BULLETIN BRAZILIAN MATHEMATICAL SOCIETY

Bull Braz Math Soc, New Series 33(2), 263-292

Minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$

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— Dedicated to IMPA on the occasion of its 50^{th} *anniversary*

Abstract. In $\mathbb{H}^2 \times \mathbb{R}$ one has catenoids, helicoids and Scherk-type surfaces. A Jenkins-Serrin type theorem holds here. Moreover there exist complete minimal graphs in $\mathbb{H}^2$ with arbitrary continuous asymptotic values. Finally, a graph on a domain of $\mathbb{H}^2$ cannot have an isolated singularity.

Keywords: minimal graph, hyperbolic plane.

Mathematical subject classification: 53A10.

1 Introduction

In this paper we consider minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$; particularly, surfaces which are vertical graphs over domains in $\mathbb{H}^2$. When a convex domain $\mathcal{D} \subset \mathbb{H}^2$ is bounded by geodesic arcs $A_1, \dots A_n$, $B_1, \dots, B_m$, together with strictly convex arcs $C_1, \dots, C_s$, we obtain necessary and sufficient conditions (in terms of the lengths of the boundary arcs of $\mathcal{D}$) which assure the existence of a unique function u defined in $\mathcal{D}$, whose graph is a minimal surface of $\mathbb{H}^2 \times \mathbb{R}$, and which takes the values $+\infty$ on the arcs $A_1, \dots A_n$, $-\infty$ on the arcs $B_1, \dots, B_m$, and arbitrary prescribed continuous data on the arcs $C_1, \dots, C_s$. In $\mathbb{R}^2 \times \mathbb{R}$, this is the theorem of Jenkins and Serrin [JS].

For example, let $\mathcal{D}$ be a domain whose boundary is a regular geodesic octagon with sides $A_1, B_1, \dots, A_4, B_4$, and suppose the interior angles are $\frac{\pi}{2}$. Our theorem yields a function u in $\mathcal{D}$, whose graph is minimal, taking the values $+\infty$ on each A_i, and $-\infty$ on each B_j. The graph of u is bounded by the eight vertical geodesics passing through the vertices of $\mathcal{D}$. Rotation of each $\mathbb{H}^2 \times \{t\}$, by π about each vertex of $\mathcal{D} \times \{t\}$, extends the graph of u to a complete embedded minimal surface in $\mathbb{H}^2 \times \mathbb{R}$ (one continues the rotation about all the vertical geodesics that arise). One can take the quotient of $\mathbb{H}^2 \times \mathbb{R}$ by various Fuchsian groups to obtain interesting quotient surfaces. For example, one can obtain an

Received 20 June 2002.

8-punctured sphere in the quotient of total curvature -12π; with four top ends and four bottom ends.

One also obtains graphs over ideal polygons with vertices at infinity. For example, consider the polygon which is the boundary of the convex hull of the n roots of unity, n even and at least four. Then, there is a minimal graph over the interior of this polygon taking the values plus and minus infinity on adjacent edges (cf. Figure 2(b)).

We prove the existence of entire minimal graphs over $\mathbb{H}^2$ (Bernstein's theorem fails here). In the model $\{0 \leq x_1^2 + x_2^2 < 1\}$ of $\mathbb{H}^2$, the asymptotic boundary of $\mathbb{H}^2 \times \mathbb{R}$ is $\{x_1^2 + x_2^2 = 1\} \times \mathbb{R}$. For any Jordan curve Γ in the asymptotic boundary of $\mathbb{H}^2 \times \mathbb{R}$ that has a simple projection on $\{x_1^2 + x_2^2 = 1\}$, there is a minimal graph over $\mathbb{H}^2$ having Γ as asymptotic boundary.

In [DN], the existence of such minimal graphs is established when Γ is the boundary value of a function with very small C^3-norm on the disk.

Finally, we prove a theorem for minimal graphs defined over a punctured disk in $\mathbb{H}^2$: the graph extends smoothly to the puncture.

2 Preliminaries

In the three dimensional manifold $\mathbb{H}^2 \times \mathbb{R}$, we take the disk model for $\mathbb{H}^2$. Let x_1, x_2 denote the coordinates in $\mathbb{H}^2$ and x_3 the coordinate in $\mathbb{R}$. The metric in $\mathbb{H}^2 \times \mathbb{R}$ is

$$d\sigma^2 = \frac{dx_1^2 + dx_2^2}{F} + dx_3^2$$

where

$$F = \left(\frac{1 - x_1^2 - x_2^2}{2}\right)^2$$

The graph of a function u defined over a domain in $\mathbb{H}^2$ has constant mean curvature H if and only if u satisfies the following equation:

$$\operatorname{div}\left(\frac{\nabla u}{\tau_u}\right) = 2H \tag{1}$$

where $\tau_u = \sqrt{1 + F|\nabla u|^2}$ and the divergence is the divergence in $\mathbb{R}^2$. We list the principal steps for the computation of (1).

The Christoffel symbols for the metric $d\sigma^2$ are the following:

$$\Gamma_{11}^1 = \Gamma_{12}^2 = \Gamma_{21}^2 = \frac{x_1}{\sqrt{F}}$$

$$\Gamma_{22}^2 = \Gamma_{12}^1 = \Gamma_{21}^1 = \frac{x_2}{\sqrt{F}}$$

$$\Gamma_{11}^2 = -\frac{x_2}{\sqrt{F}}, \ \Gamma_{22}^1 = -\frac{x_1}{\sqrt{F}}$$

The other Γ_{ij}^k are identically zero.

Let e_1, e_2, e_3, be the canonical basis of $\mathbb{R}^3$ and set $\varepsilon_1 = \sqrt{F}e_1$, $\varepsilon_2\sqrt{F}e_2$, $\varepsilon_3 = e_3$, so that ε_1, ε_2, ε_3 is an orthonormal basis for $\mathbb{H}^2 \times \mathbb{R}$. Finally let $\bar{\nabla}$ be the connection of the metric $d\sigma^2$. We have:

$$\bar{\nabla}_{\varepsilon_1}\varepsilon_1 = -x_2\varepsilon_2, \quad \bar{\nabla}_{\varepsilon_2}\varepsilon_2 = -x_1\varepsilon_1,$$
$$\bar{\nabla}_{\varepsilon_1}\varepsilon_2 = x_2\varepsilon_1, \qquad \bar{\nabla}_{\varepsilon_2}\varepsilon_1 = x_1\varepsilon_2.$$

The coordinate vector fields on the graph of u are $X_1 = \frac{1}{\sqrt{F}}\varepsilon_1 + u_1\varepsilon_3$, $X_2 = \frac{1}{\sqrt{F}}\varepsilon_2 + u_2\varepsilon_3$ and $N = \tau^{-1}(-u_1\sqrt{F}\varepsilon_1 - u_2\sqrt{F}\varepsilon_2 + \varepsilon_3)$ is the upward unit normal.

The induced metric on the graph is:

$$g_{11} = \frac{1}{F} + u_1^2, \quad g_{12} = u_1u_2, \quad g_{22} = \frac{1}{F} + u_2^2.$$

The coefficients of the second fundamental form are:

$$b_{11} = \langle \bar{\nabla}_{X_1}X_1, N\rangle = \frac{1}{\tau}\left(-\frac{x_1u_1}{\sqrt{F}} + \frac{x_2u_2}{\sqrt{F}} + u_{11}\right)$$

$$b_{12} = \langle \bar{\nabla}_{X_1}X_2, N\rangle = \frac{1}{\tau}\left(-\frac{x_2u_1}{\sqrt{F}} - \frac{x_1u_2}{\sqrt{F}} + u_{12}\right)$$

$$b_{22} = \langle \bar{\nabla}_{X_2}X_2, N\rangle = \frac{1}{\tau}\left(\frac{x_1u_1}{\sqrt{F}} - \frac{x_2u_2}{\sqrt{F}} + u_{22}\right)$$

where $\langle\ ,\ \rangle$ is the scalar product for the metric $d\sigma^2$.

Equation (1) is obtained by substituting the quantities just calculated in the following identity:

$$2H = \frac{b_{11}g_{22} + b_{22}g_{11} - 2b_{12}g_{12}}{g_{11}g_{22} - g_{12}^2}.$$

3 Catenoids and Helicoids

Catenoids. We construct a family of minimal rotational surfaces in $\mathbb{H}^2 \times \mathbb{R}$ (see also [PR]).

Let π be a vertical geodesic plane containing the origin and let γ be a curve in π. Assume γ to be a graph over the x_3 axis. Let r be the Euclidean distance between the point of γ at height t and the x_3 axis: $r = r(t)$ is a parametrization of the curve γ. Consider the surface of revolution S obtained by rotating γ about the x_3 axis. S is minimal if and only if $r = r(t)$ satisfies the following differential equation:

$$4r(t)r''(t) - 4r'(t)^2 - (1 - r(t)^4) = 0. \tag{2}$$

A first integral for equation (2) is

$$J(t) = -\frac{r'^2}{r^2} - \frac{1 + r^4}{4r^2} = -C \tag{3}$$

where C is a positive constant (*i.e.* $\frac{dJ(t)}{dt} = 0$ along the curve γ).

Hence we obtain:

$$r' = \pm\sqrt{Cr^2 - \frac{1 + r^4}{4}} \tag{4}$$

The allowed values for C and r are

$$C > \frac{1}{2}$$

$$2C - \sqrt{4C^2 - 1} \leq r^2 < 1 \leq 2C + \sqrt{4C^2 - 1}$$

i.e.

$$r_{min} = \sqrt{\frac{2C + 1}{2}} - \sqrt{\frac{2C - 1}{2}} \leq r < 1.$$

Remark that as $C \longrightarrow \frac{1}{2}$ then $r_{min} \longrightarrow 1$ hence the curve γ disappears at infinity, while as $C \longrightarrow \infty$ then $r_{min} \longrightarrow 0$.

We can write equation (4) as follows:

$$\frac{dt}{dr} = \pm\frac{2}{\sqrt{4Cr^2 - (1 + r^4)}} \tag{5}$$

In order to study the curve γ we can choose the positive sign in (5), i.e. $t > 0$. In fact by the symmetries of equation (4) and (5), the curve γ for $t < 0$ will be the reflection with respect to the plane $x_3 = 0$ of the curve γ for $t > 0$. With the choice of the positive sign, we have the following properties.

(i) $\frac{dt}{dr} > 0$ hence t is an increasing function of r.

(ii) For $r = r_{min}$ we have $\frac{dt}{dr} = \infty$, *i.e.* the tangent to the curve γ at the point $r = r_{min}$ is parallel to the x_3 axis and this is the only point where this happens.

(iii) As $r \longrightarrow 1$, we have:

$$\frac{dt}{dr} \longrightarrow \frac{\sqrt{2}}{\sqrt{2C-1}}.$$

Hence, when C varies between $\frac{1}{2}$ and $+\infty$ the asymptotic angle of γ varies between $\frac{\pi}{2}$ and 0.

(iv) $\frac{d^2t}{dr^2} < 0$, hence the concavity does not change.

(v) Consider the change of variables $w = r^2$. Equation (5) becomes

$$dt = \pm \frac{dw}{\sqrt{w(2C + \sqrt{4C^2-1} - w)(w - 2C + \sqrt{4C^2-1})}} \tag{6}$$

Hence

$$t(w) = \pm \int_{2C-\sqrt{4C^2-1}}^{w} \left(s(2C + \sqrt{4C^2-1} - s)(s - 2C + \sqrt{4C^2-1}) \right)^{-\frac{1}{2}} ds$$

This is an elliptic integral. By the properties of elliptic functions $\lim_{C\to\infty} t(w)$ is independent of w. For every value of the constant C we have $t(4C - \sqrt{4C^2-1}) = 0$, hence for the limit value $C = \infty$, the surface is a horizontal plane (doubly covered).

Using (i)-(v) we have the following theorem (see Figure 1).

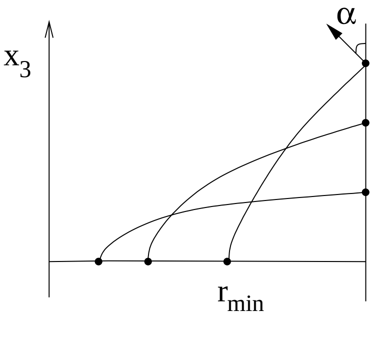


Figure 1

Theorem 1. *Let $\Gamma_\pm(t)$ be the two circles at infinity of $\mathbb{H}^2 \times \mathbb{R}$ defined by* $\{x_1^2 + x_2^2 = 1, \ \ x_3 = \pm t\}$.

Then, for each $t > 0$ there exists a rotational surface (catenoid) $C(t)$, whose asyptotic boundary is $\Gamma_+(t) \cup \Gamma_-(t)$. As $t \longrightarrow 0$, $C(t)$ converges to the doubly *covered plane $\mathbb{H}^2$ with a singularity at the origin. As $t \longrightarrow \infty$, $C(t)$ diverges to the asymptotic boundary of $\mathbb{H}^2 \times \mathbb{R}$. Furthermore for any angle $\alpha \in]0, \frac{\pi}{2}[$ there is a $C(t)$ whose asymptotic normal vector at boundary points forms an angle equal to α with the x_3 axis.*

Helicoids. Let α be a horizontal geodesic passing through the x_3 axis. Consider the surface $\mathcal{E}$ obtained by translating α vertically and rotating it around the x_3 axis. A parametrization for $\mathcal{E}$ is the following:

$$X(u, v) = (v \cos \theta(u), v \sin \theta(u), u)$$

$v \in (-1, 1)$, $u \in \mathbb{R}$ and $\theta : \mathbb{R} \longrightarrow \mathbb{R}$ is a C^2 function representing the angle between α and the x_1 axis at the level u.

$\mathcal{E}$ is a minimal surface if and only if

$$\theta_{uu} = 0.$$

Hence the solutions are:

(i) $\theta(u) = a$, $a \in \mathbb{R}$. In this case $\mathcal{E}$ is a vertical plane forming an angle a with the x_1 axis.

(ii) $\theta(u) = au$, $a \in \mathbb{R} \setminus \{0\}$. In this case the surface $\mathcal{E}$ is congruent to the Euclidean helicoid.

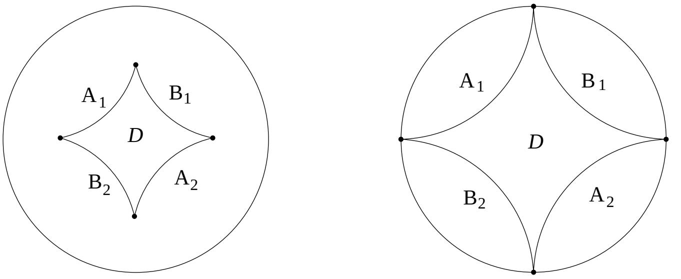
4 Scherk type surfaces

Let $\mathcal{P}$ be a regular $2k$-gon in $\mathbb{H}^2$ with (open) edges $A_1, B_1, \ldots, A_k, B_k$ (see Figure 2(a), $k = 2$). We will construct a minimal graph Σ over the domain $\mathcal{D}$ bounded by $\mathcal{P}$ such that the boundary values are alternatively $+\infty$ on the edges A_i and $-\infty$ on the edges B_i. Σ will be called a Scherk type surface. Also, the Scherk surface exists if the vertices of A_i and B_j are at infinity; so that $\mathcal{P}$ is a ideal polygon whose vertices are the $2k$ roots of unity (see Figure 2(b), $k = 2$).

Choose one edge of $\mathcal{P}$ where the desired value is $+\infty$ and call it A. Let B and C be the two geodesic arcs passing through the center of $\mathcal{P}$ and the vertices of A. Let T be the (open) triangle with sides A, B and C. Let $\Gamma(n)$ be the curve obtained by the union of the following geodesics arcs: B, C together with the arc obtained by raising A to height n, and the vertical geodesics joining the vertices of the raised A with the vertices of A.

Let Σ_n be a solution of Plateau's problem for $\Gamma(n)$. Rado's theorem is true in $\mathbb{H}^2 \times \mathbb{R}$, since vertical translation is an isometry, hence Σ_n is the graph of a function u_n defined in the triangle T and

$$u_{n|A} = +n, \quad u_{n|B} = u_{n|C} = 0.$$



(a) Geodesic 4 − gon (b) Ideal 4 − gon

Figure 2

Theorem 2. *The sequence $\{u_n\}$ converges to a minimal solution u defined in T such that*

$$u_{|A} = +\infty, \quad u_{|B} = u_{|C} = 0.$$

Moreover, the gradient of u diverges as one approaches the side A.

Proof. The sequence $\{u_n\}$ is non decreasing and positive. Hence, to show that the function u exists, we will prove that the sequence $\{u_n\}$ is uniformly bounded on compact subsets K of T.

We start by constructing a barrier over the graph of the u_n in K.

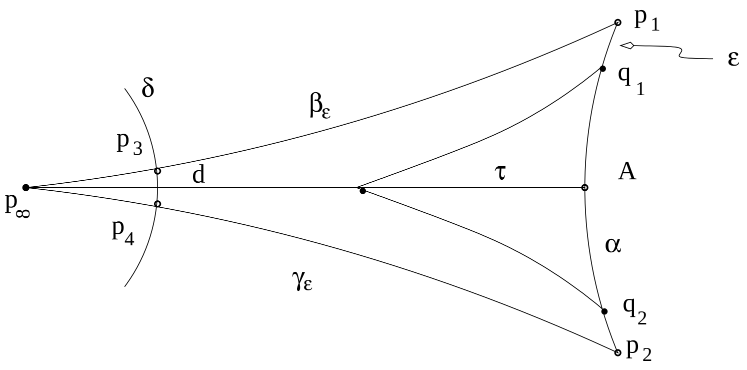


Figure 3

The following construction is represented in Figure 3.

Let α be the horizontal geodesic containing the side A. Denote by q_1 and q_2 the vertices of A. For $i = 1, 2$, let p_i the point on $\alpha \setminus A$ at a distance $\varepsilon > 0$ from q_i. Denote by a_ε the geodesic arc between p_1 and p_2. Let τ be the horizontal geodesic orthogonal to the edge A passing through the mid-point of A (in order to simplify Figure 3 we assume that τ passes through the origin of $\mathbb{H}^2$). Let p_∞ be the point at infinity of τ contained in the same halfplane as T defined by α.

Finally denote by β_ε the horizontal geodesic through p_1 and p_∞ and by γ_ε the horizontal geodesic through p_2 and p_∞.

Consider a horizontal geodesic δ orthogonal to τ and let $p_3 = \delta \cap \beta_\varepsilon$, $p_4 = \delta \cap \gamma_\varepsilon$. Call d the geodesic arc on δ between p_3 and p_4, b_ε the geodesic arc on β_ε between p_1 and p_3 and c_ε the geodesic arc on γ_ε between p_2 and p_4. We can chose $||b_\varepsilon||$ and $||c_\varepsilon||$ large enough such that the quadrilateral with edges a_ε, b_ε, c_ε, d contains the triangle T.

Let h be a positive number and call $*(h)$ each object obtained by translating vertically to height h an object of $\mathbb{H}^2 \times \{0\}$. Consider the following curves (see Figure 4):

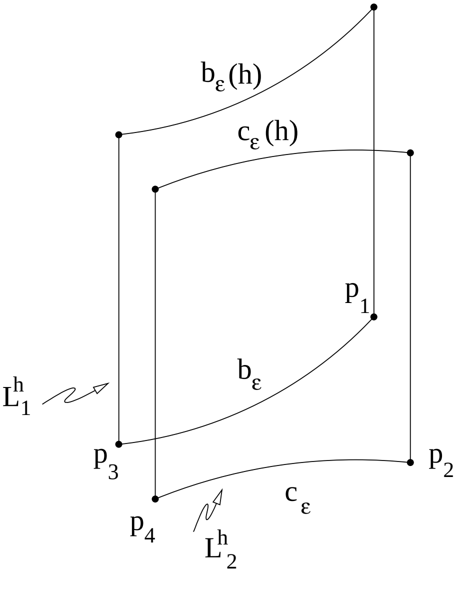


Figure 4

$$L_1^h = b_\varepsilon \cup (p_1 \times [0,h]) \cup b_\varepsilon(h) \cup (p_3 \times [0,h]),$$

$$L_2^h = c_\varepsilon \cup (p_2 \times [0,h]) \cup c_\varepsilon(h) \cup (p_4 \times [0,h]).$$

We claim that there exists a least area, hence stable, minimal annulus bounded by $L_1^h \cup L_2^h$, if $||b_\varepsilon||$ is sufficiently large. A sufficient condition is given by the Douglas criteria for the Plateau problem: if there is an annulus bounded by $L_1^h \cup L_2^h$ with area smaller than the sum of the areas of the flat geodesic domains bounded by L_1^h and L_2^h, then there exists a least area minimal annulus bounded by $L_1^h \cup L_2^h$.

By a straightforward computation we obtain that the sum of the areas of the flat geodesic domains bounded by L_1^h and L_2^h is equal to $2||b_\varepsilon||h$ (as $||c_\varepsilon|| = ||b_\varepsilon||$).

Now, consider the annulus that is the union of the four geodesic domains bounded by the following quadrilaterals (see Figure 5):

$$Q_1 = a_\varepsilon \cup (p_1 \times [0,h]) \cup a_\varepsilon(h) \cup (p_2 \times [0,h]),$$
$$Q_2 = d \cup (p_3 \times [0,h]) \cup d(h) \cup (p_4 \times [0,h]),$$
$$Q_3 = a_\varepsilon \cup b_\varepsilon \cup d \cup c_\varepsilon,$$
$$Q_4 = a_\varepsilon(h) \cup b_\varepsilon(h) \cup d(h) \cup c_\varepsilon(h).$$

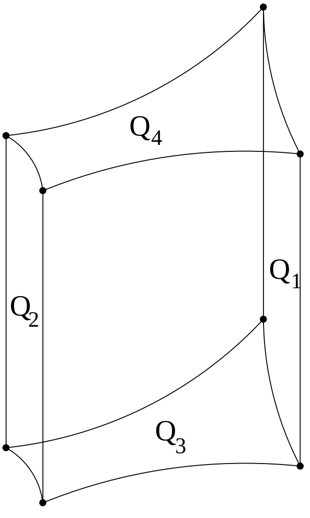


Figure 5

The area of the annulus is at most $2\pi + 2||a_\varepsilon||h$, as the area of a hyperbolic triangle is always smaller than π.

Then, in order to satisfy the Douglas condition, we need:

$$2\pi + 2||a_\varepsilon||h < 2||b_\varepsilon||h$$

that is verified as soon as we choose the edge b_ε long enough. Hence, there exists a least area minimal annulus A_ε^h bounded by L_1^h and L_2^h, for any h. By the maximum principle A_ε^h is contained in the convex hull of $L_1^h \cup L_2^h$.

For each n the annulus A_ε^h is above the surface Σ_n (the graph of u_n); by above we mean that if a vertical geodesic meets both surfaces, then the point of Σ_n is below the points of A_ε^h.

To see this, translate vertically A^h_ε to height n (so, every point of A^h_ε is above height n). Then lower the translated A^h_ε back to height zero. By the maximum principle, there is no interior contact point between A^h_ε and Σ_n before returning to the original position of A^h_ε. Moreover, if $\epsilon > 0$, the boundaries of the two surfaces do not touch. Letting $\varepsilon \longrightarrow 0$, we conclude that $A^h_0 = A^h$ is above Σ_n and, by the boundary maximum principle at each interior point of the vertical geodesics $q_1 \times [0, h]$ and $q_2 \times [0, h]$ the tangent plane to A^h is "outside" the tangent plane to Σ_n (i.e. the angle between the tangent plane to Σ_n and the geodesic plane containing either L^h_1 or L^h_2 is bigger than the angle between this last plane and the tangent plane to A^h).

The barrier A^h shows that the sequence $\{u_n\}$ is uniformly bounded on compact subsets K of T such that K is contained in the horizontal projection of A^h. The idea is to show that the horizontal projections of A^h exhaust T as $h \longrightarrow \infty$.

For $k > h$, one can use A^h as barrier to solve the Plateau problem to find a stable annulus A^k with boundary L^k_1, L^k_2. So, translating A^h vertically, one sees that the two surfaces are never tangent (neither at interior points, nor at boundary points). Hence as $k \longrightarrow \infty$, the angle the tangent plane of A^k makes along the vertical boundary segments is controlled by that of A^h.

Now, for each n let M^n be the surface A^{2n} translated down a distance n. As each M_n is stable, one has local uniform area bounds and uniform curvature estimates (see [Sc]). So, a subsequence of $\{M^n\}$ converges to a minimal surface M^∞. By the maximum principle, one can translate A^h up to $+\infty$ and down to $-\infty$ without ever touching M^∞. Then, there is some component M of M^∞ whose boundary is the union of the two vertical geodesics $q_1 \times \mathbb{R}$ and $q_2 \times \mathbb{R}$. Furthermore the distance between M and $A \times \mathbb{R}$ is bounded. In fact this distance is uniformly bounded.

Now, we have to prove that $M = A \times \mathbb{R}$.

In $\mathbb{H}^2$, consider the family of equidistant circles $\{C_t\}_{t \geq 0}$ defined as follows: $C_0 = \alpha$, each C_t is the circle equidistant from the geodesic α, whose curvature vector points towards the halfplane P_+ determined by α, containing the triangle T (see Figure 6).

The family of surfaces $C_t \times \mathbb{R}$ foliates $P_+ \times \mathbb{R}$. When t is large one has

$$(C_t \times \mathbb{R}) \cap M = \emptyset.$$

Now decrease t : By the maximum principle, one cannot have a first point of contact between M and $C_t \times \mathbb{R}$ before $t = 0$. Then $M = A \times \mathbb{R}$ and we are through.

Thus $\{u_n\}$ has a subsequence converging to a minimal solution u defined on T. The convergence is uniform on compact subsets of T. Furthermore, as we desired:

$$u_{|A} = \infty, \quad u_{|B} = u_{|C} = 0.$$

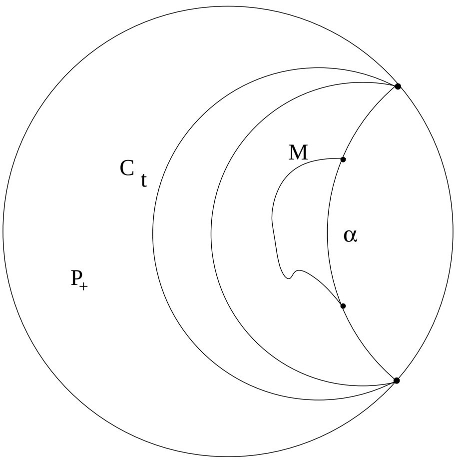


Figure 6

The last assertion of Theorem 2 follows from a more general fact proved in Lemma 1 of next section.

Remark 1. We notice that our construction can be done for any angle, smaller than π, between the two geodesic arcs where the boundary value is zero. In a $2k$-gon $\mathcal{P}$ as above, such angles are $\frac{\pi}{k}$, $k = 2, 3, \ldots$. Then, we make the symmetry of the graph of u with respect to one of the two geodesic arcs where u is zero and keep on going with such symmetries in order to close the surface. The surface thus obtained is the Scherk type surface Σ, which we were looking for.

Also one can show that the Scherk solutions u in T converge to a Scherk solution in the ideal triangle T^* obtained as limit of the triangles T when the length of the sides B and C tend to infinity. Reflection then gives a Scherk surface graph over the interior of a $2k$-gon $\mathcal{P}$.

Remark 2. When interior angles of $\mathcal{P}$ are choosen to be $\frac{\pi}{2}$, $(k > 2)$, then Σ extends to a complete embedded minimal surface in $\mathbb{H}^2 \times \mathbb{R}$. In fact the surface Σ is bounded by the $2k$ vertical geodesic through the vertices of $\mathcal{P}$ and one extends Σ by rotation of π about all the vertical geodesics that arise.

Remark 3. Let $\mathcal{P}$ be the regular $2k$-gon with $\frac{\pi}{2}$ angles. Consider the symmetry of $\mathcal{P}$ about each of its vertices. This produces $2k$ new $2k$-gons isometric to $\mathcal{P}$, each having a vertex in common with $\mathcal{P}$. Consider the hyperbolic isometries identifying alternate sides of $\mathcal{P}$ (that is the translation along the edge between the two chosen sides). The quotient of the surface Σ by these translations gives a $2k$-punctured sphere whose total curvature is $-4\pi(1-k)$.

Remark 4. There is another natural way to obtain a complete surface from Σ_n when the interior angles of $\mathcal{P}$ are equal to $\frac{\pi}{2}$ (Σ_n is the graph of u_n over the triangle T). Assume that the polygon $\mathcal{P}$ has $2k$ sides. Do the symmetry of Σ_n about all the geodesic arcs of its boundary. Then continue extending the surface by symmetry in the geodesic arcs of the boundary. This yields a complete embedded minimal surface in $\mathbb{H}^2 \times \mathbb{R}$ that is invariant by vertical translation by $2n$. The quotient of the surface by this translation gives a compact surface of genus k.

5 Jenkins-Serrin type theorems

We give necessary and sufficient conditions to solve the Dirichlet problem for the minimal surface equation in $\mathbb{H}^2 \times \mathbb{R}$, over a convex domain of $\mathbb{H}^2$, allowing infinite boundary values on some arcs of the boundary of the domain.

Let us fix some notation.

We consider an open bounded convex domain $\mathcal{D}$ whose boundary $\partial\mathcal{D}$ contains two sets of (open) geodesic arcs $A_1, \ldots, A_k$ and $B_1, \ldots, B_l$ with the property that no two A_i and no two B_i have a common endpoint. The remaining part of $\partial\mathcal{D}$ is the union of open convex arcs $C_1, \ldots, C_h$ and all endpoints.

We want to find a solution u of the minimal surface equation in $\mathcal{D}$ such that

$$u_{|A_i} = +\infty, \quad u_{|B_j} = -\infty,$$

$i = 1, \ldots, k, \quad j = 1, \ldots, l$ and u takes assigned continuous data on each arc C_s, $s = 1, \ldots, h$.

The existence of such a solution depends on a relation between the lengths of the geodesic arcs of the boundary and the perimeter of polygons inscribed in $\partial\mathcal{D}$ whose vertices are chosen among the vertices of A_i, B_j.

Let $\mathcal{P}$ be such a polygon and let

$$\alpha = \sum_{A_i \subset \mathcal{P}} ||A_i||, \quad \beta = \sum_{B_j \subset \mathcal{P}} ||B_j||, \quad \gamma = \text{Perimeter}(\mathcal{P}).$$

Theorem 3. *Let $\mathcal{D}$ be a domain as above and let $f^s : C_s \longrightarrow \mathbb{R}$ be continuous functions. If $\{C_s\} \neq \emptyset$, then the Dirichlet problem in $\mathcal{D}$ with boundary values*

$$u_{|A_i} = +\infty, \quad u_{|B_j} = -\infty, \quad u_{|C_s} = f^s$$

has a solution if and only if

$$2\alpha < \gamma, \quad 2\beta < \gamma \tag{9}$$

for each polygon $\mathcal{P}$ as above. If $\{C_s\} = \emptyset$ the result is the same except that if $\mathcal{P} = \partial\mathcal{D}$, then condition (9) *should be replaced by $\alpha = \beta$.*

If it exists, the solution is unique; in the case $\{C_s\} = \emptyset$ uniqueness is up to a constant.

Remark 5. We notice that two convex arcs C_s may have a common endpoint p; there may be a discontinuity of the data f^s at p. It will be clear from the proof of Theorem 3 that the minimal surface obtained in this case will contain the vertical segment through p, between the two limit values at p, of the continuous boundary data.

This result is analogous to that of Jenkins and Serrin for minimal graphs in $\mathbb{R}^3$ (cf. [JS]).

We prove Theorem 3 in 6 steps. Each step, especially 1 and 3, is an interesting result on its own.

Step 1. Existence when $\partial\mathcal{D}$ contains only one geodesic arc A, and one strictly convex arc C. The function $f : C \longrightarrow \mathbb{R}$ is continuous and positive.

Step 2. Existence when $\partial\mathcal{D}$ contains geodesic arcs $A_1, \ldots, A_k$ and strictly convex arcs $C_1, \ldots, C_h$. The functions $f^s : C_s \longrightarrow \mathbb{R}$ are continuous and positive.

Step 3. The same as Step 2, with $C_1, \ldots, C_h$ convex arcs (not necessarily strictly convex).

Step 4. Existence when $\partial\mathcal{D}$ contains geodesic arcs $A_1, \ldots, A_k$, $B_1, \ldots, B_l$ and convex arcs $C_1, \ldots, C_h$ with $h \geq 1$.

Step 5. Existence when $\partial\mathcal{D}$ contains only geodesic arcs $A_1, \ldots, A_k$, $B_1, \ldots, B_l$.

Step 6. Uniqueness.

Proof of Step 1. Let $u_n : \mathcal{D} \longrightarrow \mathbb{R}$ be the minimal solution with boundary values

$$u_{n|A} = +n, \quad u_{n|C} = \min(n, f)$$

(Figure 7(a)). Let us prove that u_n exists. Define $\Gamma(n)$ to be the union of the following geodesic arcs: the geodesic arc A raised to height n, the graph of the function $\min(n, f)$ and the vertical geodesic arcs joining the endpoints of the curves just described. Let $\Sigma(n)$ be the solution of the Plateau problem for the curve $\Gamma(n)$. By Rado's theorem, $\Sigma(n)$ is the graph of a function u_n defined in $\mathcal{D}$, with the desired boundary values. By the maximum principle, $\{u_n\}$ is an increasing sequence.

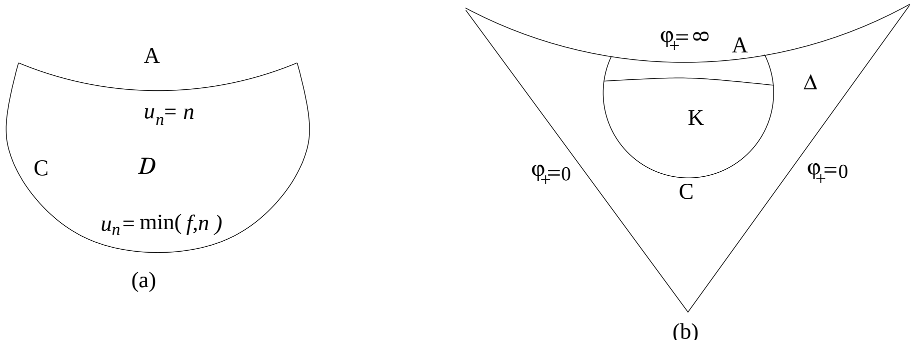


Figure 7

We now prove that the sequence $\{u_n\}$ is uniformly bounded on compact subsets of $\mathcal{D}$. Let Δ be a horizontal geodesic triangle containing $\mathcal{D}$, with sides a, b, c, such that the side a contains A in its interior. Let φ_+ be the Scherk type solution equal to $+\infty$, on a, and zero on b and c (Figure 7(b)).

Let K be a compact set in $\mathcal{D} \cup C$. On ∂K we have

$$0 \leq u_n \leq \max_{K \cap C} f + \varphi_+$$

By the maximum principle, the previous inequality holds in K. Hence $\{u_n\}$ is uniformly bounded in K, and $\{u_n\}$ converges to a minimal solution u in every compact subset of $\mathcal{D} \cup C$. As $\{u_n\}$ is an increasing sequence, u takes the right boundary values.

Remark 6. Let C be a strictly convex arc and denote by $\mathcal{C}(C)$ the (open) convex hull of C. Let u be a minimal solution in $\mathcal{C}(C)$ with bounded values on C. As a result of the previous proof, u is bounded on every compact set of $\mathcal{C}(C)$ depending only on the values of u on C and on the distance of the compact set to the boundary of $\mathcal{C}(C)$.

Assertion. If u is a minimal solution that is unbounded on C, then u is unbounded in $\mathcal{C}(C)$.

This assertion implies that for solving the Dirichlet problem one can not assign infinite data on a strictly convex arc of the boundary of the domain.

For the proof of the assertion we use a modification of the argument of step 1. Let A be the geodesic arc in the boundary of $\mathcal{C}(C)$. For each $n \in I\!N$, define the function $u_n = \min\{n, u\}$. Let Δ be a horizontal geodesic triangle containing $C(C)$ with sides a, b, c, such that the side a contains A in its interior. Let φ_+ be the Scherk type solution equal to $+\infty$, on a, and zero on b and c.

Let K be a compact set in $C(C) \cup C$. On ∂K we have

$$\min_{K \cap C} u_n \leq u_n \leq \max_{K \cap C} u_n + \varphi_+$$

By the maximum principle, the previous inequality holds in K. Letting $n \longrightarrow \infty$, one sees that u is unbounded on $C(C)$.

For the proof of Step 2, we need several preliminary results.

The first depends on the fact that the tangent plane to the graph of a minimal solution is almost vertical at points near to a geodesic arc of the boundary where the solution diverges to infinity. Let us be more precise.

Denote by S the graph of a minimal solution $u : \mathcal{D} \longrightarrow \mathbb{R}$ and let

$$(\nu)_u = ((\nu_1)_u, (\nu_2)_u, (\nu_3)_u)$$

be the inward unit conormal to the boundary of S.

Let $(x_1(s), x_2(s), x_3(s))$ be an arc length parametrization of the boundary of S. A straightforward computation yields:

$$(\nu_3)_u = -\frac{\partial x_1}{\partial s}\frac{u_2}{\tau} + \frac{\partial x_2}{\partial s}\frac{u_1}{\tau}.$$

Then $|(\nu_3)_u| < 1$ and $(\nu_3)_u$ is integrable on arcs of $\partial \mathcal{D}$ regardless of the boundary behaviour of u on such arcs. The behaviour of the flux of $(\nu_3)_u$ on geodesic arcs of the boundary is established in the following Lemma.

Lemma 1. *Let $\mathcal{D}$ be a domain and let A be a geodesic arc of the boundary of $\mathcal{D}$.*

(i) Let $u : \mathcal{D} \longrightarrow \mathbb{R}$ be a minimal solution such that $u_{|A} = \infty$. Then

$$\int_A (\nu_3)_u ds = ||A||.$$

(ii) Let $\{u_n\}$ be a sequence of minimal solutions in $\mathcal{D}$ continuous in $\mathcal{D} \cup A$. If $\{u_n\}$ diverges uniformly to infinity on compact subsets of A and remains uniformly bounded in compact subsets of $\mathcal{D}$, then

$$\lim_{n \longrightarrow \infty} \int_A (\nu_3)_n ds = ||A||,$$

where $(\nu_3)_n$ is the third component of the unit conormal to the boundary of the graph of u_n.

If $\{u_n\}$ diverges uniformly to infinity in compact subsets of $\mathcal{D}$ and remains uniformly bounded on compact subsets of A, then

$$\lim_{n\longrightarrow\infty}\int_A (\nu_3)_n ds = -||A||.$$

Proof. (i) First we prove that the tangent plane to S at points $(z, u(z))$ with z next to A is almost vertical. Let $N = (N_1, N_2, N_3)$ be the upward unit normal to S, then $|(\nu_3)_u| = \sqrt{1 - N_3^2}$ at boundary points. We extend ν_3 to the interior points of $\mathcal{D}$ by setting $|(\nu_3)_u| = \sqrt{1 - N_3^2}$ and choosing the sign that makes ν_3 continuous at the boundary (where it is already defined).

At points where the tangent plane is almost vertical, N_3 approaches zero, hence $(\nu_3)_u$ approaches one. In other words the tangent plane at points $(z, u(z))$ with z next to A is almost vertical if and only for any $\varepsilon > 0$ there is a neighborhood of A in $\mathcal{D}$ such that

$$|(\nu_3)_u| > 1 - \varepsilon \tag{10}$$

at each point of the neighborhood.

A minimal graph is stable, so one has Schoen's curvature estimates for the surface S : let p be a point of S and let $\mathcal{D}(p, R)$ be a disk contained in S centered at p of intrinsic radius R, then

$$|\mathcal{A}(q)| \le \kappa \;\; \forall q \in \mathcal{D}\left(p, \frac{R}{2}\right) \tag{11}$$

where $\mathcal{A}$ is the second fundamental form of S and κ is an absolute constant (see [Sc]).

Now, assume by contradiction that there is a sequence of points $\{z_m\}$ in $\mathcal{D}$ approaching A (i.e. $u_n(z_m) \to \infty$ as $m \to \infty$) such that (10) does not hold. Then, there is a radius R independent on m such that $\mathcal{D}(p_m, R) \subset S$, where $p_m = (z_m, u(z_m))$. Hence, by the curvature estimate (11), around each p_m the surface S is a graph over a disk $D(p_m, r)$ of the tangent plane at p_m, and the graph has bounded distance from the disk $D(p_m, r)$. The radius of the disk depends only on R, hence it is independent of m. It is clear that, if z_m is close enough to A, then the horizontal projection of $D(p_m, r)$ and thus of the surface S is not contained in $\mathcal{D}$. Contradiction. Hence (10) holds in a neighborhood of A.

Now, fix $\varepsilon > 0$ and let $\delta \le \varepsilon$. Let q_1, q_2 be the points of A at distance δ from the endpoints of A and call A_δ the subarc of A bounded by q_1, q_2. We construct a neighborhood of A_δ in $\mathcal{D}$ (see Figure 8). Let τ be a horizontal geodesic orthogonal to A passing through the mid-point of A. We can assume that τ passes through the origin. For $i = 1, 2$, let α_i be a horizontal geodesic through q_i forming an angle of $\frac{\pi}{2}$ with A. Finally, let β be the horizontal geodesic orthogonal to τ, at distance δ from A and call $q_3 = \beta \cap \alpha_1$, $q_4 = \beta \cap \alpha_2$. Denote by Q_δ the geodesic quadrilateral having vertices at points q_1, q_2, q_3, q_4.

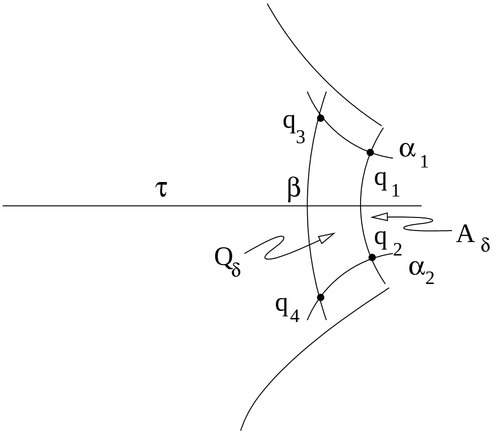


Figure 8

The form $(\nu_3)_u ds$ is exact, hence:

$$0 = \int_{A_\delta} (\nu_3)_u ds + \int_{Q_\delta \setminus A_\delta} (\nu_3)_u ds.$$

Then, if ε is small enough, using (10) we obtain:

$$\int_{A_\delta} (\nu_3)_u ds \geq -2\varepsilon + (1-\varepsilon)||A_\delta||.$$

Letting ε (and so δ) tend to zero yields:

$$\int_A (\nu_3)_u ds \geq ||A||.$$

The opposite inequality is obvious, so (i) follows.

For the proof of (ii) one makes the obvious modifications of the arguments in (i). □

Let us prove another useful result.

Lemma 2. *Let $u : \mathcal{D} \longrightarrow \mathbb{R}$ be a minimal solution continuous on a convex arc C of the boundary of $\mathcal{D}$. Then:*

$$\int_C (\nu_3)_u ds < ||C||.$$

Proof. It is enough to prove the result for a closed subarc of C, say $\tilde{C}$. Then we can assume that u is defined in a convex set $\tilde{\mathcal{D}}$ with continuous boundary data. Denote by v the solution of the Dirichlet problem in $\tilde{\mathcal{D}}$ such that:

$$v_{|\partial\tilde{\mathcal{D}}\setminus\tilde{C}} = u, \quad v_{\tilde{C}} = u + a$$

where a is a constant to be fixed later. Let $w = v - u$, then

$$w_{|\partial\tilde{\mathcal{D}}\setminus\tilde{C}} = 0, \quad w_{|\tilde{C}} = a$$

By the maximum principle w is uniformly bounded in $\tilde{\mathcal{D}}$.

Using Stokes' theorem (together with a standard approximation argument at the points of discontinuity) we obtain:

$$\int_{\partial\tilde{\mathcal{D}}} w[(\nu_3)_u - (\nu_3)_v]ds = \int\int_{\tilde{\mathcal{D}}} \left[w_1 \left(\frac{v_1}{\tau_v} - \frac{u_1}{\tau_u} \right) + w_2 \left(\frac{v_2}{\tau_v} - \frac{u_2}{\tau_u} \right) \right] dx_1 dx_2.$$

The argument of the last integral is equal to the following expression:

$$\left(\frac{\tau_u + \tau_v}{2} \right) \left[\left(\frac{u_1}{\tau_u} - \frac{v_1}{\tau_v} \right)^2 + \left(\frac{u_2}{\tau_u} - \frac{v_2}{\tau_v} \right)^2 + \frac{1}{F} \left(\frac{1}{\tau_u} - \frac{1}{\tau_v} \right)^2 \right].$$

Hence, it is non negative and not identically zero in $\tilde{\mathcal{D}}$. Then we have:

$$a \int_{\tilde{C}} [(\nu_3)_u - (\nu_3)_v]ds > 0.$$

choosing alternatively $a = \pm 1$ we obtain the result. □

Remark 7. We point out that the results of Lemma 1 and 2 hold for non convex domains as well.

Proof of Step 2. We prove that the first condition in (9) is sufficient and necessary for existence. We start by sufficiency.

Let $u_n : \mathcal{D} \longrightarrow \mathbb{R}$ be the minimal solution with the following boundary values

$$u_{n|A_i} = +n, \quad u_{n|C_s} = \min(n, f^s).$$

By Remark 6, $\{u_n\}$ is uniformly bounded in compact sets contained in each of the convex hulls $C(C_s)$, $s = 1, \ldots, h$. Hence, passing to a subsequence, $\{u_n\}$ converges on compact subsets of

$$\cup_{s=1}^h C(C_s)$$

to a minimal solution u defined in an open set $\mathcal{U}$ containing $\cup_{s=1}^h C(C_s)$. Furthermore $\{u_n\}$ diverges uniformly on compact subsets of $\mathcal{D} \setminus \mathcal{U}$ and u is a countinuous function with values in $\mathbb{R} \cup \infty$.

Let $\mathcal{V} = \mathcal{D} \setminus \mathcal{U}$. We claim that $\mathcal{V} = \emptyset$. We start by showing that $\partial\mathcal{V}$ has a very special structure, when $\mathcal{V}$ is not empty.

Lemma 3. *With the notation above, one has:*

(i) $\partial\mathcal{V}$ *consists only of geodesic chords of* $\mathcal{D}$ *and parts of the boundary of* $\mathcal{D}$*;*

(ii) *two chords of* $\partial\mathcal{V}$ *cannot have a common endpoint;*

(iii) *the endpoints of chords of* $\partial\mathcal{V}$ *are among the vertices of the geodesic arcs* A_i*;*

(iv) *a component of* $\mathcal{V}$ *cannot consist only of an interior chord of* $\mathcal{D}$.

Proof. It is clear by Remark 6 that each arc of $\partial\mathcal{V}$ must be geodesic and that no vertex of $\partial\mathcal{V}$ lies in $\mathcal{D}$, then (i) follows. Now assume by contradiction that (ii) does not hold. Let K_1, K_2 be two arcs of $\partial\mathcal{V}$ having a common endpoint $q \in \partial\mathcal{D}$. Choose two points $q_1 \in K_1$ and $q_2 \in K_2$ such that the triangle T with vertices q, q_1, q_2 lies in $\mathcal{D}$. We have:

$$\int_{\partial T} (\nu_3)_n ds = 0$$

where $(\nu_3)_n$ is defined as in Lemma 1 at interior points of $\mathcal{D}$. The triangle T may be either in $\mathcal{U}$ or in $\mathcal{V}$. Assume the former is true, then, by the first equality in (ii) of Lemma 1, choosing correctly the orientation, we have:

$$\lim_{n\to\infty} \int_{\overline{qq_1}} (\nu_3)_n ds = ||\overline{qq_1}||, \quad \lim_{n\to\infty} \int_{\overline{q_2q}} (\nu_3)_n ds = ||\overline{qq_2}||. \tag{12}$$

Here $\overline{*}$ indicates the geodesic arc between two points and $(\nu_3)_n$ is defined as in Lemma 1 at interior points of $\mathcal{D}$.

On the other hand:

$$\left| \int_{\overline{q_1q_2}} (\nu_3)_n ds \right| \leq ||\overline{q_1q_2}||. \tag{13}$$

(12) and (13) together with the triangle inequality give a contradiction.

If $T \subset \mathcal{V}$, we make the same reasoning using the second equality in (ii) of Lemma 1.

(iii) and (iv) are proved with analogous arguments, using Lemma 1. We leave this to the reader. □

Now, we come back to the proof of Step 2. Assume by contradiction that $\mathcal{V}$ is not empty. The convex hull of each C_i is contained in $\mathcal{U}$, and each component of $\mathcal{V}$ is bounded by a geodesic polygon $\mathcal{P}$, whose vertices are among the endpoints of the A_i. Denote by $\hat{A}_i$ those edges of A_i that are contained in $\mathcal{P}$. In the notation of Theorem 3, $||\mathcal{P}|| = \gamma$, $\sum ||\hat{A}_i|| = \alpha$.

For each u_n we have:

$$0 = \int_{\mathcal{P}} (\nu_3)_n ds = \int_{\cup \hat{A}_i} (\nu_3)_n ds + \int_{\mathcal{P} \setminus \cup \hat{A}_i} (\nu_3)_n ds.$$

By (ii) of Lemma 1, we infer:

$$\lim_{n \to \infty} \int_{\mathcal{P} \setminus \cup \hat{A}_i} (\nu_3)_n ds = -(\gamma - \alpha).$$

For every n, $|(\nu_3)_n| < 1$, so

$$\left| \int_{\cup \hat{A}_i} (\nu_3)_n ds \right| \leq \sum ||\hat{A}_i|| = \alpha.$$

Hence $\alpha \geq \gamma - \alpha$, that contradicts the assumed conditions.

We are left with the proof of the necessity of the condition $2\alpha < \gamma$. Let u be the minimal solution with the given boundary values and let $\mathcal{P}$ be a polygon as in the hypothesis of Theorem 3. We have:

$$\int_{\cup \hat{A}_i} (\nu_3)_u ds + \int_{\mathcal{P} \setminus \cup \hat{A}_i} (\nu_3)_u ds = 0.$$

Furthermore $|(\nu_3)_u| < 1$ on $\mathcal{P} \setminus \cup \hat{A}_i$, hence

$$\left| \int_{\mathcal{P} \setminus \cup \hat{A}_i} (\nu_3)_u ds \right| < \gamma - \alpha$$

and by (i) of Lemma 1, we have:

$$\int_{\cup \hat{A}_i} (\nu_3)_u ds = \alpha$$

Hence $2\alpha < \gamma$. □

Proof of Step 3. Let $\{u_n\}$ be defined as in Step 2. First we prove that $\{u_n\}$ is bounded at some point of $\mathcal{D}$. Assume that this is not the case, then $\mathcal{V} = \mathcal{D}$ and we have:

$$0 = \int_{\cup A_i} (\nu_3)_n ds + \int_{\cup C_i} (\nu_3)_n ds.$$

(ii) of Lemma 1 implies

$$\lim_{n \to \infty} \int_{\cup C_i} (\nu_3)_n ds = -\sum ||C_i|| \leq -(\gamma - \alpha).$$

Here $\gamma = ||\partial\mathcal{D}||$ and $\alpha = \sum ||A_i||$.

On the other hand, as $|(\nu_3)_n| < 1$ for every n, we have

$$\left| \int_{\cup A_i} (\nu_3)_n ds \right| < \sum ||A_i|| = \alpha.$$

Then $\alpha \geq \gamma - \alpha$, a contradiction.

Hence the sequence $\{u_n\}$ is bounded at some point of $\mathcal{D}$. In fact we will prove that there is a disk in $\mathcal{D}$ of radius independent on n where each u_n is uniformly bounded.

Up to an isometry, we can assume that $\{u_n\}$ is bounded at the origin $\sigma \in \mathbb{H}^2$. We remark that by the maximum principle each u_n is positive in the domain of definition.

Let $m_n = u_n(\sigma)$. We assert that the gradient of u_n at σ is bounded depending only on the constant m_n. In order to prove it, we will compare the gradient of u_n with that of a Scherk type surface.

Up to a rotation of x_1, x_2 coordinates, we can assume that

$$\frac{\partial u_n}{\partial x_1}(\sigma) > 0, \quad \frac{\partial u_n}{\partial x_2}(\sigma) = 0.$$

Let $\Delta(n)$ be a geodesic triangle contained in $\mathcal{D}$ with edges a, b, c such that the x_1 axis bisects the edge a orthogonally and $\Delta(n)$ is symmetric with respect to the x_1 axis (see Figure 9).

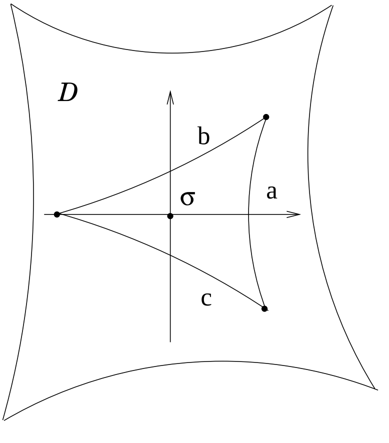


Figure 9

Let $\varphi_{\Delta(n)}$ denote the Scherk type surface over $\Delta(n)$ with value $+\infty$ on a, value 0 on b, c and $\varphi_{\Delta(n)}(\sigma) = m_n$ (we allow translations of $\Delta(n)$ along the x_1 axis in order to find such a Scherk type surface). Define $C(m_n) = |\nabla\varphi_{\Delta(n)}(\sigma)|$. We claim that:

$$|\nabla u_n(\sigma)| \le C(m_n). \tag{14}$$

In fact, assume by contradiction that (14) does not hold. Then, the symmetries of $\varphi_{\Delta(n)}$ imply:

$$\frac{\partial u_n}{\partial x_1}(\sigma) > \frac{\partial \varphi_{\Delta(n)}}{\partial x_1}(\sigma), \quad \frac{\partial u_n}{\partial x_2}(\sigma) = \frac{\partial \varphi_{\Delta(n)}}{\partial x_2}(\sigma) = 0.$$

Now, we move $\Delta(n)$ by hyperbolic translations along the x_1 axis, pushing the edge a towards σ. As $\varphi_{\Delta(n)}$ and $\frac{\partial\varphi_{\Delta(n)}}{\partial x_1}$ diverge as one approaches the side a, there is a position of $\Delta(n)$ such that:

$$u_n(\sigma) < \varphi_{\Delta(n)}(\sigma)$$

and

$$\frac{\partial u_n}{\partial x_1}(\sigma) = \frac{\partial \varphi_{\Delta(n)}}{\partial x_1}(\sigma), \quad \frac{\partial u_n}{\partial x_2}(\sigma) = \frac{\partial \varphi_{\Delta(n)}}{\partial x_2}(\sigma) = 0.$$

Define $w = \varphi_{\Delta(n)} - u_n$. We have:

$$w(\sigma) = \chi > 0, \quad \nabla w(\sigma) = 0.$$

Then, there are at least four level lines of $w = \chi$ through σ ([CM],[Se]). These level lines divide every small neighborhood of σ in at least four domains in which w is alternately greater than and less than χ. We prove that this yields a contradiction (our argument is analogous to [Se], we give it for the sake of completeness).

Let G be the subset of $\overline{\Delta(n)}$ whose points are at distance less than ε from the boudary of $\Delta(n)$. The function u_n has bounded continuous gradient in $\overline{\Delta(n)}$, hence, using the form of the graph of $\varphi_{\Delta(n)}$, one has that the set G is divided into two components by the conditions

$$w > \chi, \quad w < \chi,$$

for suitably small ε.

The first component is adjacent to edge a, while the second is adjacent to b and c and the components themselves are separated by two level lines $w = \chi$ exiting from the vertices of a. By the maximum principle, each component of the set $w > \chi$ must extend to the boundary of $\Delta(n)$. It follows that the set $w > \chi$ consists of one component. Then, any two regions near σ where $w > \chi$ can be joined by a simple Jordan arc C^+ along which $w > \chi$. Analogously any two regions near σ where $w < \chi$ can be joined by a simple Jordan arc C^- along

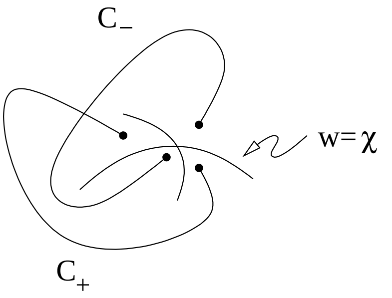


Figure 10

which $w < \chi$. Then, the curves C_+ and C_- must intersect (see Figure 10). This is a contradiction.

Now let $M = \sup_{n\in IN} u_n(\sigma)$. As $m_n \leq M$, with the same reasoning used for proving (14), one has that $C(m_n) \leq C(M)$. Hence the sequence $\{u_n\}$ has uniformly bounded gradient at σ.

Now, by Schoen's curvature estimates (see (11)), one has that in a neighborhood of the point $p_n = (\sigma, u_n(\sigma))$ the surface is a graph of bounded height and slope over a disk $D(p_n, R)$ of the tangent plane to the surface at p_n, of radius R independent of n. As $|\nabla u_n(\sigma)|$ is uniformly bounded, the projection of each $D(p_n, R)$ on the horizontal plane contains a disk of fixed radius and $\{u_n\}$ is uniformly bounded there. Then, there exists an open set $\mathcal{U}$ in which $\{u_n\}$ converges uniformly. Now we can apply the same reasoning as in Step 2, in order to prove that $\mathcal{U} = \mathcal{D}$. □

Proof of Step 4. By the previous arguments we can find a minimal solution $u^+ : \mathcal{D} \longrightarrow \mathbb{R}$ such that

$$u^+{}_{|A_i} = \infty, \quad u^+{}_{|B_j} = 0, \quad u^+{}_{|C_s} = \max\{0, f^s\}.$$

Furthermore we can find a minimal solution $u^- : \mathcal{D} \longrightarrow \mathbb{R}$ such that

$$u^-{}_{|A_i} = 0, \quad u^-{}_{|B_j} = -\infty, \quad u^-{}_{|C_s} = \min\{0, f^s\}.$$

Then, define for each s:

$$(f^s)_n = \begin{cases} -n & \text{if } f^s < -n \\ f^s & \text{if } |f^s| \leq n \\ n & \text{if } f^s > n \end{cases}$$

and let $u_n : \mathcal{D} \longrightarrow \mathbb{R}$ be the minimal solution such that

$$u_{n|A_i} = n, \quad u_{n|B_j} = -n, \quad u_{n|C_s} = (f^s)_n.$$

By the maximum principle:

$$u^- \leq u_n \leq u^+ \text{ in } \mathcal{D}.$$

Hence, the sequence $\{u_n\}$ is uniformly bounded on compact subsets of $\mathcal{D}$ and there exists a subsequence converging to a minimal solution that takes the prescribed boundary values.

This proves existence under condition (9). The necessity of condition (9) is proved as in Step 2. □

Proof of Step 5. We remark that in this case the number of edges A_i is equal to the number of edges B_j, say k. We need to construct some auxiliary sets and minimal solutions.

Let $v_n : \mathcal{D} \longrightarrow \mathbb{R}$ be the minimal solution such that

$$v_{n|A_i} = n, \quad v_{n|B_i} = 0.$$

For $c \in]0, n[$, we introduce the following subsets of $\mathcal{D}$:

$$E_c = \{v_n > c\} \cap \mathcal{D}, \quad F_c = \{v_n < c\} \cap \mathcal{D}.$$

Let E_c^i be the component of E_c whose closure contains the edge A_i and let F_c^j be the component of F_c whose closure contains the edge B_j. By the maximum principle $E_c = \cup_{i=1}^k E_c^i$ and $F_c = \cup_{i=1}^k F_c^i$. We choose c close enough to n such that the E_c^i are disjoint and we define:

$$\mu(n) = \limsup\{c \in]0, n[\;\mid\; E_c^i \cap E_c^j = \emptyset \;\; i \neq j\}.$$

Of course there is at least one pair i, j such that

$$\overline{E^i}_{\mu(n)} \cap \overline{E^j}_{\mu(n)} \neq \emptyset$$

and this implies that for any given $F^i_{\mu(n)}$, the set $F^j_{\mu(n)}$ is disjoint from it.

For each n, we define the following minimal solution in $\mathcal{D}$:

$$u_n = v_n - \mu(n).$$

In order to prove that the sequence $\{u_n\}$ is uniformly bounded on compact subsets of $\mathcal{D}$, let us define two auxiliary minimal solutions in $\mathcal{D}$.

Let ${u_i}^+$ and ${u_i}^-$ be the minimal solutions in $\mathcal{D}$ with the following boundary values:

$$ {u_i}^+{}_{|A_i} = \infty, \quad {u_i}^+{}_{|\partial\mathcal{D}\setminus A_i} = 0 $$

$$ {u_i}^-{}_{|B_j} = -\infty, \quad j \neq i, \quad {u_i}^-{}_{|\partial\mathcal{D}\setminus \cup_{j\neq i} B_j} = 0 $$

For every $i = 1, \dots, k$, ${u_i}^+$ and ${u_i}^-$ exist by previous steps.

Finally, for any $z \in \mathcal{D}$ we define:

$$ u^+(z) = \max_{1\le i\le k} \{{u_i}^+(z)\}, \quad u^-(z) = \min_{1\le i\le k} \{{u_i}^-(z)\}. $$

We claim that at any point of $\mathcal{D}$:

$$ u^- \le u_n \le u^+. \tag{15} $$

Let $p \in \mathcal{D}$ such that $u_n(p) > 0$, then p belongs to $E^i_{\mu(n)}$ for some i. On $\partial E^i_{\mu(n)}$ one has $u_n \le u_i^+$, then this inequality holds in $E^i_{\mu(n)}$, and

$$ u_n(p) \le u_i^+(p) \le u^+(p). $$

Since u^- is non positive, the left inequality in (15) is obvious at the point p.

The proof of (15) at points where u_n is negative is analogous, using the set $F^i_{\mu(n)}$.

Hence $\{u_n\}$ has a subsequence converging to a minimal solution $u : \mathcal{D} \longrightarrow \mathbb{R}$. Let us prove that u takes the right boundary values.

Recall that:

$$ u_{n|A_i} = n - \mu(n), \quad u_{n|B_i} = -\mu(n), $$

so we must prove that the sequences $\{\mu(n)\}$ and $\{n - \mu(n)\}$ both diverge to infinity. We prove it for the sequence $\{\mu(n)\}$; the proof will be analogous for the latter sequence.

The assumption that $\{\mu(n)\}$ does not diverge will give a contradiction to the hypothesis $\alpha = \beta$. By contradiction, take a subsequence (still denoted by $\{u_n\}$) such that $\mu(n)$ tends to a finite limit μ_0. Then:

$$ u_n \longrightarrow \infty \quad \text{on} \quad A_i, \quad u_n \longrightarrow -\mu_0 \quad \text{on} \quad B_i. $$

Then, for the limit function u we have:

$$ u_{|A_i} = \infty, \quad u_{|B_i} = -\mu_0. $$

Let $(\nu_3)_u$ be the unit inward conormal to the boundary of the graph of u. We finally obtain:

$$ \alpha = \int_{\cup A_i} (\nu_3)_u ds = -\int_{\cup B_i} (\nu_3)_u ds > -\beta, $$

where the first equality is given by (i) of Lemma 1. This is a contradiction.

The necessity of the condition $\alpha = \beta$ is proved as in Step 2. □

Proof of Step 6 (Uniqueness). Let u and v be two minimal solutions assuming values $+\infty$ on each A_i, $-\infty$ on each B_j and the same data on each geodesic arc C_s.

Let M be a large constant and define:

$$\psi = \begin{cases} -M & \text{if } u - v < -M \\ u - v & \text{if } |u - v| < M \\ M & \text{if } u - v > M \end{cases}$$

Let $0 < \delta < \varepsilon$ and denote by $\mathcal{D}_{\delta\varepsilon}$ the subset of $\mathcal{D}$ whose distance from $\partial\mathcal{D}$ is greater than δ and whose distance from the vertex of each A_i and B_j is greater than ϵ. It is clear that the boundary Γ of $\mathcal{D}_{\delta\varepsilon}$ consists of bounded arcs $\tilde{A}_i$, $\tilde{B}_j$, $\tilde{C}_s$ adjacent to the A_i, B_j, C_s and circular arcs adjacent to the vertices of A_i, B_j. Let $(\nu_3)_u$ and $(\nu_3)_v$ be defined as in Lemma 1 for the functions u and v respectively. Consider the following integral:

$$\int_\Gamma \psi[(\nu_3)_u - (\nu_3)_v]ds.$$

For δ small enough, we have:

$$\int_{\cup\tilde{C}_s} \psi[(\nu_3)_u - (\nu_3)_v]ds \leq 2\int_{\cup\tilde{C}_s} |\psi| \leq 2\varepsilon \sum ||C_s||.$$

We recall that next to boundary arcs where the solution is infinity, $|\nu_3|$ is almost one, hence:

$$\begin{aligned}\int_{\cup\tilde{A}_i} \psi[(\nu_3)_u - (\nu_3)_v]ds &= \int_{\cup\tilde{A}_i} \psi[(\nu_3)_u - 1]ds - \int_{\cup\tilde{A}_i} \psi[(\nu_3)_v - 1]ds \\ &\leq 2\varepsilon M \sum ||\tilde{A}_i||\end{aligned}$$

In the same way we obtain:

$$\int_{\cup\tilde{B}_j} \psi[(\nu_3)_u - (\nu_3)_v]ds \leq 2\varepsilon M \sum ||\tilde{B}_j||$$

By summing the previous inequalities, we infer:

$$\begin{aligned}\int_\Gamma \psi[(\nu_3)_u - (\nu_3)_v]ds \leq 2\varepsilon \sum(||C_s|| + M||\tilde{A}_i|| + M||\tilde{B}_j||) + \\ + 4\pi\sinh(\varepsilon)M(k + l)\end{aligned} \tag{16}$$

where the last term is the contribution of the circular arcs next to vertices.

On the other hand, integrating by parts, we obtain:

$$\int_\Gamma \psi[(\nu_3)_u - (\nu_3)_v]ds = \int\int \left[\psi_1\left(\frac{v_1}{\tau_v} - \frac{u_1}{\tau_u}\right) + \psi_2\left(\frac{v_2}{\tau_v} - \frac{u_2}{\tau_u}\right)\right] dx_1 dx_2$$

where the double integral is taken over the set $D_{\delta\varepsilon} \cap \{|u - v| < M\}$. The argument of the last integral is non-negative and it is zero only at points where $\nabla u = \nabla v$ (see Lemma 2).

Then, letting $\varepsilon \longrightarrow 0$ in (16), we obtain:

$$\psi_1\left(\frac{v_1}{\tau_v} - \frac{u_1}{\tau_u}\right) + \psi_2\left(\frac{v_2}{\tau_v} - \frac{u_2}{\tau_u}\right) = 0$$

at points where $|u - v| < M$. Hence $\nabla u = \nabla v$ in $\mathcal{D}$, since M can be taken arbitrarily large.

It follows that $u = v + const$ in $\mathcal{D}$. If the family of boundary convex arcs $\{C_s\}$ is empty, this proves the result. If not, the constant must be zero by the boundary condition on C_s. □

6 Existence of complete minimal graphs

Theorem 4. *Let Γ be a continuous Jordan curve in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$, that is a vertical graph. Then, there exists a minimal vertical graph on $\mathbb{H}^2$ having Γ as asymptotic boundary. The graph is unique.*

Proof. In the model $D = \{0 \le x_1^2 + x_2^2 < 1\}$ for $\mathbb{H}^2$, the curve Γ is a graph over the circle $x_1^2 + x_2^2 = 1$. Consider an exhaustion of D by disks D_n centered at the origin, of Euclidean radius $1 - \frac{1}{n}$. For each n, let Γ_n be a vertical C^2 graph over ∂D_n converging to Γ as $n \longrightarrow \infty$. We choose the curves Γ_n contained in the convex hull of Γ. The curves Γ_n may be taken as the trace on $\partial D_n \times \mathbb{R}$ of the function whose graph is a C^2 extension of Γ inside D. Let M_n be the Plateau solution with boundary Γ_n; by Rado's theorem M_n is a vertical graph of a C^2 function $v_n : D_n \longrightarrow \mathbb{R}$. The sequence $\{v_n\}$ is uniformly bounded on compact subsets of D, hence there is a subsequence converging to a minimal solution $v : D \longrightarrow \mathbb{R}$, uniformly on compact subsets of D. Let M be the graph of the function v. We have only to prove that the asymptotic boundary of M is Γ. By definition of M, one has that $\Gamma \subset \partial_\infty M$. In order to prove the converse, we show that any point $p \notin \Gamma$ is not contained in $\partial_\infty M$. Let p such a point and assume that it lies below Γ (the reasoning is analogous, when p lies above Γ). We construct a surface that separates the point p from the M_n's, with mean curvature vector pointing upwards (a barrier, see Figure 11).

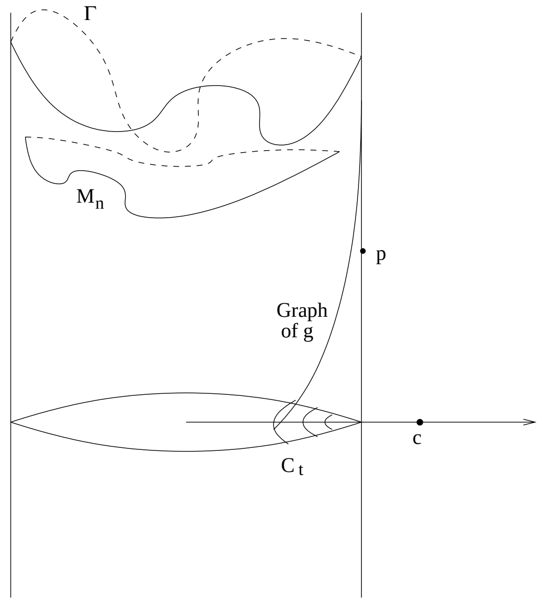


Figure 11

We can assume that the first two coordinates of the point p are $(1, 0)$. Consider a family of circles C_t in the plane $\{x_3 = 0\}$, with Euclidean radius t, centered at a fixed point $(c, 0, 0)$, $c > 1$. If $t > c - 1$, such circles intersect the hyperbolic plane $\{x_3 = 0\}$ in equidistant circles with curvature $\kappa_t = \sqrt{1 - (\sin\beta_t)^2}$, where β_t is the angle between C_t and ∂D. Each C_t divides the hyperbolic plane into two components: the curvature vector of C_t points towards the component containing the origin. Consider the function g defined as follows:

$$g(x_1, x_2) = \exp(\alpha(t_0 - t)) - k, \quad (x_1, x_2) \in C_t, \quad t \in [t_0, c - 1]$$

α, t_0, k, positive constants to be fixed later. The function g is constant on each C_t and

$$g_{|C_{t_0}} = -k, \quad g(1, 0) = \exp(\alpha(t_0 - c + 1)) - k,$$

Using equation (1), one obtains that the mean curvature of the graph of g with respect to the upward unit normal vector is:

$$H(x_1, x_2) = \frac{F\alpha}{2\tau^2}\exp(\alpha(t_0 - t))\left\{\frac{\alpha^2\sqrt{F}\exp(\alpha(t_0 - t))}{t}[\sqrt{F} + x_1 c - 1] + \alpha - \frac{1}{t}\right\}.$$

We can choose c and t_0 such that the mean curvature H is positive. Then we choose α and k such that $g(1, 0)$ is larger than the third coordinate of the point p

and such that the graph of g does not interesect the curves Γ_n, for n large. Then, as for n large, the boundary of M_n lies above the graph of g, so does the surface M_n (by the maximum principle). Hence the asymptotic boundary of M (that is the limit of the sequence $\{M_n\}$) does not contain the point p.

Uniqueness follows by the maximum principle.

Theorem 4 is proved. □

7 Isolated singularities

Theorem 5. *Let $\mathcal{D}_\rho^*$ be a punctured disk of radius ρ in $\mathbb{H}^2$. Let $u : \mathcal{D}_\rho^* \longrightarrow \mathbb{R}$ be a C^2 function such that the graph of u is minimal in $\mathbb{H}^2 \times \mathbb{R}$. Then u extends C^2 to the puncture.*

Proof. First we prove that u is bounded on $\mathcal{D}_\rho^*$. Up to isometry, we can take $\mathcal{D}_\rho^*$ centered at the origin of $\mathbb{H}^2$. Let $\varepsilon < \rho$ and consider the annulus $A_\varepsilon = \mathcal{D}_\rho \setminus \mathcal{D}_\varepsilon$. Let C_ε the half catenoid that is a graph over $\mathbb{H}^2 \setminus \mathcal{D}_\epsilon$ with waist on $\partial\mathcal{D}_\varepsilon$. The function u is bounded on A_ε, hence there is a vertical translation of C_ε that does not touch the graph of u. Now translate vertically C_ε towards the graph of u. By the maximum principle the first contact point is on $\partial\mathcal{D}_\rho \times \mathbb{R}$.

Letting $\epsilon \longrightarrow 0$, C_ε tends to a plane, hence:

$$\min_{\partial\mathcal{D}_\rho} u \leq u \leq \max_{\partial\mathcal{D}_\rho} u$$

Now, let $v : \mathcal{D}_\rho \longrightarrow \mathbb{R}$ be the minimal solution with boundary values $u_{|\partial\mathcal{D}_\rho}$. We will show that $u \equiv v$ in $\mathcal{D}_\rho^*$.

Consider the form θ defined in $\mathcal{D}_\rho^*$ by:

$$\theta = (u - v)\left(\frac{u_1}{\tau_u} - \frac{v_1}{\tau_v}\right) dx_2 - \left(\frac{u_2}{\tau_u} - \frac{v_2}{\tau_v}\right) dx_1$$

We have:

$$\int_{\partial\mathcal{D}_\varepsilon} \theta = \int_{\partial A_\varepsilon} \theta = \int_{A_\varepsilon} d\theta$$

where the first equality depends on the fact that $u \equiv v$ on $\partial\mathcal{D}_\rho$ and the last equality is by Stokes' theorem.

The form θ is bounded on $\mathcal{D}_\rho^*$ because u, $|u_i|{\tau_u}^{-1}$, $|v_i|{\tau_v}^{-1}$ are bounded for $i = 1, 2$. Then:

$$\int_{\partial A_\varepsilon} \theta \longrightarrow 0 \text{ as } \varepsilon \longrightarrow 0 \tag{17}$$

As in Lemma 2, we obtain that $d\theta$ is non negative and it is 0 if and only if $u_i = v_i$, for $i = 1, 2$. Letting $\varepsilon \longrightarrow 0$ we obtain $\nabla u \equiv \nabla v$ and so $u \equiv v$ on $\mathcal{D}_\rho^*$. □

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