On Hypersurfaces Embedded in Euclidean Space with Positive Constant H_r Curvature

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ABSTRACT. We consider hypersurfaces M embedded in a half-space $\mathbb{R}^{n+1}_+$ with positive constant r^{th} symmetric function of the principal curvatures (H_r-surfaces). For such H_r-surfaces, $1 < r \leq n$, with strictly convex boundary in $\partial\mathbb{R}^{n+1}_+$ we show that, if H_r is small enough in terms of the geometry of the boundary of M, then M is topologically a disk. When $r = 2$, we also prove a compactness theorem for certain classes of H_2-surfaces.

1. INTRODUCTION

Let M be an embedded hypersurface of $\mathbb{R}^{n+1}$ and let $(\kappa_1(x), \dots, \kappa_n(x))$ be the set of its principal curvatures at the point $x \in M$. For $1 \leq r \leq n$ we consider the r^{th} symmetric function of the principal curvatures of M i.e.,

$$\binom{n}{r} H_r(x) = \sum_{1 \leq i_1 < \cdots < i_r \leq n} \kappa_{i_1}(x) \cdots \kappa_{i_r}(x).$$

H_1, H_n are the mean curvature and the Gauss-Kronecker curvature of M respectively, while H_2 is the scalar curvature of M. When H_r is constant, we say that M is a H_r-surface. In the following we always consider $H_r > 0$.

We obtain some results about the structure of the set of H_r-surfaces embedded in $\mathbb{R}^{n+1}_+ = \{x_{n+1} \geq 0\}$ with strictly convex boundary in $\partial\mathbb{R}^{n+1}_+$. The case of H_1 in $\mathbb{R}^3$ is studied in [RR] and [S]. The case of H_1-surfaces in hyperbolic space is studied in [NS] and [S]. One of our main tools is Alexandrov reflection technique. This idea was first introduced in [A], and we refer to [RR] for an explanation of Alexandrov method adapted to our situation. We just remark that Alexandrov reflection is based on Hopf maximum principle (cf. [H]), which is a consequence of ellipticity of the equation satisfied by our hypersurfaces (cf. [K], [N]). $H_2 > 0$ yields an elliptic equation on any hypersurface. $H_r > 0$, $r > 2$, yields an elliptic equation on a compact hypersurface with boundary in a hyperplane (cf. Section 2).

The paper is divided into two parts. In the first one we restrict to H_2-surfaces and we obtain a compactness result, in the second one we study the topology of H_r-surfaces, $r > 1$.

From now on we will say surface and plane instead of hypersurface and hyperplane.

In order to state the compactness result, we need the following definition.

Definition 1.1. *Let C be a positive constant. We say that the surface M is C-*admissible *if the following inequality holds at each point of M:*

$$C(x) = \sum_{i=1}^{n} \mathrm{k}_i(x) - \max_{1\le i\le n} \mathrm{k}_i(x) \ge C.$$

We will see that C-admissibility allows us to prove curvature estimates.

Denote by P the plane $\{x_{n+1} = 0\}$ and by σ the origin of $\mathbb{R}^{n+1}$.

Theorem 1.1. *Let $\{M_m\} \subset \mathbb{R}^{n+1}_+$ be embedded compact H_2-surfaces with scalar curvature equal to one, and*

$$\partial M_m \subset B(\sigma, r_m) = \Big\{(x_1, \dots, x_{n+1}) \mid \sum_{i=1}^{n+1} x_i^2 \le r_m^2\Big\},$$

with r_m a sequence converging to zero. Assume that there exists a constant $C > 0$ such that all M_m are C-admissible. Then, there is a subsequence of M_m which converges either to the origin σ, or to the sphere $S \subset \mathbb{R}^{n+1}_+$ of radius one tangent to P at σ. In the first case the surfaces converge as subsets, and in the second case the convergence is smooth on compact subsets of $\mathbb{R}^{n+1} \setminus \sigma$.

Theorem 1.2. *Let $\{M_m\} \subset \mathbb{R}^{n+1}_+$ be properly embedded non compact H_2-surfaces with scalar curvature equal to one and $\partial M_m \subset B(\sigma, r_m)$ with r_m a sequence converging to zero. Assume that there exists a constant $C > 0$ such that all M_m are C-admissible, and that the M_m are contained in a vertical cylinder of $\mathbb{R}^{n+1}_+$ outside some compact set. Then, there is a subsequence of M_m which converges to the stack of spheres of radius one tangent to P at σ. The convergence is smooth on compact subsets of $\mathbb{R}^{n+1}_+ \setminus x_{n+1}$-axis.*

In the second part of this paper we shall investigate the topology of compact embedded H_r-surfaces in $\mathbb{R}^{n+1}_+$ with strictly convex boundary Γ in $\partial\mathbb{R}^{n+1}_+$. We show that, if $H_r > 0$ is sufficiently small in terms of the geometry of Γ, then M is topologically a disk. The same result for H_1-surfaces in $\mathbb{R}^3$ is established in [RR]. An important point of the proof in [RR] is a rescaling by homotheties, followed by a compactness theorem (the analogous of our Theorem 1.1). In [S] one can find a different proof of the result in $\mathbb{R}^3$ without using homotheties and the compactness theorem. With this technique the second author was able to prove the same result for H_1-surfaces in hyperbolic 3-space (cf. [S]). Our proof concerning H_r-surfaces in $\mathbb{R}^{n+1}_+$ is mainly influenced by the latter compactness-free technique.

Theorem 1.3. *Let* Γ *be a strictly convex codimension one submanifold of* $\partial\mathbb{R}^{n+1}_+$. *There is a number* $h(\Gamma)$, *depending only on the geometry of* Γ, *such that whenever* $M \subset \mathbb{R}^{n+1}_+$ *is a compact embedded* H_r*-surface bounded by* Γ *and* $0 < H_r < h(\Gamma)$, *then* M *is topologically a disk. Furthermore, either* M *is a graph over* Ω, *or* $M \cap (\Omega \times [0,\infty))$ *is a graph over* Ω *and* $M \setminus (M \cap (\Omega \times [0,\infty)))$ *is a graph over a part of* $\Gamma \times \mathbb{R}_+$ *with respect to the lines orthogonal to* $\Gamma \times \mathbb{R}_+$.

2. A Compactness Theorem for H_2-surfaces

We start by recalling some properties of H_r, $1 \le r \le n$, on a compact surface M with boundary in a plane. Then, we restrict to the case $r = 2$. We refer to [Re] and [Ro] for a general discussion about H_r.

First we prove that M has a strictly convex point. Englobe M with a very big sphere. As ∂M is compact and contained in a plane, we can find such a sphere tangent to M at an interior point and M contained in the ball bounded by the sphere. The tangency point is a strictly convex point. As it is proved in [K] and [N], $H_r > 0$ yields an elliptic equation on a surface M with a strictly convex point. Hence we can use Hopf maximum principle. Furthermore, ellipticity implies that $\partial H_r/\partial\kappa_j > 0$ at every point of M. As

$$\binom{n}{r}\sum_{1\le j\le n}\frac{\partial H_r}{\partial \kappa_j} = (n-r+1)\binom{n}{r-1}H_{r-1},$$

we have that H_{r-1} is positive at every point of M. By induction, we obtain that $H_i > 0$ on M for every $i = 1,\dots,r$. Then it is true that (cf. [HLP]):

$$H_1 \ge H_2^{1/2} \ge \cdots \ge H_r^{1/r}, \tag{2.1}$$

and we can orient M by its mean curvature vector.

Height and area estimates. Let $1 < r \le n$ and $H_r > 0$. Height estimates for H_r-graphs are obtained in [Ro]: the maximum height over the plane P of any embedded compact H_r-surface in $\mathbb{R}^{n+1}_+$ with boundary in P is $2(H_r)^{-1/r}$. For the sake of completeness, we recall the proof.

Let $M \subset \mathbb{R}^{n+1}_+$ be a H_r-graph over a compact domain $\Omega \subset P$, with $\partial M = \partial\Omega$. Let $a_r = (H_r)^{1/r}$. Define $\varphi = a_r X_{n+1} + N_{n+1}$, where X_{n+1} and N_{n+1} are the last coordinates of the position and the normal vector respectively ($\vec{N}$ is chosen to point downward). Let L_{r-1} be the linearized operator associated with H_r. L_{r-1} is elliptic and we have (cf. [Re]):

$$L_{r-1}(X_{n+1}) = r\binom{n}{r}H_r N_{n+1},$$
$$L_{r-1}(N_{n+1}) = -\left[n\binom{n}{r}H_1H_r - (r+1)\binom{n}{r+1}H_{r+1}\right]N_{n+1}.$$

Using (2.1) we obtain:

$$L_{r-1}(\varphi) = L_{r-1}(a_rX_{n+1} + N_{n+1}) \ge 0,$$
$$\varphi\big|_{\partial M} = N_{n+1} \le 0.$$

Hence, by ellipticity of L_{r-1}, $\varphi \le 0$ on M. So, $a_rX_{n+1} \le -N_{n+1} \le 1$. By Alexandrov reflection technique, we obtain that the maximum height of any compact embedded H_r-surface in $\mathbb{R}^{n+1}_+$ with boundary in P is $2(H_r)^{-1/r}$, as desired.

Now let us obtain area estimates (depending only on H_r on a compact subset of $\operatorname{int}(\mathbb{R}^{n+1}_+)$). Let $\varepsilon > 0$ and let $M(\varepsilon)$ denote the part of M above $P(\varepsilon) = \{x_{n+1} = \varepsilon\}$. Assume $M(\varepsilon)$ is not empty. Since $L_{r-1}(\varphi) \ge 0$ and $\varphi = a_r\varepsilon/4 + N_{n+1} \le a_r\varepsilon/4$ on $\partial M(\varepsilon/4)$, we have on $M(\varepsilon/4)$, $a_rX_{n+1} + N_{n+1} \le a_r\varepsilon/4$. Then, on $M(\varepsilon/2) \subset M(\varepsilon/4)$ we have $-N_{n+1} \ge a_r\varepsilon/4$.

Let $\Omega(\varepsilon) = \{x \in \Omega \mid u(x) \ge \varepsilon\}$, where M is the graph of the function u. The above estimate of N_{n+1} yields an estimate of the gradient of u in $\Omega(\varepsilon/2)$, hence area estimates for $M(\varepsilon/2)$.

We can use this estimate to show that the distance between $\Omega(\varepsilon)$ and $\partial\Omega(\varepsilon/2)$ is larger than some positive constant δ depending only on H_r and ε. Thus for each $p \in \Omega(\varepsilon)$ we have that $D(p,\delta) \subset \Omega(\varepsilon/2)$, where $D(p,\delta)$ is the disk in P of radius δ centered at p, and that we control $|u|$ and $|\nabla u|$ on $D(p,\delta)$.

In the following lemma we establish purely interior *a-priori* estimates for C-admissible graphs of positive constant scalar curvature. A general discussion about the problem can be found in [N1].

Lemma 2.1 (Curvature estimates). *Let Ω be a domain in $\mathbb{R}^n$ and let $u : \Omega \longrightarrow \mathbb{R}$ be a C^2 function such that the graph of u is a C-admissible surface with positive constant scalar curvature S. Then, for any point $p \in \operatorname{int}(\Omega)$ we have*

$$\big|\max_{1\le i\le n} \kappa_i(p)\big| \le \frac{\alpha(S,R,n)}{C},$$

where R is the maximum radius of a disk centered at p, contained in Ω, and $\alpha(S,R,n)$ is a positive constant depending only on S, R, and n.

Proof. The proof of curvature estimates is inspired by [CNS].
Denote by M the graph of the function u. Let

$$f(\kappa_1,\dots,\kappa_n) = \Big(\sum_{1\le i<j\le n} \kappa_i\kappa_j\Big)^{1/2}.$$

On the surface M the value of the function f is $\binom{n}{2}^{1/2} S^{1/2}$. Denote by $f_i = \partial f/\partial \kappa_i$ the partial derivatives of f with respect to the principal curvatures of M. We remark that f is a concave function of $(\kappa_1, \dots, \kappa_n)$ (cf. [CNS1]). Since $\sum \kappa_i \geq nH_2^{1/2} > 0$, to estimate $|\max_{1\leq i\leq n} \kappa_i(p)|$ it suffices to estimate the maximum of the principal curvatures at p. Assume that there exists a disk $D(p,R)$ of radius R, centered at p, contained in Ω. By gradient estimates, we have a bound for $k = 2\max_{D(p,R)} w$, where $w = (1+|\nabla u|^2)^{1/2}$. Set

$$\tau = \frac{1}{w}, \quad a = \frac{1}{k} = \frac{1}{2}(\min_{D(p,R)} \tau).$$

Then

$$\frac{1}{\tau - a} \leq \frac{1}{a} = k.$$

Let ζ in $C_0^\infty(D(p,R))$ with $0 \leq \zeta \leq 1$, $\zeta \equiv 1$ in $D(p,R/2)$, and satisfying

$$|D\zeta|^2,\ |D^2\zeta| \leq \frac{C_1}{R^2}. \tag{2.2}$$

Set

$$M := \max_{D(p,R)} \left(\zeta \frac{1}{\tau - a} \kappa_i(x)\right),$$

where the maximum is also taken over all principal curvatures κ_i. We can assume $M > 0$ and it is achieved at some point $x^0 \in D(p,R)$. Set $w(x^0) = W$. It suffices to prove that

$$M \leq \frac{\alpha(S,R,n)}{C}.$$

It is convenient to use new coordinates, describing the surface by $v(y)$, where y are tangential coordinates to the surface at the point $(x^0, u(x^0))$.

Namely, let $e_1, \dots, e_{n+1}$ denote the unit vectors in the directions of the axes, and introduce new orthonormal vectors: $\varepsilon_1, \dots, \varepsilon_n, \varepsilon_{n+1}$, where $\varepsilon_{n+1} = W^{-1}(-u_1, \dots, -u_n, 1)$ is the normal at x^0 and ε_1 corresponding to the tangential direction at x^0 with largest principal curvature. We represent the surface near $(x^0, u(x^0))$ by tangential coordinates $y_1, \dots, y_n$ and $v(y)$ (summation is from 1 to n): $x_j e_j + u(x) e_{n+1} = x_j^0 e_j + u(x^0) e_{n+1} + y_j \varepsilon_j + v(y) \varepsilon_{n+1}$, thus $\nabla v(0) = 0$. Set $\omega = (1+|\nabla v|^2)^{1/2}$.

Then the normal curvature in the ε_1-direction is:

$$\kappa = \frac{v_{11}}{(1+v_1^2)\omega}.$$

In the y-coordinates we have the normal:

$$\vec{N} = -\frac{1}{\omega} v_j \varepsilon_j + \frac{1}{\omega} \varepsilon_{n+1},$$

and

$$\tau = \frac{1}{w} = \vec{N} \cdot e_{n+1} = \frac{1}{\omega W} - \frac{1}{\omega} \sum a_j v_j, \tag{2.3}$$

where $a_j = \varepsilon_j \cdot e_{n+1}$, so $\sum a_j^2 \le 1$.

At the point $y = 0$, since the y_1-direction is a direction of principal curvature, we have $v_{1j}(0) = 0$ for $j > 1$. By rotating the $\varepsilon_2, \dots, \varepsilon_n$, we may achieve that $\{v_{ij}(0)\}$ is a diagonal matrix. Note that in the y coordinates inequalities (2.2) still hold.

At the point $y = 0$ the function:

$$\log\left(\zeta \frac{1}{\tau - a} \frac{v_{11}}{(1 + v_1^2)\omega}\right) \tag{2.4}$$

takes its maximum, hence:

$$\frac{v_{11i}}{v_{11}} + \frac{\zeta_i}{\zeta} - \frac{\tau_i}{\tau - a} - \frac{2v_1 v_{1i}}{1 + v_1^2} - \frac{\omega_i}{\omega} = 0, \quad \forall i, \tag{2.5}$$

$$\frac{v_{11ii}}{v_{11}} - \frac{v_{11i}^2}{v_{11}^2} + \left(\frac{\zeta_i}{\zeta}\right)_i - \left(\frac{\tau_i}{\tau - a}\right)_i - 2v_{1i}^2 - v_{ii}^2 \le 0, \quad \forall i. \tag{2.6}$$

From (2.3), we find at $y = 0$, $i = 1, \dots, n$,

$$\tau_i = -a_i v_{ii}, \quad \tau_{ii} = -a_j v_{jii} - \frac{v_{ii}^2}{W}. \tag{2.7}$$

The principal curvatures of the surfaces (in y-coordinates) are the eigenvalues of the matrix (cf. [CSN2]):

$$a_{i\ell} = \frac{1}{\omega}\left\{v_{i\ell} - \frac{v_i v_j v_{j\ell}}{\omega(1+\omega)} - \frac{v_\ell v_k v_{ki}}{\omega(1+\omega)} + \frac{v_i v_\ell v_j v_k v_{jk}}{\omega^2(1+\omega)^2}\right\}.$$

Hence, at the origin, the matrix $a_{i\ell} = v_{i\ell}$ is diagonal, and for every $j = 1, \dots, n$:

$$\frac{\partial a_{i\ell}}{\partial y_j} = v_{i\ell j}, \quad \frac{\partial^2 a_{i\ell}}{\partial y_1^2} = v_{i\ell 11} - v_{11}^2(v_{i\ell} + \delta_{i1} v_{1\ell} + \delta_{1\ell} v_{1i}). \tag{2.8}$$

As indicated in [CNS1], the concave function $f(\kappa)$ can be written as a concave function F of the symmetric matrix $A = \{a_{i\ell}\}$, and at $y = 0$:

$$\frac{\partial F}{\partial a_{i\ell}} = \frac{\partial f}{\partial \kappa_i}\delta_{i\ell} = f_i \delta_{i\ell}. \tag{2.9}$$

By differentiating the equation $F(a_{i\ell}) = \binom{n}{2}^{1/2} S^{1/2}$ with respect to y_1, we obtain:

$$\frac{\partial F}{\partial a_{i\ell}} \frac{\partial a_{i\ell}}{\partial y_1} = 0.$$

Differentiating once more with respect to y_1 and using the concavity of F, we have:

$$0 \le \frac{\partial F}{\partial a_{i\ell}} \frac{\partial^2 a_{i\ell}}{\partial y_1^2}.$$

Using (2.8) and (2.9), we infer at $y = 0$:

$$f_i v_{iij} = 0, \quad \forall j \tag{2.10}$$

and

$$0 \le f_i(v_{ii11} - v_{11}^2 v_{ii}) - 2 f_1 v_{11}^3. \tag{2.11}$$

Replacing (2.5), (2.6), and (2.10) in (2.11), and using (2.2) and (2.7), we obtain:

$$\frac{a\zeta^2}{\tau - a} \sum f_i v_{ii}^2 \le \sum f_i \left(\frac{C_1}{R^2} + \frac{C_1 \zeta}{R} \frac{|v_{ii}|}{\tau - a} \right). \tag{2.12}$$

By Young's inequality we have for every i:

$$\frac{C_1 \zeta}{R(\tau - a)} f_i |v_{ii}| \le \frac{a\zeta^2}{2(\tau - a)} f_i v_{ii}^2 + \frac{C_1^2}{2R^2 a(\tau - a)} f_i.$$

Replacing last inequality in (2.12) we obtain (with a different constant C_1):

$$\frac{\zeta^2}{(\tau - a)} \sum f_i v_{ii}^2 \le \frac{1}{a^2} \frac{C_1}{R^2} \frac{1}{\tau - a} \sum f_i. \tag{2.13}$$

C-admissibility implies that:

$$\sqrt{2n(n-1)S} \sum f_i v_{ii}^2 \ge C \sum v_{ii}^2. \tag{2.14}$$

By substituting (2.14) in (2.13), we obtain:

$$\frac{\zeta^2 C}{(\tau - a)\sqrt{2n(n-1)S}} \sum v_{ii}^2 \le \frac{1}{a^2}\frac{C_1}{R^2}\frac{1}{\tau - a}\sum f_i.$$

v_{11} is the maximal principal curvature at 0, hence:

$$\frac{\zeta^2}{\tau - a} C v_{11}^2 \le \frac{1}{a^2}\frac{C_1}{R^2}\frac{1}{\tau - a} n^2 v_{11}.$$

Then, the following estimate holds for M (remember that $|\nabla u|$ is bounded depending on S and R):

$$M = \frac{\zeta}{\tau - a} v_{11} \le \frac{C_1 n k^3}{C R^2} = \frac{\alpha(S,R,n)}{C}. \qquad \square$$

Let R_s be the radius of the n-dimensional sphere of constant r-th curvature equal to H_r: $R_s = (H_r)^{-1/r}$.

Definition 2.1. *M is a small H_r-surface if there exists a ball $B(p,R)$, centered at some point $p \in \mathbb{R}^{n+1}_+$, of radius $R < R_s$, such that $M \subset B(p,R)$. Otherwise we say that M is a large H_r-surface.*

Remark 2.1. If M is a small H_r-surface, then $M \subset \bigcap_\alpha B_\alpha$, where B_α denotes the family of balls $B(q,\rho)$ of radius $\rho \le R_s$, centered at $q \in \mathbb{R}^{n+1}_+$, and $\partial M \subset B(q,\rho)$.

Proof of Theorem* 1.*1. The idea of the proof is the same as in [RR].

First we assume that the surfaces M_m have boundary in the plane P.

It follows from height estimates that all the M_m are contained in a fixed compact subset of $\mathbb{R}^{n+1}$. Let $r > 0$ and let Q be a vertical plane outside the compact set containing the M_m. For m large $\partial M_m \subset D(\sigma, r)$, so, using Alexandrov reflection technique, one can parallely translate Q until it meets $\partial D(\sigma, r)$, and the part of M_m swept out by Q is a graph over a part of Q. Therefore, one has uniform area and curvature estimates for this part of M_m. Alexandrov reflection with horizontal planes gives that the part of each M_m above $\{x_{n+1} = 1\}$ is a vertical graph, so one has uniform area and curvature estimates for the $M_m(1+\varepsilon)$, $\varepsilon > 0$. Standard compactness techniques yield a subsequence, which we also call M_m, that converges on compact subsets of $\mathbb{R}^{n+1} \setminus I$, where $I = \{0\} \times [0,1]$. The limit is either empty, or a compact surface M of scalar curvature one, properly embedded in $\mathbb{R}^{n+1} \setminus I$ (embeddedness follows because the part of M contained in each of the halfspaces $\{\alpha_1 x_1 + \cdots + \alpha_n x_n > 0 \mid \sum \alpha_i^2 = 1,\ x_{n+1} > 0\}$ is a graph).

If the limit is empty, then for m large M_m is uniformly closed to I. Thus M_m is a small H_2-surface and, as $\partial M_m \subset B(r_m)$, it follows that $M_m \subset B(r_m)$. So M_m converges to σ.

Now, we assume that M_m converges to a surface M properly embedded in $\mathbb{R}^{n+1} \setminus I$. If one does Alexandrov reflection, vertical planes can be moved up to

$\partial D(r)$ for each $r > 0$, and the symmetries of M with respect to these hyperplanes lie in the compact domain enclosed by M and the plane P (since this holds for M_m, m large). This works up till $r = 0$ by continuity, and so M is a rotational surface about the vertical line through 0, and each component of M has multiplicity one. By the classification of rotational H_2-surfaces (cf. [L]), M can be a Delaunay surface, a stack of spheres, or a sphere. M has height at most two, hence it must be the sphere of radius one tangent to the plane P at σ.

Finally, we show that the convergence is uniform on compact subsets of $\mathbb{R}^{n+1} \setminus \sigma$. Given $\varepsilon > 0$, there exists $r > 0$ so that for m large,

$$M_m \cap \left\{D(\sigma, r) \times \left]\tfrac{3}{2}, \infty\right[\right\} = M_m \cap \{D(\sigma, r) \times]2-\varepsilon, 2+\varepsilon[\},$$

and this part is a graph above $D(\sigma, r)$. Coming down and making Alexandrov reflection with horizontal planes $P(t)$ from $t = 2$ to $t = 1$, we see that

$$M_m \cap \{D(\sigma, r) \times]\varepsilon, 1-\varepsilon[\} = \emptyset.$$

So, we have uniform estimates for M_m on compact sets of $\mathbb{R}^{n+1} \setminus \sigma$.

When the boundary of M_m is not planar, the proof works as well, with the following change. Instead of using vertical planes, we do Alexandrov reflection with ε-tilted planes, i.e., planes Q whose unit normal vector $\vec{n}(\varepsilon)$ satisfies

$$\langle \vec{n}(\varepsilon), (0, \dots, 0, 1)\rangle = \varepsilon.$$

Given ε and ρ positive, we choose r_m small enough so that Alexandrov reflection works with ε-tilted planes $Q + t\vec{n}(\varepsilon)$, t coming from infinity, up till the plane reaches $B(\rho)$. Then, we take $\varepsilon \to 0$ and we get the assertion of the theorem. □

Remark 2.2. We are not able to prove the same theorem for H_r-surfaces, $r > 2$, because in the general case, we don't have curvature estimates (cf. [N1] for a general discussion about this matter). As the referee suggested, the classification of rotational H_r-surfaces ($r > 2$) can be found in [P].

Remark 2.3. From Theorem 1.1 we infer the following result. For any positive integer k and ε, $\delta > 0$, there exists $\rho = \rho(k, \varepsilon, \delta) > 0$ such that any large surface M with scalar curvature equal to one and $\partial M \subset B(\rho)$ satisfies that $M \setminus B(\delta)$ is a radial graph of a function u defined over a domain of the sphere S, with $|u|_{C^k} < \varepsilon$. In fact, if this statement were false, we could construct a sequence M_m that contradicts Theorem 1.1.

Remark 2.4. One proves Theorem 1.2 as in [RS] with the same changes, with respect to the mean curvature case, as in Theorem 1.1.

3. The Topology of H_r-surfaces

Let M be a compact embedded H_r-surface in $\mathbb{R}^{n+1}_+$ with strictly convex boundary Γ in $\partial\mathbb{R}^{n+1}_+ = P$. Let $\mathcal{B}$ be the compact component of $\mathbb{R}^{n+1}_+$ bounded by M, and let Ω be the compact domain in P such that $\partial\Omega = \partial M$. M can be oriented by its mean curvature vector, which points toward $\mathcal{B}$.

Before proving Theorem 1.3 we state two lemmas.

Lemma 3.1. *M has the following properties.*

(i) *Each point $q \in M$ at maximal distance d from P, is contained in the vertical solid cylinder over Ω.*

(ii) *$M \cap \Omega \times [d/2, d]$ is a graph over a domain in P.*

(iii) *$M \setminus M \cap (\Omega \times \mathbb{R}_+)$ is a graph over a part of $\Gamma \times \mathbb{R}_+$ with respect to the lines orthogonal to $\Gamma \times \mathbb{R}_+$.*

(iv) *If M is contained in $\Omega \times \mathbb{R}_+$, then M is a graph over Ω.*

Lemma 3.2. *There exists a number $r > 0$, depending only on the principal curvatures of Γ, and a point $p \in \Omega$, depending on M, such that $M \cap D(p, r) \times \mathbb{R}_+$ is a graph over $D(p, r)$.*

The proofs of the lemmas are analogous to the case $r = 1$ (cf. [RR]). Essentially they are applications of Alexandrov reflection technique.

We recall that the radius R_s of the n-dimensional sphere of constant r-th curvature equal to H_r is $(H_r)^{-1/r}$, and the radius R_c of the n-dimensional cylinder of constant r-th curvature equal to H_r is equal to $(n/(n-r)H_r)^{-1/r}$. We remark that $R_c < R_s$. From the height estimates in Section 2, we know that $d \leq 2(H_r)^{-1/r}$.

***Proof of Theorem* 1.3.** We know that the maximum distance of M from P is $d \in [0, 2R_s]$. In order to prove Theorem 1.3, we will distinguish two different cases.

In the first one, we assume that $d \in [0, 2R_c[$. If H_r is small enough in terms of the geometry of the boundary Γ, we work out that M must be a small H_r-surface, then we conclude that M is a graph over a domain in P; in particular this implies that M is topologically a disk.

In the second case, $d \in [2R_c, 2R_s]$, we obtain the more interesting behaviour of M: large H_r-surfaces have no topology if H_r is small enough in terms of the geometry of the boundary Γ.

Let 2ω be the circumscribed diameter of Ω. We can assume that $\Omega \subset D(\sigma, \omega)$, where $D(\sigma, \omega)$ is the disk in P centered at the origin, of radius ω.

FIRST CASE: $d < 2R_c$.

(1A) We will show that $M \subset D(\sigma, \omega + R_c) \times [0, 2R_c[$.

Let C be a cylinder of radius R_c and horizontal axis α in the plane $\{x_{n+1} = R_c\}$, let $\tilde{C}$ be one of the two half-cylinders obtained by cutting C with the vertical plane containing α ($\partial\tilde{C} \subset \{x_{n+1} = 0\} \cup \{x_{n+1} = 2R_c\}$). Since M is compact, we can choose $\tilde{C}$ in a position to be disjoint from M and where M is lying on the convex side of $\tilde{C}$. Now, we start to translate $\tilde{C}$ horizontally towards M. By the maximum principle, as $\tilde{C}$ approaches M by translation, the first contact of $\tilde{C}$ with

M cannot be at an interior point of M. Therefore no accident will occur before $\partial\bar{C}$ reaches $\Gamma \subset D(\sigma,\omega)$. We repeat this process taking as axis of C each direction of P exiting from σ. This implies that M stays in $D(\sigma,\omega+R_c)\times[0,2R_c[$.

(1B) The next step is to see that, for H_r sufficiently small, M is contained in the ball of radius R_s centered at σ. We choose H_r such that:

$$\omega < R_s - R_c = H_r^{-1/r}\left[1-\left(\frac{n-r}{n}\right)^{1/r}\right].$$

Denote by σ' the first n coordinates of the origin. We consider the half-sphere S_+ of radius R_s passing through the point $(\sigma', 2R_c+R_s)$ and $\partial S_+ \subset \{x_{n+1} = 2R_c\}$. S_+ is disjoint from M because of (1A). We translate S_+ vertically towards M. We continue to denote by S_+ the translation. By the maximum principle, S_+ does not touch M at an interior point before it arrives on P; this means M is below S_+ when ∂S_+ is on P. Therefore M is contained in the ball $B(\sigma,R_s)$; in particular M is a small H_r-surface.

(1C) At last we prove that $M\subset\Omega\times[0,d]$ provided that $(H_r)^{1/r}$ is smaller than the smallest value of the principal curvatures of Γ. Then Lemma 3.1-(iv) implies that M is a graph over Ω. We take the half-sphere S_+ or radius R_s centered at σ, $S_+\subset\mathbb{R}^{n+1}_+$. We translate S_+ horizontally; by the maximum principle $\mathrm{int}(S_+)$ cannot touch $\mathrm{int}(M)$. So we can move S_+ until it reaches Γ. The condition on the curvature of Γ ensures that we can touch every point of Γ. Therefore M is contained in the cylinder over Ω, and the first part of Theorem 1.3 is established.

Second Case: $2R_c \le d \le 2R_s$. The part of M over the plane $\{x_{n+1} = d/2\}$ and the part of M lying outside of the solid cylinder $\Omega\times\mathbb{R}_+$ are graphs, because of Lemma 3.1-(ii) and Lemma 3.1-(iii) respectively. It follows that each connected component of those parts of M is topologically a disk. Hence, we need to understand the topology of M in $\Omega\times[0,d/2]$.

Let $\ell(\Gamma)$ and $K(\Gamma)$ such that $0<\ell(\Gamma)\ll d/2$ and $2\omega<K(\Gamma)$ be fixed numbers depending only on the geometry of Γ. It will be clear in (2D) how we choose $\ell(\Gamma)$ and $K(\Gamma)$. First, we show that no point of M is in $\Omega\times[\ell(\Gamma),d/2]$. Actually, we obtain even more: no point of M is contained in $D(\sigma,K(\Gamma))\times[\ell(\Gamma),d/2]$ (this will be achieved in (2C)). Then, we use the latter to figure out that no interior point of M is in $\Omega\times[0,\ell(\Gamma)]$. This implies that all parts of M are graphs and therefore M is topologically a disk.

(2A) The goal is to show that there exists a point $p\in M\cap\{x_{n+1}=d-\ell(\Gamma)\}$ such that the distance between p and the x_{n+1}-axis is at least

$$\lambda = \sqrt{2\ell(\Gamma)H_r^{-1/r}-\ell^2(\Gamma)}.$$

From Lemma 3.1-(ii) it follows that the part of M over $\{x_{n+1}=d-\ell(\Gamma)\}$, say M', is a graph of height $\ell(\Gamma)$ with boundary $\partial M'$ on $\{x_{n+1}=d-\ell(\Gamma)\}$.

By contradiction, suppose that $\partial M'$ is contained in a disk in the plane $\{x_{n+1} = d - \ell(\Gamma)\}$ centered at $(\sigma', d - \ell(\Gamma))$ of radius strictly smaller than λ. Consider the n-dimensional sphere of radius R_s centered at $(\sigma', d + \ell(\Gamma) - R_s)$ and denote by S the part of that sphere above the plane $\{x_{n+1} = d\}$. S is disjoint from M. All the points of $\partial M'$ have a distance from the x_{n+1}-axis less than λ. By the choice of λ and by the maximum principle, we can move S vertically towards M' without contact point until ∂S arrives on $\{x_{n+1} = d - \ell(\Gamma)\}$. So, we can continue to move S and the first contact with M' must occur at an interior point of S with a point of $\partial M'$. This means that the height of M' is strictly less than $\ell(\Gamma)$, which gives a contradiction.

(2B) We want to prove that the compact domain in $\{x_{n+1} = d - \ell(\Gamma)\}$ bounded by $\partial M'$ contains the horizontal disk centered at $(\sigma', d - \ell(\Gamma))$ of radius $K(\Gamma)$. We sketch the proof; the same argument for $r = 1$ is outlined with more details in [S].

Using the notation of (2A), let $q \in \partial M'$ be a point at maximal distance $d_{\max} \geq \lambda$ from the x_{n+1}-axis. Now, consider H_r-surfaces with H_r so small that

$$\lambda = \sqrt{2\ell(\Gamma)H^{-1/r} - \ell^2(\Gamma)} > K(\Gamma) + 2\omega.$$

This implies $d_{\max} - 2\omega > K(\Gamma)$. Next, let $\{V(t)\}_{t\geq 0}$ be a family of parallel vertical planes such that the x_{n+1}-axis is contained in $V(0)$. Apply the Alexandrov reflection technique to M with these planes. For t large, $V(t)$ is disjoint from M. Now, if we approach M by $V(t)$, there will be a first contact point of some $V(t)$ with M. One continues to decrease t and considers the symmetries of the part of M swept out by $V(t)$, with respect to $V(t)$. These symmetries are in $\mathcal{B}$, the compact component of $\mathbb{R}^{n+1}_+$ bounded by $M \cup \Omega$. By the maximum principle, no accident can occur before reaching Γ, this means at least until $V(\omega)$. We now turn our attention to the point $q \in \partial M'$ and we examine the set $I(q)$ of the reflected images of q (when t varies from ∞ to ω). The set $I(q)$ is either empty or it is still contained in the plane $\{x_{n+1} = d - \ell(\Gamma)\}$, and $I(q) \setminus \partial I(q)$ is in the interior of $\mathcal{B}$. We repeat the process for all possible families $V(t)$ defined as above, and we find that the union of the corresponding $I(q)$ contains the disk of radius $d_{max} - 2\omega$ centered at $(\sigma', d - \ell(\Gamma))$. Therefore the disk of radius $K(\Gamma)$ centered at the same point is in $\mathcal{B}$.

(2C) Consider the family of planes $P(t) = \{x_{n+1} = t\}$ and apply the Alexandrov reflection technique. For $t > d$, $P(t)$ is disjoint from M. We can decrease t until the first accident occurs and this will happen when the image of an interior point of M touches Γ, hence for $t \leq d/2$. The disk of radius $K(\Gamma)$ in the plane $P(d - \ell(\Gamma))$ centered at the x_{n+1}-axis is in the interior of $\mathcal{B}$. By decreasing t from d to $d/2$, all the symmetries of this disk with respect to $P(t)$ are also contained in the interior of $\mathcal{B}$, and therefore no point of M is in $D(\sigma, K(\Gamma)) \times [\ell(\Gamma), d - \ell(\Gamma)]$.

(2D) Let $r > 0$ and $p \in \Omega$ be given by Lemma 3.2. Let C be the unique half-catenoid contained in $\mathbb{R}^{n+1}_+$ with axis orthogonal to P, meeting P orthogonally in the circle $C_0 = \partial D(p, \rho)$, where $\rho < r$ and ρ is sufficiently small such that the principal curvatures of C_0 are larger than the largest value of the principal curvatures of Γ. The latter condition allows us to translate C_0 in Ω so as to touch every point of Γ. Note that C is a graph over the non compact component of $P \setminus \Gamma$. Let $\Sigma = C \cap (P \times [0, \ell(\Gamma)]$ and let C_1 be the circle of Σ in $\{x_{n+1} = \ell(\Gamma)\}$. Let $E = \{\vec{v} \in P \mid C_0 + \vec{v} \subset \Omega\}$ and choose $K(\Gamma)$ so that the disk contained in the plane $\{x_{n+1} = \ell(\Gamma)\}$ centered at the point $(\sigma, \ell(\Gamma))$ of radius $K(\Gamma)$ contains $C_1 + \vec{v}$ for all $\vec{v} \in E$.

Let $\vec{a} = (0, \ldots, 0, \ell(\Gamma))$. (2C) implies $\Sigma + \vec{a} \subset \operatorname{int}(\mathcal{B})$ and by Lemma 3.2, $\partial\Sigma + t\vec{a} \subset \operatorname{int}(\mathcal{B})$ for $0 \leq t \leq \ell(\Gamma)$. It follows that $\Sigma \subset \operatorname{int}(\mathcal{B})$. Otherwise, when one translates $\Sigma + \vec{a}$ towards Σ, there would be a first point of contact of $\Sigma + t\vec{a}$ with M. At this contact point $\Sigma + t\vec{a}$ lies on the side of M to which the mean curvature vector of M points. This is impossible since the contact point is an interior point of both surfaces and $\Sigma + t\vec{a}$ is a minimal surface (i.e., $H_1 = 0$).

We know that the boundary of $\Sigma + \vec{v} \subset \operatorname{int}(\mathcal{B})$ for each $\vec{v} \in E$. Hence $\Sigma + \vec{v} \subset \operatorname{int}(\mathcal{B})$ for each $\vec{v} \in E$, by a reasoning similar as above: the family $\Sigma + t\vec{v}$ for $0 \leq t \leq 1$ can not have first point of interior contact with M as t goes from 0 to 1. Our choice of C_0 guarantees that for each $q \in \Gamma$, there is a $\vec{v} \in E$ such that $C_0 + \vec{v}$ is tangent to Γ at q and Σ is orhogonal to P at q. So, we have proved that $\Omega \times [0, \ell(\Gamma)] \subset \operatorname{int}(\mathcal{B})$, as desired.

Collecting the results obtained in (A2)-(D2) we have that $\Omega \times [0, d/2] \subset \operatorname{int}(\mathcal{B})$. Therefore $M \cap (\Omega \times [0, \infty))$ is a graph over Ω. The part of M outside of $\Omega \times [0, \infty)$ is also a graph, so M is topologically a disk, and Theorem 1.3 is proved. □

Corollary 3.1. *Let Γ be as in Theorem 1.3. Then, there exists a positive number $V(\Gamma)$, depending only on the extreme values of Γ, such that any H_r-surface $M \subset \mathbb{R}^{n+1}_+$ bounded by Γ which encloses a compact set W in $\mathbb{R}^{n+1}_+$ with $\operatorname{Vol}(W) > V(\Gamma)$ is topologically a disk.*

Proof. From height estimates we have that $M \subset B(r + 2(H_r)^{-1/r})$, where $r > 0$ is chosen such that $\Gamma \subset D(r)$. So, W is contained in the same ball, and

$$\operatorname{Vol}(W) \leq c(r + 2(H_r)^{-1/r})^{n+1}, \tag{3.1}$$

for a positive constant c depending on n. Thus, if $\operatorname{Vol}(W)$ is big enough, we will have $H_r < h(\Gamma)$, and the result follows from Theorem 1.3. □

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