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Some Remarks on Compact Constant Mean Curvature Hypersurfaces in a Halfspace of $\mathbb{H}^{n+1}$

Barbara Nelli and Beate Semmler

We consider embedded compact hypersurfaces M in a halfspace of hyperbolic space with boundary ∂M in the boundary geodesic hyperplane P of the halfspace and with non-zero constant mean curvature. We prove the following. Let $\{M_n\}$ be a sequence of such hypersurfaces with ∂M_n contained in a disk of radius r_n centered at a point $\sigma \in P$ such that $r_n \to 0$ and that each M_n is a large H-hypersurface, $H > 1$. Then there exists a subsequence of $\{M_n\}$ converging to the sphere of mean curvature H tangent to P at σ. In the case of small H-hypersurfaces or $H \leq 1$, if we add a condition on the curvature of the boundary, there exists a subsequence of $\{M_n\}$ which are graphs. The convergence is smooth on compact subset of $\mathbb{H}^3 \setminus \sigma$.

KEYWORDS: Killing and Geodesic Graph, Mean Curvature, Large Hypersurfaces, Alexandrov Reflection Technique

MATH. SUBJ. CLASSIFICATION: 53 A 10, 53 C 42

1 Introduction

Let P be a hyperplane in hyperbolic space $\mathbb{H}^{n+1}$ and let $\mathbb{H}^{n+1}_+$ be one of the two halfspaces determined by P; we consider compact embedded H-hypersurfaces (i.e. hypersurfaces of constant mean curvature H) $M \subset \mathbb{H}^{n+1}_+$, with boundary $\partial M = \Gamma$ a codimension one embedded submanifold of P.

There is little known about the geometry and the topology of such M in terms of that of Γ. In the Euclidean case of dimension three, some interesting results are obtained in [5]. Let M be a compact embedded H-surface in $\mathbb{R}^3_+ = \{x_3 \geq 0\}$ and $\partial M = \Gamma \subset \{x_3 = 0\}$ a convex Jordan curve; Rosenberg and Ros proved, for example, that if H is sufficiently small (in terms of the curvature of Γ), then M has genus zero.

In this paper, we make some progress on understanding this situation in hyperbolic space. Our main result is a compacity theorem for H-hypersurfaces.

After submitting this paper, the second author has obtained in [6] a result on the genus of a compact H-surface in $\mathbb{H}^3$, similar to that mentioned above.

We warmly thank Harold Rosenberg for suggesting the problem to us and for useful discussions.

2 The Main Result

In the following an H-surface will always be a hypersurface of $\mathbb{H}^{n+1}$ of constant mean curvature H.

DEFINITION 2.1 A *Geodesic Graph* is a graph in the following system of coordinates: Let Ω be a domain in a hyperplane P and let u be a real function that each point $p \in \Omega$ associated with the point on the geodesic γ_p through p, orthogonal to P, at hyperbolic distance $u(p)$ from P.

DEFINITION 2.2 Let $H > 1$ be a constant; we say that an H-surface M is *small* if there exists a ball of mean curvature bigger than H that contains M. We say that M is *large* if it is not small.

REMARK 2.3 If M is a small H-surface, then $M \subset \cap_\alpha B_\alpha$ where B_α denotes the family of balls $B(q, \rho)$ of radius $\rho \leq \text{arctanh}\frac{1}{H}$, centered at $q \in \mathbb{H}^{n+1}$, and $\partial M \subset B(q, \rho)$.

THEOREM 2.4 *Let $\{M_m\}$ be a sequence of H-surfaces in $\mathbb{H}^{n+1}_+$ such that $\partial M_m = \Gamma_m \subset D(r_m) \subset P$, where $D(r_m)$ are the disks of radius r_m centered at a point σ, and $r_m \longrightarrow 0$.*

Let $k_{\partial M_m}$ be the smallest value of the mean curvature of ∂M_m.

(i) $H > 1$. If the M_m are large H-surfaces, then there exists a subsequence of M_m that converges to the sphere S_H of radius $\text{arctanh}(\frac{1}{H})$ tangent to P at σ; the convergence is smooth on compact subsets of $\mathbb{H}^{n+1} \setminus \sigma$. If $k_{\partial M_m} > H$, then there exists a subsequence converging to S_H as above or there is a subsequence which are geodesic graphs over Ω_m ($\Omega_m \subset P$, $\partial\Omega_m = \Gamma_m$).

(ii) $H \leq 1$. There exists a subsequence of M_m that converges smoothly on compact subsets of $\mathbb{H}^{n+1} \setminus \sigma$. If $k_{\partial M_m} > 1$, then each M_n of the subsequence is a geodesic graph over Ω_m.

3 Basic Properties

Let us give some definitions.

DEFINITION 3.1 *Geodesic Cylinder.* Let Ω be a domain in a geodesic hyperplane P. For each $p \in \Omega$ let γ_p be the unique geodesic through p orthogonal to P. The *geodesic cylinder* over Ω is the set (see Figure 1)

$$C(\Omega) = \bigcup_{p \in \Omega} \gamma_p$$

DEFINITION 3.2 *Killing Cylinder.* Let Ω be a domain in a geodesic hyperplane P. Let $p \in \Omega$ and let γ_p be the unique geodesic through p orthogonal to P; let q be any point of Ω and $\eta_p(q)$ the orbit through q of the hyperbolic translation along γ_p (*i.e.* the integral curve of the Killing vector field associated with the hyperbolic translation). The *Killing cylinder over Ω with respect to γ_p* is the set (see Figure 2)

$$K(\Omega, \gamma_p) = \bigcup_{q \in \Omega} \eta_p(q)$$

DEFINITION 3.3 A *Killing Graph with respect to a geodesic γ_p* is a graph in the following system of coordinates: let Ω be a domain in P and let u be a real function that each point $q \in \Omega$ associated with the point on $\eta_p(q)$ at hyperbolic distance $u(q)$ from P.

REMARK 3.4 Let P_{γ_p} be the family of hyperplanes orthogonal to γ_p. We remark that for each $q \in \Omega$ the orbit $\eta_p(q)$ is orthogonal to each P_{γ_p} and a Killing graph over P is a Killing graph over each P_{γ_p}.

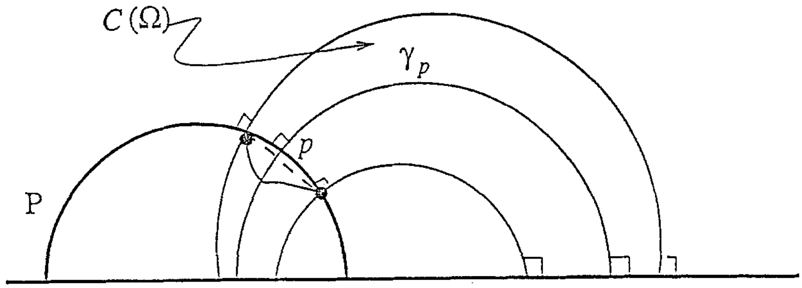

Figure 1

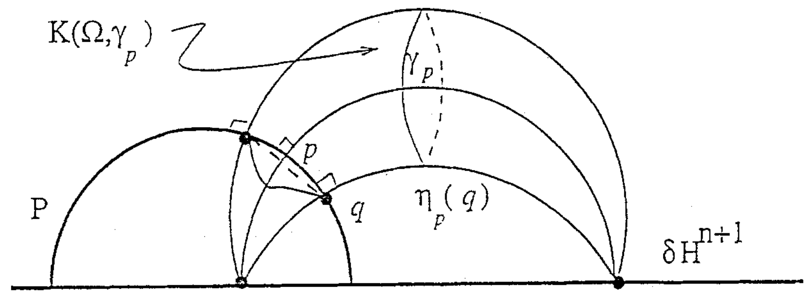

Figure 2

This last notion of a Killing graph arises naturally from the Alexandrov reflection technique (cf. [1]). This technique will be one of our main tools, so let us explain an application of it. We work in the upper halfspace model:

$$P = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1}_+ \mid \sum_{i=1}^{n+1} x_i^2 = 1\}$$

and

$$\mathbb{H}^{n+1}_+ = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1}_+ \mid \sum_{i=1}^{n+1} x_i^2 \geq 1\}$$

Let M be a compact embedded H-surface $M \subset \mathbb{H}^{n+1}_+$, with boundary $\partial M = \Gamma$ a codimension one embedded submanifold of P. Let $\mathcal{B}$ be the compact component of $\mathbb{H}^{n+1}_+$ bounded by M and by the domain $\Omega \subset P$ such that $\partial\Omega = \partial M$. We orient M by its mean curvature vector $\mathbf{H}$ and Γ by the orientation of M.

Let $q \in M$ be a point at maximal distance d from P and let γ be the geodesic through q orthogonal to P. Consider the family $P_\gamma(t) \subset \mathbb{H}^{n+1}_+$ of hyperplanes orthogonal to γ parametrized such that $t = \text{dist}(P_\gamma(t), P)$. We want to prove that the part of M lying above $P_\gamma(\frac{d}{2})$ is a Killing graph with respect to γ. If $t > d$, $P_\gamma(t) \cap M = \emptyset$, and $q \in P_\gamma(d) \cap M$; when we decrease t slightly , the part of M (that we denote by $M(t)^+$) above $P_\gamma(t)$ is a Killing graph over P and no point of $M(t)^+$ has the normal vector orthogonal to the Killing direction at the point. Let $M(t)^-$ be the hyperbolic reflection with respect to $P_\gamma(t)$ of $M(t)^+$; $M(t)^-$ is contained in $\mathcal{B}$, has mean curvature H and its mean curvature vector is the reflection of the mean curvature vector of $M(t)^+$. Now, continue to decrease t and consider the first $P_\gamma(\tau)$ where one of the following conditions fails to hold:

1) $\text{int}(M(\tau)^-) \subset \text{int}\mathcal{B}$,

2) $M(\tau)^+$ is a Killing graph over $P_\gamma(\tau)$ and no point of $M(\tau)^+$ has the normal vector orthogonal to the Killing direction at the point.

If 1) fails first, one applies the maximum principle to M and $M(\tau)^-$ at the point where they touch to conclude that $P_\gamma(\tau)$ is a plane of symmetry of M. If 2) fails first, then the point where the normal vector to $M(\tau)^+$ becomes orthogonal to the Killing direction is on $\partial(M(\tau)^+) \subset P_\gamma(\tau)$; hence we can apply the boundary maximum principle to $M(\tau)^-$ and to the part of M below $P_\gamma(\tau)$ to conclude that $P_\gamma(\tau)$ is a plane of symmetry of M.

Both cases are impossible for $\tau \geq \frac{d}{2}$ since ∂M is on P, so the result follows.

In the following lemma we establish some basic properties of an H-surface.

LEMMA 3.5 *Let M be a compact embedded H-surface of $\mathbb{H}^{n+1}_+$, with boundary $\partial M = \Gamma$ a codimension one embedded submanifold of P. Let $\mathbf{H}$ be the mean curvature vector of M and $\mathbf{Y}$ the unit vector field orthogonal to P, pointing towards $\mathbb{H}^{n+1}_+$. Then:*

(i) $\mathbf{H}$ points towards $\mathcal{U} = \mathcal{B} \cup [C(\Omega) \cap (\mathbb{H}^{n+1} \setminus \mathbb{H}^{n+1}_+)]$.

(ii) Each point $q \in M$ at maximal distance from P is contained in the geodesic cylinder $C(\Omega)$.

(iii) Let γ be any geodesic orthogonal to P passing through a point of Ω; if M is contained in $K(\Omega,\gamma)$ then M is a Killing graph with respect to γ.

(iv) If for each point $q \in \partial M$, $\langle \mathbf{H}(q), \mathbf{Y}(q)\rangle < 0$, then M is a Killing graph with respect to any geodesic orthogonal to P passing through a point of Ω; therefore M is a geodesic graph.

Proof. (i) Consider the family of hyperplanes $\{P(t)\}_{t\in\mathbb{R}_+} \subset \mathbb{H}^{n+1}_+$ obtained from P by hyperbolic translations along the vertical geodesic orthogonal to P, parametrized such that $P(0) = P$; if t is big enough, then $P(t) \cap M = \emptyset$; decrease t and consider the first plane $P(\tau)$ that touches M. At the first point of contact between $P(\tau)$ and M, $\mathbf{H}$ points down, *i.e.* towards $\mathcal{U}$, hence the same is true at each point of M.

(ii) Let $d = d(q, P)$ and let γ be the geodesic through q orthogonal to P. Suppose, by contradiction, that $q \notin C(\Omega)$ so $\gamma \cap P$ is not in Ω. Consider the family $P_\gamma(t)$ of hyperplanes orthogonal to γ, such that $P_\gamma(0) = P$; we can do Alexandrov reflection up to the hyperplane $P_\gamma(\frac{d}{2})$ without an accident and the symmetry of the part of M above $P_\gamma(\frac{d}{2})$ with respect to $P_\gamma(\frac{d}{2})$ is contained in $\mathcal{B}$ so the points where it touches P are in Ω. This is a contradiction, since the symmetry of q is on $\gamma \cap P$, which is not in Ω.

(iii) In this case we can do Alexandrov reflection with the family of hyperplanes orthogonal to γ till P without any accident, so M is a Killing graph over P.

(iv) Let γ be a geodesic through a point of Ω and let $P_\gamma(t)$ be the family of hyperplanes orthogonal to γ. The hypothesis implies that near ∂M, M is contained in $K(\Omega,\gamma)$; we claim that $M \subset K(\Omega,\gamma)$. If not, we would find a Killing orbit η in $\partial K(\Omega,\gamma)$ that contains at least two points of M. Applying Alexandrov technique with the family $P_\gamma(t)$ we find a τ such that the symmetry of the part of M above $P_\gamma(\tau)$ with respect to $P_\gamma(\tau)$ touches M at an interior point (that lies on η), so that $P_\gamma(\tau)$ is a hyperplane of symmetry for M. This is a contradiction. Applying (iii), the first result follows. This means that on each γ there is only one point of M, hence M is a geodesic graph too.

□

Estimates of height, area and curvature for a geodesic graph are a basic tool in the proof of our main result. In order to obtain them, it is more convenient to work in the Minkowski model for $\mathbb{H}^{n+1}$. Consider $\mathbb{R}^{n+2}$ with the Lorentz metric

$$Q(x_1,\ldots,x_{n+2}) = -dx_1^2 + \sum_{i=2}^{n+2} dx_i^2$$

The hypersurface of $\mathbb{R}^{n+2}$

$$\mathbb{H}^{n+1} = \{(x_1,\ldots,x_{n+2}) \in \mathbb{R}^{n+2} \mid x_1 > 0\ ,\ Q(x_1,\ldots,x_{n+2}) = -1\}$$

with the Riemannian metric induced by Q is the Minkowski model for $\mathbb{H}^{n+1}$. Let M be an H-surface, let $\mathbf{X}$ be its position vector and let n be the unit normal to M such that $\langle \mathrm{n}, \mathbf{H}\rangle = -H \le 0$. Then we have (cf. [4]):

$$\Delta\mathbf{X} = n\mathbf{X} - nH\mathrm{n},$$

$$\Delta \mathrm{n} = nH\mathrm{X} - |A|^2\mathrm{n},$$

where Δ is the Laplacian on M and A is the second fundamental form of M. Now suppose that M is a geodesic graph over a domain $\Omega \subset P = \{x_{n+2} = 0\} \cap \mathbb{H}^{n+1}$ with boundary in this hyperplaneplane of $\mathbb{H}^{n+1}$; so

$$\mathrm{X}(\rho) = (x_1 \cosh\rho, \ldots, x_n \cosh\rho, \sqrt{\left(x_1^2 - \sum_{i=2}^{n} x_i^2 - 1\right)} \cosh\rho, \sinh\rho)$$

$$\rho_{|\partial\Omega} = 0$$

where ρ is the distance from the plane P. From previous equations we obtain

$$\Delta x_{n+2} = n x_{n+2} - nH n_{n+2},$$

$$\Delta n_{n+2} = nH x_{n+2} - |A|^2 n_{n+2}$$

where $n_{n+2} \geq 0$. This yields

$$\Delta(H x_{n+2} - n_{n+2}) = n_{n+2}(|A|^2 - nH^2) \geq 0$$

By the maximum principle,

$$H x_{n+2} - n_{n+2} \leq (H x_{n+2} - n_{n+2})_{|\partial M} \leq 0 \tag{1}$$

In the case $H > 1$, (1) and Alexandrov reflection yield height estimates so that the maximum distance between M and P is smaller than $\operatorname{arctanh}(\frac{1}{H})$ (cf. Lemma 3.3 of [4]).

To obtain height estimates in the case $H \leq 1$, we proceed in the following way.

Consider the upper halfspace model and let P be a vertical hyperplane; $\partial M \subset P$. Let E_θ be the equidistant sphere with respect to P contained in $\mathbb{H}^{n+1}_+$ such that $\operatorname{dist}(P, E_\theta) = \operatorname{arctanh}(\sin\theta)$. E_θ forms an angle θ with P. The mean curvature vector of E_θ points towards P and $H(E_\theta) = \sin\theta$, so by the maximum principle M cannot touch E_θ for $\theta \geq \arcsin H$ (since the contact is at an interior point). This implies that the maximum of the distance between M and P is smaller than $\operatorname{arctanh} H$.

Now, we come back to the Minkowski model and use (1) to obtain area estimates for M *far from* its boundary.

Let δ be a small positive constant and let M_δ be the part of M at distance from P greater than or equal to $\operatorname{arcsinh}\delta$. For a geodesic graph we have

$$n_{n+2} = \frac{(\cosh\rho)^2}{\sqrt{(\cosh\rho)^2 + (\sum_{i=1}^{n} \rho_i x_i)^2 - \rho_1^2 + \sum_{i=2}^{n} \rho_i^2}}$$

where $\rho_i = \frac{\partial\rho}{\partial x_i}$, $i = 1, \ldots, n$. Furthermore, if g_{ij} are the coefficients of the induced metric on M, we have

$$\sqrt{\det\{g_{ij}\}} = \frac{(\cosh\rho)^{n+1}}{n_{n+2}\sqrt{x_1^2 - \sum_{i=2}^{n} x_i^2 - 1}}$$

We remark that the area element of Ω is

$$dA_\Omega = \frac{dx_1 \dots dx_n}{\sqrt{x_1^2 - \sum_{i=2}^n x_i^2 - 1}},$$

hence

$$\text{Area}(M_\delta) = \int_\Omega \frac{(cosh\rho)^{n+1} dA_\Omega}{n_{n+2}} \leq C_1(H) \frac{\text{Area}(\Omega)}{H\delta}$$

where $C_1(H)$ is a positive constant depending on H (by height estimates) and the last inequality follows from (1).

As we remark in the Appendix, in order to obtain curvature estimates for M on compact subsets of Ω we only need to estimate the gradient of ρ. By (1) we have

$$\sqrt{(\cosh\rho)^2 + (\sum_{i=1}^n \rho_i x_i)^2 - \rho_1^2 + \sum_{i=2}^n \rho_i^2} \leq \frac{(\cosh\rho)^2}{H \sinh\rho}$$

so, if $x_{n+2} > \text{arcsinh}\delta$, since we have height estimates (*i.e.* estimates for $cosh\rho$), we easily obtain

$$|\nabla\rho| = \sqrt{\sum_{i=1}^n \rho_i^2} \leq \frac{C_2(H)}{\delta}$$

4 Proof of the Main Result

In this section we give the proof of Theorem 2.4.

We will use the upper halfspace model, and we fix the notations of Section 3 ; so we take

$$D(r_m) = \{\sum_{i=1}^n x_i^2 \leq \tanh^2 r_m, \ \sum_{i=1}^{n+1} x_i^2 = 1\}$$

and $\sigma = (0, \dots, 0, 1)$.

(i) $H > 1$. First we prove that *almost all* parts of M_m are geodesic graphs.

It follows from height estimates that all the M_m are contained in a fixed compact set K of $\mathbb{H}^{n+1}$. Consider a point of M_m at maximal distance from P and let γ_m be the unique geodesic orthogonal to P passing through that point. Let $P_{\gamma_m}(t) \subset \mathbb{H}_+^{n+1}$ be the family of planes orthogonal to γ_m, parametrized such that t is the distance between $P_{\gamma_m}(t)$ and P (see Figure 3).

By the Alexandrov reflection technique we have that the part of M_m above $P_{\gamma_m}(\text{arctanh}\frac{1}{H})$ is a Killing graph with respect to γ_m over a domain Ω_m in the plane

$P_{\gamma_m}(\text{arctanh}\frac{1}{H})$ and the hypothesis of Lemma 3.5 (iv) are satisfied, hence it is a geodesic graph over Ω_m. Then, one has uniform area and curvature estimates for the part of M_m above the plane $P_{\gamma_m}(\text{arctanh}(\frac{1}{\text{H}})+\delta)$, $\delta>0$ (for each m the area of $P_{\gamma_m}(\text{arctanh}(\frac{1}{\text{H}})+\delta)\cap \text{K}$ is bounded by the geometry of K).

Let $r>0$, then for m large, $\partial M_m \subset D(r)$. For each geodesic β on P with initial point σ we consider the family of planes orthogonal to β; let Q^r_β be the plane of this family tangent to $\partial D(r)$ (see Figure 3). Denote by $M_m(\beta,r)$ the part of M lying in the halfspace determined by Q^r_β, which does not contain ∂M. The Alexandrov reflection technique shows that $M_m(\beta,r)$ is a Killing graph with respect to β over a domain in Q^r_β. We prove that $M_m(\beta,r)$ is a geodesic graph.

Let $\mathbf{Y}$ be the unit vector field orthogonal to Q^r_β, pointing towards the halfspace determined by Q^r_β that contains $M_m(\beta,r)$ and let $\mathbf{H}$ be the mean curvature vector of $M_m(\beta,r)$. We claim that for each $p\in\partial M_m(\beta,r)$, $\langle \mathbf{Y}(p),\mathbf{H}(p)\rangle<0$. The fact that $M_m(\beta,r)$ is a Killing graph over a domain in Q^r_β and (i) of Lemma 3.5 guarantee that $\langle \mathbf{Y}(p),\mathbf{H}(p)\rangle\le 0$, so the only problem may be that there exists a point $p\in\partial M_m(\beta,r)$ such that $\langle \mathbf{Y}(p),\mathbf{H}(p)\rangle=0$. The reflection of $M_m(\beta,r)$ with respect to Q^r_β would be tangent to M_m at such a point; the fact that each point of $\partial M_m(\beta,r)$ is an interior point of M_m gives a contradiction by the maximum principle. So, we can use (iv) of Lemma 3.5 to conclude that $M_m(\beta,r)$ is a geodesic graph over a domain in Q^r_β.

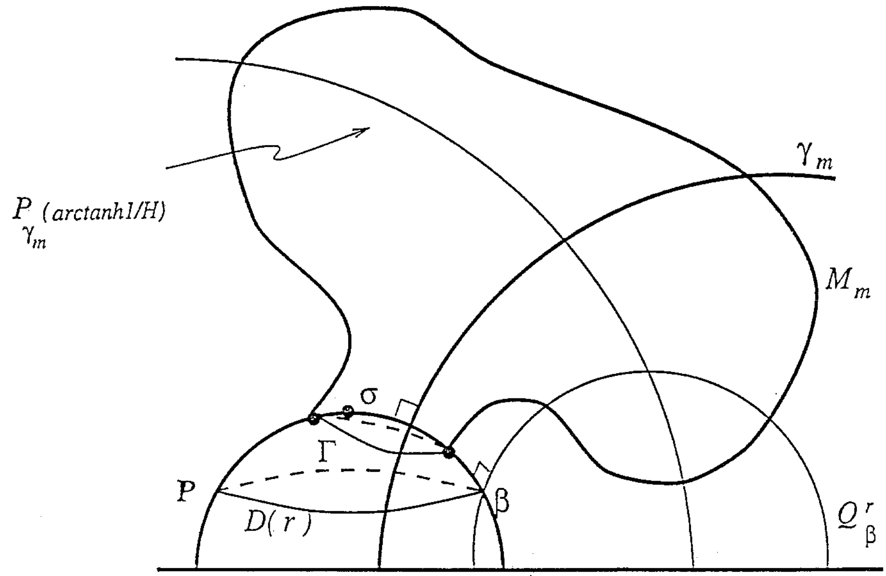

Figure 3

We remark that there are some parts of M_m which are not necessarily geodesic graphs, so we do not have uniform area and curvature estimates, and for this reason we do not have convergence in $\mathbb{H}^{n+1}$. When $m \longrightarrow \infty$, $\Gamma_m \longrightarrow \sigma$, *i.e.* r goes to 0; hence γ_m converges to a vertical geodesic through σ (in the following we will denote it by γ) and Q^r_β converges to a vertical plane containing σ (here we use (ii) of Lemma 3.5). Then the parts of M_m that are not necessarily geodesic graphs are near the segment $I = [1, \operatorname{arctanh}(\frac{1}{H})]$ on γ, hence a standard compacity technique yields a subsequence (which we also call) M_m that converges on compact subsets of $\mathbb{H}^{n+1} \setminus$ I. The limit is either empty or a compact immersed surface M of mean curvature H.

First we analyse the case of an empty limit.

In this case, for m large, M_m is uniformly close to I. Then M_m is a small H-surface, and this is impossible.

Now, suppose that $k_{\partial M_m} > H$. Consider the codimension one halfsphere $S^+ \subset \mathbb{H}^{n+1}_+$ centered at σ of radius $\operatorname{arctanh}(\frac{1}{\mathrm{H}})$. We make a hyperbolic translation along γ such that the image of S^+ is disjoint from K. Then, by decreasing the Killing coordinate, we come back down; for m large, Γ_m is near σ, so the image of S^+ cannot touch M_m before it arrives at P again, *i.e.* M_m is below S^+. If M_m is not a geodesic graph then, by (iv) of Lemma 3.5 there exists a point $q \in \partial M_m$ such that $< \mathbf{H}(q), \mathbf{Y}(q) > \geq 0$, where $\mathbf{H}$ is the mean curvature of M_m. So, by the fact that $k_{\partial M_m} > H$ we can do isometries of S^+, moving its center on P, until S^+ touches M_m either at an interior point (case of strict inequality) or at the boundary point q, being tangent to it (case of equality). In both cases, by the maximum principle, we obtain $M_m = S^+$ which is impossible. Hence M_m is a geodesic graph over Ω_m.

Next we assume that $\{M_m\}$ converges to M on $\mathbb{H}^{n+1} \setminus I$.

For each $r > 0$ the planes Q^r_β can be moved up to $\partial D(r)$ and the symmetries of M with respect to these planes do not touch M (since this holds for M_m, m large). By continuity, this works up till $r = 0$ and so M is a rotational surface about γ which can have self-intersections at most on γ. M is compact so it is not a Delaunay surface; M is not a stack of spheres, since it has height at most $2\operatorname{arctanh}(\frac{1}{\mathrm{H}})$. Hence, by [2], M is the sphere S_H of radius $\rho = \operatorname{arctanh}(\frac{1}{\mathrm{H}})$.

We have only to prove that the convergence is uniform on compact subsets of $\mathbb{H}^{n+1} \setminus \sigma$.

Let $\varepsilon > 0$, there exists $r > 0$ such that, for m large

$$M_m \cap C\{D(r), [\frac{3}{2}\rho + 1, \infty)\} = M_m \cap C\{D(r), [2\rho + 1 - \varepsilon, \infty)\} \tag{2}$$

where $C\{D(r), [a, b]\}$ is defined in the following way. Let $P(t) \subset \mathbb{H}^{n+1}_+$ be the family of planes orthogonal to γ parametrized such that $P(0) = P$; $C\{D(r), [a, b]\}$ is the slice of $C(D(r), \gamma)$ between $P(a)$ and $P(b)$. Furthermore, the intersection in (2) is a Killing graph above $D(r)$. Coming down with planes $P(t)$ from $t = 2\rho - \varepsilon$ to $t = \rho$, we obtain by Alexandrov reflection that

$$M_m \cap C\{D(r), [2(t - \rho) + \varepsilon, 2\rho - \varepsilon]\} = \emptyset$$

So, in particular for $t = \rho + 1$

$$M_m \cap C\{D(r), [\varepsilon, 2\rho - \varepsilon]\} = \emptyset$$

This gives uniform area and curvature estimates for M_m on compact subsets of $\mathbb{H}^{n+1} \setminus \sigma$, not just on compact subsets of $\mathbb{H}^{n+1} \setminus I$. The result follows.

(ii) $H \leq 1$. As in the previous case, *almost all* parts of M_m are geodesic graphs and we have height, area and curvature estimates, so there exists a subsequence of $\{M_m\}$ that converges smoothly on compact subsets of $\mathbb{H}^{n+1} \setminus \sigma$.

We prove that the limit is empty. Here, it is more convenient to assume that $M_m \subset \mathbb{H}^{n+1}_-$. Let

$$a_m = \inf\{x_{n+1} \mid (x_1, \dots, x_{n+1}) \in \partial M_m\}, \quad b_m = \inf\{x_{n+1} \mid (x_1, \dots, x_{n+1}) \in M_m\}$$

Since $\Gamma_m \longrightarrow \sigma$, we have that $\lim_{m \longrightarrow \infty} a_m = 1$. If $b_m < a_m$ the horosphere $\{x_{n+1} = b_m\}$ touches M_m at an interior point and this is a contradiction by the maximum principle; hence

$$M_m \subset \{(x_1, \dots, x_{n+1}) \in \mathbb{H}^{n+1}_- \mid a_m \leq x_{n+1} \leq 1\}$$

and this implies that the limit is empty.

If $k_{\partial M_m} > 1$, the result follows by the same argument as in (i), using a compact halfsphere of curvature smaller than $k_{\partial M_m}$.

□

REMARK 4.1 The first result of *(i)* and *(ii)* in Theorem 2.4 remains true, if one assumes that $\Gamma_m \subset B(r_m)$, a ball of radius r_m centered at σ (for example Γ_m contained in a horosphere). Here M_m is an H-surface such that $M_m \cap \mathbb{H}^{n+1} \setminus B(r_m) \subset \mathbb{H}^{n+1}_+$ and $\Gamma_m \longrightarrow \sigma$ as $m \longrightarrow \infty$. One does the same argument as in Theorem 2.4 with geodesic planes ε-tilted from planes orthogonal to P. For $\varepsilon > 0$ one can choose r_m small enough so that Alexandrov reflection works with ε-tilted planes coming from ∞, up till the plane reaches $\partial B(r_m)$.

5 A Topological Result

Using the previous arguments it is very easy to prove a result about the topology of H-surfaces.

THEOREM 5.1 *Let M be a compact H-surface embedded in $\mathbb{H}^{n+1}_+$ such that $\Gamma = \partial M \subset P$. Then*

(i) if $H > 1$ and if M is a small surface and $k_\Gamma > H$, then M is a geodesic graph.

(ii) If $H \leq 1$ and if $k_\Gamma > 1$, then M is a geodesic graph.

In both cases M is topologically a disk of dimension n.

Proof. (i) Since $k_\Gamma > H$, we can find an n-disk D^H on P such that ∂D^H is a codimension two sphere of mean curvature H and $\Gamma \subset D^H$. Consider the unique codimension one compact sphere S^H of mean curvature H such that $P \cap S^H = \partial D^H$, *i.e.* the centers of S^H and D^H coincide and $S^H \cap \mathbb{H}_+^{n+1}$ is a halfsphere (see Figure 4). Since M is small, the ball with boundary S^H contains M (cf. Remark 2.3). Then, we are in the same situation as in Theorem 2.4 (case of empty limit) and we can conclude by the same argument as in the proof of Theorem 2.4..

(ii) $M \subset \mathbb{H}_+^{n+1}$ is compact, hence there exists a compact sphere $S_{H'}$ with center on P, mean curvature $H' \leq k_\Gamma$, such that the ball bounded by $S_{H'}$ still contains M. We conclude by the same argument as in (i).

□

REMARK 5.2 Barbosa and Sa Earp in [2] proved that an H-hypersurface immersed in $\mathbb{H}^{n+1}$ satisfying the hypothesis (i) and (ii) of Theorem 5.1 is a graph in the system of coordinates: hyperplane - orthogonal horocycles. They used an analytic method.

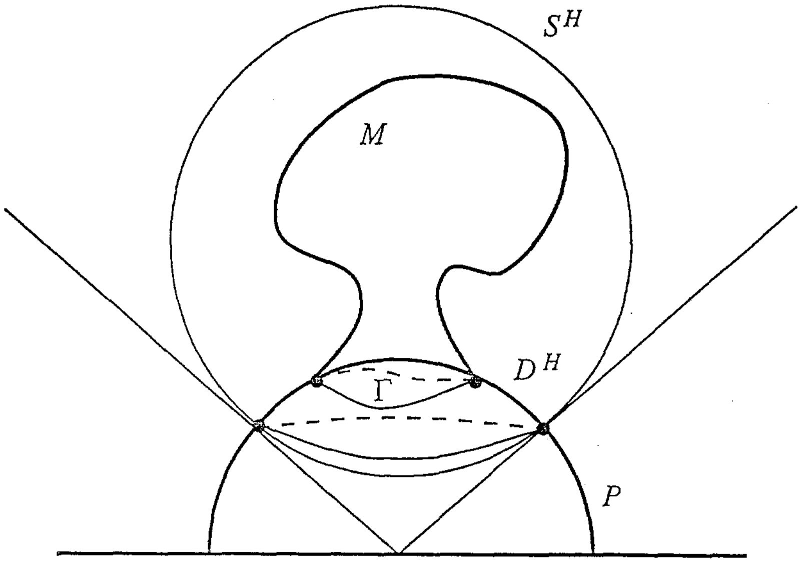

Figure 4

6 Appendix

In the Minkowski model, the equation of a geodesic graph over the hyperplane $\{x_{n+2} = 0\}$, of mean curvature H is:

$$\operatorname{div}\left(\frac{\nabla_h \rho}{T_\rho}\right) = n \cosh \rho \left(- H + \frac{\sum_{i=1}^{n} x_i \rho_i + \sinh \rho \cosh \rho}{(\cosh \rho) T_\rho} \right)$$

where ρ is the hyperbolic distance from the plane $\{x_{n+2} = 0\}$,

$$\nabla_h \rho = \begin{pmatrix} (x_1^2 - 1)\rho_1 + \sum_{i=2}^{n+1} x_1 x_i \rho_i \\ \vdots \\ (x_j^2 + 1)\rho_j + \sum_{i \neq j} x_j x_i \rho_i \\ \vdots \end{pmatrix}$$

$j = 2, \ldots, n+1$, $T_\rho^2 = (\cosh\rho)^2 + |\nabla_h \rho|^2$ and div is the standard divergence in $\mathbb{R}^n$.

For the solution of such equation, Schauder theory guarantees that $C^{2,\alpha}$ estimates depend only on C^0 and C^1 estimates (cf. [3]). As T_ρ is bounded away from 0, $C^{2,\alpha}$ estimates for ρ imply curvature estimates for its graph.

For the sake of completeness, we give the equation of a geodesic graph of mean curvature H in the upper halfspace model. If $\{x_n = 0\}$ is the plane where the graph is defined, the position vector is

$$\mathrm{X}(\rho) = (x_1, \ldots, x_{n-1}, x_{n+1} \tanh\rho, \frac{x_{n+1}}{\cosh\rho})$$

and the unit normal vector to the graph is

$$\mathrm{n} = \frac{x_{n+1}}{\cosh\rho W_\rho}\left(x_{n+1}\rho_1, \ldots, x_{n+1}\rho_{n-1}, x_{n+1}\rho_{n+1} - 1, \sinh\rho + \frac{x_{n+1}\rho_{n+1}}{\cosh\rho}\right)$$

where $\rho = \rho(x_1, \ldots, x_{n-1}, x_{n+1})$, $\rho_i = \dfrac{\partial\rho}{\partial x_i}$, and $W_\rho^2 = (\cosh\rho)^2 + x_{n+1}^2 |\nabla\rho|^2$.

Hence the equation of a geodesic graph is

$$\mathrm{div}\left(\frac{\nabla\rho}{W_\rho}\right) = \frac{n}{x_{n+1}^2 \cosh\rho}\left(H - \frac{\sinh\rho}{W_\rho}\right)$$

where $H = \langle \mathrm{H}, \mathrm{n} \rangle$.

References

[1] ALEXANDROV, A.D. : Uniqueness theorems for surfaces in the large. V. Vestnik Leningrad Univ., 13 No.19 A.M.S. (Series 2) **21** (1958) 412–416.

[2] DE MIRANDA GOMEZ, J. : Sobre hipersuperficies com curvatura media constante no espaco hiperbolico. PhD thesis, IMPA (1985).

[3] GILBARG, D. and TRUDINGER, N.S. : Elliptic Partial Differential Equations of Second Order. Springer-Verlag (1983).

[4] KOREVAAR, N. and KUSNER,R. and MEEKS III, W.H. and SOLOMON, B. : Constant mean curvature surfaces in hyperbolic space. American Journal of Mathematics **114** (1992) 1–43.

[5] Ros, A. and Rosenberg, H. : Constant mean curvature surfaces in a half-space of $\mathbb{R}^3$ with boundary in the boundary of the half-space. To appear in Journal of Differential Geometry (1997).

[6] Semmler, B. : Blow up theorems for compact constant mean curvature surfaces. Preprint (1997).

Barbara Nelli
Dipartimento di Matematica
Universitá di Pisa
via Buonarroti, 2, Pisa - ITALY
e-mail: nelli@dm.unipi.it

Beate Semmler
U.F.R. de Mathématiques
Université Paris 7
2, place Jussieu
75251 Paris Cedex 05 - FRANCE
e-mail: semmler@math.jussieu.fr

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