}$.

The other homogeneous 3-manifolds (beside the space forms) are the Berger spheres, Heisenberg space and $\widetilde{PSL}(2,\mathbb{R})$. Since the Berger spheres are compact, the question is interesting in the last two spaces: Heisenberg space and $\widetilde{PSL}(2,\mathbb{R})$.

Another interesting question in Heisenberg space is whether the only embedded compact H-surfaces are the rotational spheres of constant mean curvature.

Theorem 3.1 has the following straightforward consequence.

Corollary 3.2. *([28]) A simply connected H-surface properly embedded in $\mathbb{H}^2 \times \mathbb{R}$, $H > \frac{1}{\sqrt{3}}$ is a rotational sphere.*

The proof of Theorem 3.1 follows from the fact that for $H > \frac{1}{\sqrt{3}}$, a properly embedded H-surface in $\mathbb{H}^2 \times \mathbb{R}$ with finite topology and one end is contained in a vertical cylinder (Theorem 1.2 in [28]).

Theorem 3.1 in $\mathbb{H}^2 \times \mathbb{R}$ does not hold without the one end hypothesis. In fact, there are examples of constant mean curvature cylinders lying in the tubular neighborhood of a horizontal geodesic (cf. [24]).

4 A Halfspace Theorem for $H = \frac{1}{2}$

D. Hofmann e W. Meeks proved a beautiful theorem on minimal surfaces, the so-called "Halfspace Theorem" in [18]: there is no non planar, complete, minimal surface properly immersed in a halfspace of $\mathbb{R}^3$. A halfspace theorem for minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ is false, in fact there are many minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ that have bounded third coordinates ([27], [40]). It is natural to investigate about halfspace type results for surfaces of constant mean curvature $H = \frac{1}{2}$ in $\mathbb{H}^2 \times \mathbb{R}$. We are able to prove the following result.

Theorem 4.1. *([30]) Let S be a simply connected rotational surface with constant mean curvature $H = \frac{1}{2}$. Let Σ be a complete surface with constant mean curvature $H = \frac{1}{2}$, different from a rotational simply connected one. Then, Σ can not be properly immersed in the mean convex side of S.*

In [20] L. Hauswirth, H. Rosenberg and J. Spruck prove a halfspace type theorem for surfaces on one side of a horocylinder.

The result in [20] is different in nature from our result because in [20], the "halfspace" is one side of a horocylinder, while for us, the "halfspace" is the mean convex side of the rotational simply connected surface.

Let us state a conjecture (Strong Halfspace Theorem) that would generalize Theorem 2.1 to surfaces with constant mean curvature $H = \frac{1}{2}$.

Conjecture. ([30]) *Let Σ_1, Σ_2 be two complete properly embedded surfaces with constant mean curvature $H = \frac{1}{2}$, different from the rotational simply connected one. Then Σ_i can not lie in the mean convex side of Σ_j, $i \neq j$.*

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