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Some Remarks on Embedded Hypersurfaces in Hyperbolic Space of Constant Curvature and Spherical Boundary

BARBARA NELLI AND HAROLD ROSENBERG

Abstract: We consider embedded hypersurfaces M in hyperbolic space with compact boundary C and some r^{th} mean curvature function H_r a positive constant. We investigate when symmetries of C are symmetries of M. We prove that if $0 \leq H_r \leq 1$ and C is a sphere then M is a part of an equidistant sphere. For $r = 1$ (H_1 is the mean curvature) we obtain results when C is convex.

Key words: *Symmetries of hypersurfaces of constant curvature, Alexandrov reflection, flux formula*

MSC 1991: 53 A 10, 53 C 42

1. Introduction

It is unknown whether an embedded compact surface M in $\mathbb{R}^3$ of constant mean curvature H (an H-surface) with boundary a circle C, is spherical. One of our results in this paper establishes this fact in hyperbolic space if $0 \leq H \leq 1$; more generally, we prove this for such hypersurfaces in hyperbolic space having some r-th mean curvature function, H_r, a constant between zero and one.

There are partial results on the problem concerning mean curvature ($r = 1$) in $\mathbb{R}^n$. If M is transverse along C to the hyperplane containing $\partial M = C$, then M is spherical. More generally, if one assumes C to be convex, the transversality hypothesis, together with constant curvature, implies that M has all the symmetries of C [BMRS]. Barbosa has proved that if M is immersed in a cylinder of radius $\frac{1}{|H|}$ of $\mathbb{R}^n$ (assuming $H \neq 0$), then M is spherical [B], and Barbosa and Jorge have proved that if M is stable, then M is spherical [BJ].

There are examples of compact, non-spherical immersed H-surfaces in $\mathbb{R}^3$ with boundary a circle [K]. It is conjectured that if M is an embedded H-surface in $\mathbb{R}^3$ with circle boundary or immersed and of genus zero, then M has to be spherical [BMRS].

In $\mathbb{H}^3$, Rosenberg and Spruck have proved that any Jordan curve in a horosphere bounds exactly two embedded compact surfaces with H_2 a prescribed constant between zero and one. Moreover, any such immersed surface is embedded [RS]. In particular, an immersed solution to the circle boundary problem is spherical in this case.

2. Embedded Submanifolds with Constant H_r

Theorem 2.1. *Let C be a sphere in $\mathbb{H}^{m+1}$ of codimension two and let M be a compact embedded hypersurface in $\mathbb{H}^{m+1}$ with $\partial M = C$. Assume some symmetric curvature H_r of M to be constant and $0 \leq H_r \leq 1$. Then M is spherical (part of an equidistant sphere).*

Proof. We shall work in the half-space model of hyperbolic space, that is,

$$\mathbb{H}^{m+1} = \{(x_1, \dots, x_{m+1}) \in \mathbb{R}^{m+1} \mid x_{m+1} > 0\}$$

with the hyperbolic metric, i.e. the Euclidean metric divided by x_{m+1}. For each $c > 0$, $L(c) = \{x_{m+1} = c\}$ is a horosphere of curvature one (each H_r is one) and the mean curvature vector of $L(c)$ points vertically upward. Each line in $L(c)$ is a horocycle of curvature one and curvature vector vertically upward.

Now suppose that C is a codimension two sphere in $\mathbb{H}^{m+1}$. After an ambient isometry, we can assume $C \subset L(1)$. Let M be a compact hypersurface, $\partial M = C$, and H_r constant on M, $0 < H_r \leq 1$. The equation H_r a positive constant is an elliptic equation, thus it satisfies an interior and boundary maximum principle [H]: two such (connected) hypersurfaces that touch at an interior point (i.e., they are tangent at an interior point of both hypersurfaces, and one hypersurface is locally on one side of the other) and whose mean curvature vectors have the same direction at this point, are equal. This also holds if the point where they touch is on the boundary of both hypersurfaces and the boundaries are tangent at the point.

Now we use the maximum principle to prove that M is above $L(1)$. Suppose this were not the case. Let $p \in M$ be a lowest point of M, i.e., $c = x_{m+1}(p) \leq x_{m+1}(q)$ for all $q \in M$, and $c < 1$. M touches $L(c)$ at p and is locally above $L(c)$ at p. The mean curvature vector of $L(c)$ points up at p, hence so does the mean curvature vector of M at p. Each principal curvature of M at p (with respect to the upward pointing normal $\mathbf{n}$) is at least one. For, if $\mathbf{X}$ is a unit horizontal vector at p, the vertical plane at p generated by $\mathbf{X}$ and $\mathbf{n}$ (which is both a hyperbolic and Euclidean plane) intersects M in a curve γ tangent to $\mathbf{X}$ at p and above the horizontal line through p with $\mathbf{X}$ as tangent. This horizontal line has curvature one, so the curvature of γ is at least one at p (with respect to $\mathbf{n}$). This shows that all principal curvatures of M at p are at least one, hence $H_r \geq 1$. By hypothesis, $H_r \leq 1$, so $H_r = 1$ and we can apply the maximum principle to conclude $M \subset L(c)$. Since $\partial M = C \subset L(1)$, this is a contradiction. Notice that the above argument shows that if $x_3(p) = 1$ for some p in the interior of M, then $M \subset L(1)$. Hence, we can assume $M \cap L(1) = C$.

Now we prove that M is transverse to $L(1)$ along C. Suppose this were not so and let $p \in C$ be a point where the tangent space of M is horizontal. For each tangent vector $\mathbf{X}$ to M at p, the plane generated by $\mathbf{n}$ and $\mathbf{X}$ intersects M in a curve γ, with endpoint p and tangent to $\mathbf{X}$ at p. Since γ is above the horocycle through p with $\mathbf{X}$ as tangent, the curvature of γ is at least one at p. Thus, for all vectors $\mathbf{X}$ in a half-space of $\mathrm{T}_p(M)$, the normal curvatures defined by $\mathbf{X}$ are at least one. Hence, the principal curvatures of M at p are at least one and $H_r \geq 1$. Then $H_r = 1$ and $M \subset L(1)$ by the boundary maximum principle.

Now that we know that M is above $L(1)$ and transverse to $L(1)$ along ∂M, we can do Alexandrov reflection with vertical hyperplanes to prove M has all symmetries of ∂M. Hence, M is a rotational hypersurface and by [G] M is umbilical. Thus M is contained in an equidistant sphere [S].

Alexandrov reflection is a well-known technique, but, for expository reasons, we will describe the proof.

Let D be the compact domain in $L(1)$ bounded by C. $\Sigma = M \cup D$ divides $\mathbb{H}^{m+1}$ into two components A and B; let B denote the compact component.

Consider a vertical hyperplane P disjoint from Σ (so $P \subset A$) and move P parallel to itself (say, to the right) until it touches Σ at a first point q (if there is no ambiguity, we will simply refer to a hyperplane as a plane). Now, when moving P a little more to the right from q, to a plane $P(t)$, the part of M on the left of $P(t)$ (which we denote by $M(t)^-$) is a graph (with respect to the horizontal) over a domain in $P(t)$ and no point of $M(t)^-$ has a horizontal tangent space. Here we use the fact that M is transverse to $L(1)$ along C, for the point q may be on C.

Let $M(t)^+$ be the symmetry of $M(t)^-$ through $P(t)$ (in this model, the hyperbolic symmetry through $P(t)$ equals the Euclidean symmetry). $M(t)^+$ is contained in B and has the same constant H_r. Further, the mean curvature vector of $M(t)^+$ is the symmetry of the mean curvature vector of $M(t)^-$.

Now continue moving $P(t)$ to the right, and consider the first parallel plane $P(\tau)$ where one of the following conditions fails to hold:

1) $\text{int}(M(\tau)^+) \subset \text{int}(B)$,
2) $M(\tau)^-$ is a graph over a part of $P(\tau)$ and no point of $M(\tau)^-$ has a horizontal tangent space.

If 1) fails first, one applies the maximum principle to M and $M(\tau)^+$ at the point where they touch to conclude that $P(\tau)$ is a plane of symmetry of M. If 2) fails first, then the point p where the tangent space of $M(\tau)^-$ becomes horizontal is on $\partial(M(\tau)^-) \subset P(\tau)$, and one can apply the boundary maximum principle to $M(\tau)^+$ and the part of M to the right of $P(\tau)$, to conclude that $P(\tau)$ is a plane of symmetry of M. Thus, for each vertical plane P, some parallel translate of P is a plane of symmetry of M, and M is spherical.

To complete the proof of Theorem 2.1, we consider the case $H_r = 0$. We will prove that M equals the totally geodesic disk N with $\partial N = C$.

Let $N(t)$, $0 \leq t < 1$, be a family of (non-compact) equidistant spheres in $\mathbb{H}^{m+1}$ satisfying (cf. Figure 1)

- each $N(t)$ has positive H_r, $C \cap N(t) = \emptyset$, and at the highest point of $N(t)$, the mean curvature vector points upward,
- $N(t)$ varies continuously with t and as $t \longrightarrow 1$, $N(t)$ converges to the totally geodesic hyperplane $N(1)$ containing N; and
- $N(0)$ is below M, i.e.

$$\sup_{p \in N(0)} x_{m+1}(p) < \inf_{q \in M} x_{m+1}(q).$$

Observe that no $N(t)$ can touch M for $t < 1$. Otherwise, at the first point p where $N(t)$ touches M, M would be locally above $N(t)$ at p, hence, the mean curvature vectors of M and $N(t)$ at p point in the same direction. Thus, all the normal curvatures of M at p are as large as those of $N(t)$. Since H_r of $N(t)$ is greater than zero, this contradicts H_r of M being equal to zero. Thus M is above $N(1)$.

Now one proves that M is below $N(1)$ by a similar argument as above.

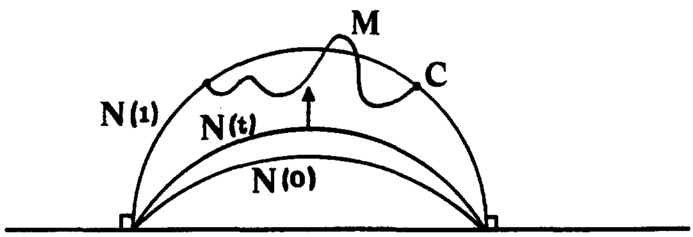

Fig. 1.

Let $\epsilon > 0$; we construct a family of spheres $S_\epsilon(t)$, $0 \le t \le 1$, satisfying:

- $S_\epsilon(0)$ is an equidistant sphere with center on the vertical geodesic passing through the center of C such that M is inside $S_\epsilon(0)$ and the exterior angle between $S_\epsilon(0)$ and $\{x_{m+1} = 0\}$ is $\alpha_\epsilon(0) = \frac{\pi}{2} - \epsilon$ (cf. Figure 2); observe that $\alpha_\epsilon(0) < \frac{\pi}{2}$ guarantees that the mean curvature vector of $S_\epsilon(0)$ points inside,
- $S_\epsilon(t)$, $0 < t \le 1$, are obtained from $S_\epsilon(0)$ by a homothety from the Euclidean center of $S_\epsilon(0)$ and $S_\epsilon(1) \cap N = C$. Denote by $\alpha_\epsilon(t)$ the exterior angle between $S_\epsilon(t)$ and $\{x_{m+1} = 0\}$; by construction $\alpha_\epsilon(t) < \frac{\pi}{2}$ for $t \in [0,1]$, so the mean curvature vector of each $S_\epsilon(t)$ points inside.

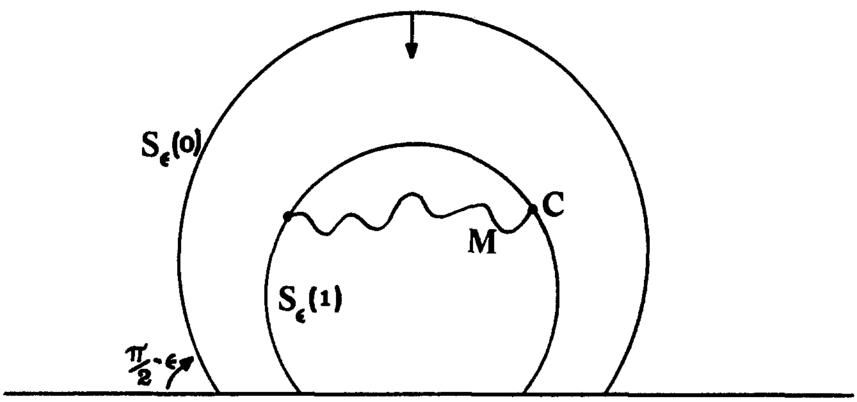

Fig. 2.

As before, no $S_\epsilon(t)$ can touch M and $S_\epsilon(1) \cap M = C$, hence, M is below $S_\epsilon(1)$. Now observe that, as $\epsilon \longrightarrow 0$, $\alpha_\epsilon(t) \longrightarrow \frac{\pi}{2}$ and $S_\epsilon(1) \longrightarrow N(1)$, so M is below $N(1)$. Hence, $M = N$ and Theorem 2.1 is proved. □

3. Submanifolds of Constant Mean Curvature

Theorem 3.1. *Let $M \subset \mathbb{H}^{m+1}$ be a compact embedded codimension one submanifold of constant mean curvature H. Assume $C = \partial M$ is a convex submanifold of a geodesic hyperplane N and that M is transverse to N along C. Then M has all the symmetries of C, in particular, M is spherical if C is a sphere.*

Remark 3.2. The idea of the proof of Theorem 3.1 is given in [BMRS] where the theorem is proved in $\mathbb{R}^{m+1}$; we adapt this to hyperbolic space. In [R] the second author proves Theorem 3.1 in $\mathbb{R}^{m+1}$ for H_r a positive constant. We have not been able to establish this in hyperbolic space; the flux formula (Theorem 7.2 of [R]) is true in $\mathbb{R}^{m+1}$, but we do not know if it holds in $\mathbb{H}^{m+1}$, except for the mean curvature. If this flux formula holds for a given r, then Theorem 3.1 as well; the proof is the same.

Lemma 3.3. (A flux formula) *Let* $\mathbf{Y}$ *be a Killing vector field of* $\mathbb{H}^{m+1}$, M *and* D *compact hypersurfaces immersed in* $\mathbb{H}^{m+1}$, $\partial M = \partial D$, M *of constant non-zero mean curvature. Assume* $\Sigma = M \cup D$ *is an oriented cycle,* M *oriented by its mean curvature vector and* $\mathbf{n_D}$ *the normal orienting* D*. Then*

$$\int_D \langle \mathbf{Y}, \mathbf{n_D} \rangle = \frac{1}{mH} \int_{\partial M} \langle \mathbf{Y}, \nu \rangle.$$

Here ν *is the inward pointing conormal to* M *along* ∂M.

Proof. The cycle Σ bounds an oriented immersed domain Ω in $\mathbb{H}^{m+1}$. The divergence of the induced vector field $\mathbf{Y}$ on Ω is zero, so, by the divergence theorem,

$$\int_D \langle \mathbf{Y}, \mathbf{n_D} \rangle = -\int_M \langle \mathbf{Y}, \mathbf{n_M} \rangle.$$

Let $\delta_{\mathbf{Y}}(A(M))$ denote the first variation of the area (volume) of M with $\mathbf{Y}$ as variation. The first variation formula of the area is

$$\delta_{\mathbf{Y}}(A(M)) = \int_M \operatorname{div}_M(\mathbf{Y}).$$

Since $\mathbf{Y}$ is a Killing vector field, the first variation of area is zero.

Consequently, we have

$$\begin{aligned} 0 &= \int_M \operatorname{div}_M(\mathbf{Y}) = \int_M \operatorname{div}_M(\mathbf{Y}^T) + \int_M \operatorname{div}_M(\mathbf{Y}^\perp) \\ &= -\int_{\partial M} \langle \mathbf{Y}, \nu \rangle - \int_M mH \langle \mathbf{Y}, \mathbf{n_M} \rangle, \end{aligned}$$

where $\mathbf{Y}^T$ and $\mathbf{Y}^\perp$ are the tangent and the normal parts of $\mathbf{Y}$ with respect to M. Since H is constant, the flux formula is established. □

Proof of Theorem 3.1. Let D be the domain in N bounded by $C = \partial M$. As in the Euclidean case we begin by proving that M is contained in one of the half-spaces of $\mathbb{H}^{m+1}$ determined by N.

We can assume $N = \{(x_1, \ldots, x_{m+1}) \mid \sum_{i=1}^{m+1} x_i^2 = 1,\ x_{m+1} > 0\}$ and that, in a neighbourhood of C, M is contained in the outer component U of $\mathbb{H}^{m+1} \setminus N$ (i.e., where some points have $x_{m+1} > 1$). Then we will prove $M \subset U$. Without loss of generality, we can assume M to be transverse to N.

Suppose on the contrary that $M \cap N$ has other components than C. First, one proves that not all components can be in D. The proof of this uses the flux formula and is the same as in the Euclidean case; we refer the reader to [BMRS] for details. Next one observes that if E is a component of $M \cap (N \setminus D)$, then E cannot be null homologous in $N \setminus D$. One sees this as follows.

Let γ be a geodesic in N starting at a point of D, of infinite length and $\gamma \cap E \neq \emptyset$. Let $F(t)$ be a family of geodesic hyperplanes of $\mathbb{H}^{m+1}$, $0 \leq t < \infty$, such that for each $q \in \gamma$, there exists exactly one $F(t)$ such that $F(t)$ intersects γ orthogonally at q. Parametrize so that $F(0)$ contains the initial point of γ (cf. Figure 3).

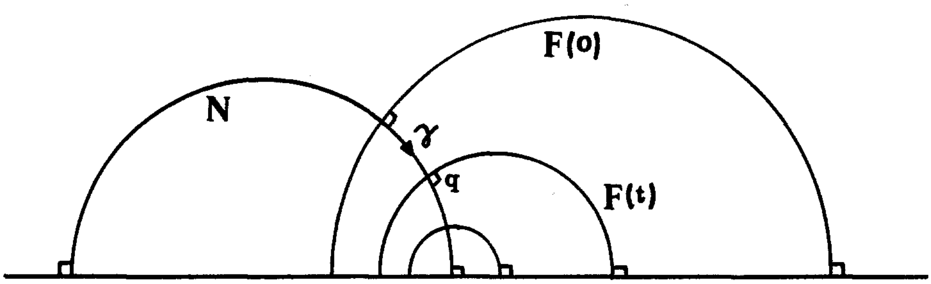

Fig. 3.

Notice that γ intersects C in exactly one point and γ intersects E in at least two points. Each $F(t)$ is orthogonal to N, so (hyperbolic) symmetry through $F(t)$ leaves N and U (the outer component of $\mathbb{H}^{m+1} \setminus N$) invariant.

Now one does Alexandrov reflection of M with the family $F(t)$. For t large, M is disjoint from $F(t)$. As t decreases from ∞ to zero, there will be a first point of contact of some $F(t)$ with M. One continues to decrease t and considers the symmetries of M through the planes $F(t)$. Since γ intersects E in at least two points, there will be some plane $F(\tau)$ such that the symmetry of M through $F(\tau)$ will touch M at an interior point. This occurs at an interior point since C is convex and $\gamma \cap C$ is one point. Thus, M is invariant under symmetry through $F(\tau)$, which is impossible (for M would then be part of an embedded closed manifold of constant mean curvature, hence, a sphere. But a sphere cannot meet N in two or more components). This shows that E cannot be homologous to zero in $N \setminus D$.

The same Alexandrov reflection technique shows that there cannot be more than one component E in $M \cap (N \setminus D)$. For, if E_1, E_2 are two such components, each of them bounds a domain in N that contains D, so γ meets each E_1, E_2 in at least one point. So Alexandrov reflection gives a plane of symmetry as before. Hence, there is at most one component E of $M \cap (N \setminus D)$ and E bounds a domain in N containing D. This case is ruled out exactly as in [BMRS] using the flux formula. Thus, we have $M \cap N = C$ and $M \subset U$.

Now we show that M has all the symmetries of C. Let $F = F(0)$ be a hyperplane of symmetry of C and N. Let γ be a geodesic in N starting at $F(0)$, γ orthogonal to $F(0)$, and γ of infinite length. Let $F(t)$ be a family of hyperplanes, $0 \leq t < \infty$, starting at $F(0)$ and such that each $q \in \gamma$ meets exactly one $F(t)$ and orthogonally.

Let $\Sigma = M \cup D$ and do Alexandrov reflection with Σ and the $F(t)$ exactly as in the proof of Theorem 2.1. Then M has the symmetries of C and Theorem 3.1 is proved. □

Finally we generalize a result in [NS].

Theorem 3.4. *Let M be a compact embedded hypersurface in $\mathbb{H}^{m+1}$ such that $\partial M = C$ is a sphere of codimension 2 and radius ρ, and the mean curvature of M is $H = \coth \rho$. Then M is part of a sphere of radius ρ.*

Proof. After an ambient isometry, we can assume that C is contained in the horosphere $L(1) = \{x_{m+1} = 1\}$ and centered at $(0, \dots, 1)$. Let $\mathbf{Y}$ be the Killing vector field in $\mathbb{H}^{m+1}$ defined by $\mathbf{Y}(x) = (x_1, \dots, x_{m+1})$.

Let D be the hyperbolic m-disk bounded by C ($D \subset \{\sum_{i=1}^{m+1} x_i^2 = (\cosh \rho)^2\}$) and orient $M \cup D$ so that the flux formula holds,

$$\int_D \langle \mathbf{Y}, \mathbf{n_D} \rangle = \frac{1}{mH} \int_{\partial M} \langle \mathbf{Y}, \boldsymbol{\nu} \rangle,$$

with the notation of Lemma 3.3.

Let D' be the m-disk contained in $L(1)$, bounded by C.

As $\mathbf{Y}$ is a Killing vector field, we have

$$\int_D \langle \mathbf{Y}, \mathbf{n_D} \rangle = \int_{D'} \langle \mathbf{Y}, \mathbf{n_{D'}} \rangle.$$

The second term is equal to $\mathrm{vol}(D') = m^{-1}(\sinh \rho)\mathrm{vol}(C)$ (as the induced metric on $L(1)$ is the Euclidean metric and the Euclidean radius of D' is $\sinh \rho$), hence we have

$$mH \int_D \langle \mathbf{Y}, \mathbf{n_D} \rangle = (\cosh \rho)\mathrm{vol}(C).$$

Now

$$-(\cosh \rho)\mathrm{vol}(C) \le \int_C \langle \mathbf{Y}, \boldsymbol{\nu} \rangle \le (\cosh \rho)\mathrm{vol}(C)$$

as $|\mathbf{Y}| = \cosh \rho$ on C, and equality is possible if and only if $\langle \mathbf{Y}, \boldsymbol{\nu} \rangle = \pm \cosh \rho$ on C, i.e., $\mathbf{Y} = \pm(\cosh \rho)\boldsymbol{\nu}$ on C.

Thus the flux formula yields $\boldsymbol{\nu} = \pm \mathbf{n_D}$ on C; this means that M is orthogonal to the hyperplane N.

By Theorem 3.1, M is a sphere and Theorem 3.4 is proved. □

Added in Proof. The authors recently received a preprint of Barbosa and Sa Earp related to our paper. They proved our Theorem 2.1 in the case $r = 1$ and M immersed; their technique is different than ours.

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B. NELLI
Dipartimento di Matematica
via Buonarroti 2
56127 Pisa
Italy
e-mail: nelli@dm.unipi.it

H. ROSENBERG
Département de Mathématiques
Université de Paris VII
2, Place Jussieu
75251 Paris
France

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