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# THE STATE OF THE ART OF "BERNSTEIN'S PROBLEM"

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**Abstract**

Whether the only minimal stable complete hypersurfaces in $\mathbb{R}^{n+1}$, $3 \leq n \leq 7$, are hyperplanes, is an open problem. We describe its historical motivations and our results obtained by exploring it.

## 1 Historical Background

A minimal hypersurface in $\mathbb{R}^{n+1}$ is a critical point of the area functional with respect to compact support variations. In 1776, Meusnier discovered a geometrical interpretation of a minimal surface in terms of the mean curvature: a minimal surface has mean curvature $H \equiv 0$ at each point. Meusnier's result holds in any dimension.

Consider a hypersurface of $\mathbb{R}^{n+1}$ described as a graph of a $C^2$ function $f : \Omega \longrightarrow \mathbb{R}$, $\Omega \subset \mathbb{R}^n$. In this case the area functional is

$$A(f) = \int_\Omega \sqrt{1 + |\nabla f|^2} dv,$$

where $dv$ is the volume element on $\mathbb{R}^n$. By definition, the graph of $f$ is minimal if and only if $f$ is a critical point of $A$. It is straightforward that a minimal graph is a minimum of $A$.

Furthermore, the graph of $f$ is minimal if and only if $f$ satisfies the *Minimal Surface Equation:*

$$(1 + |\nabla f|^2) \sum_{i=1}^{n} f_{ii} - \sum_{i,j=1}^{n} f_i f_j f_{ij} = 0 \qquad (1)$$

The following result is known as Bernstein's Theorem:

*A complete minimal* $C^2$ *graph in* $\mathbb{R}^3$ *is a plane.*

Bernstein's Theorem was proved by Bernstein in [2] (see also [16] for a different approach). Then he stated Bernstein's conjecture:

*If* $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ *is a solution of the minimal surface equation* (1) *in* $\mathbb{R}^n$ *then* $f$ *is a linear function.*

Bernstein's conjecture has been a longstanding problem and it was proved to be true for $n \leq 7$ and false for $n \geq 8$. We recall the main steps of the proof.

A minimal hypersurface is *stable* if it is a local minimum of the area functional with respect to compactly supported deformations. A minimal graph is a global minumum for the area functional, hence it is stable.

Fleming [9] gave a new proof of Bernstein's Theorem, using geometric measure theory. He showed that the existence of a non flat complete minimal graph in $\mathbb{R}^3$, yields the existence of a minimal stable non trivial cone in $\mathbb{R}^3$. He then proved that such a cone does not exist.

Fleming's results were extended by Almgreen [1] in $\mathbb{R}^4$ and by Simons [17] in $\mathbb{R}^{n+1}$, for $n \leq 6$. In the same paper, however, Almgreen founded non flat minimal stable cones of codimension one in $\mathbb{R}^{2m}$, for $m \geq 4$. De Giorgi [6] extended the non existence result to $\mathbb{R}^{n+1}$, $n \leq 7$, proving that the existence of a complete minimal graph in $\mathbb{R}^{n+1}$ implies the existence of a minimal stable cone in the lower dimensional space $\mathbb{R}^n$.

Finally Bombieri, De Giorgi and Giusti [3] showed the existence of complete minimal graphs in $\mathbb{R}^{n+1}$ for $n \geq 8$. In fact they constructed a calibration from Simons' minimal cones [17] and deduced from it the existence of complete minimal graphs.

## 2 Some Extensions of Bernstein's Problem

Fleming's proof of Bernstein's Theorem extends to area minimizing hypersurfaces in $\mathbb{R}^{n+1}$, $n \leq 6$, but not to minimal stable hypersurfaces. So, it is quite natural to ask the following Bernstein's problem in the parametric case:

*Does it exist a complete, minimal, stable, non planar hypersurface in* $\mathbb{R}^{n+1}$?

The discussion about the parametric Bernstein's problem will be the subject of the rest of this paper.

Let us come back to the non parametric case, for a while, in order to recall the result of Heinz [11]. Heinz studied the solution of (1) on a disk in $\mathbb{R}^2$ centered at $p$, of radius $R$. He proved that there exists an universal constant $C$ such that

$$|A|(p) \leq \frac{C}{R^2}$$

where $|A|$ is the norm of the second fundamental form of the graph.

Heinz' result implies Bernstein's Theorem.

Schoen, Simon and Yau [18] extended Heinz' estimate to area minimizing hypersurfaces of $\mathbb{R}^{n+1}$, up to dimension $n = 5$. Their result implies Bernstein's conjecture for area minimizing hypersurfaces in the appropriate dimension.

When $n = 2$, Do Carmo and Peng [7] and independently Fischer-Colbrie and Schoen [10], proved that a stable minimal surface immersed in $\mathbb{R}^3$ is a plane.

On the contrary, when $n \geq 8$, the graphs in [3] are stable, minimal, complete, non planar hypersurfaces in $\mathbb{R}^{n+1}$.

So, we are left with the following parametric Bernstein's problem:

*Are the hyperplanes the only minimal, stable, complete hypersurfaces in* $\mathbb{R}^{n+1}, 3 \leq n \leq 7$?

Before describing our contributions to the parametric Bernstein's problem, we recall some other partial results.

We denote by $B(p, R)$ the ball in $\mathbb{R}^{n+1}$ centered at $p$ of radius $R$. Schoen, Simon and Yau [18] proved the following very general result about minimal stable hypersurfaces.

**SSY-Theorem** *For any* $n \in \mathbb{N}$ *and* $q \in \left[0, 4 + \sqrt{\frac{8}{n}}\right[$, *there exists a constant* $\beta(q, n)$ *satisfying the following condition: if* $M$ *is a stable minimal hypersurface of* $\mathbb{R}^{n+1}$ *then, for any* $R > 0$, $p \in M$,

$$\int_{B(p,\frac{R}{2})\cap M} |A|^q \leq \beta(q,n)R^{-q}vol(B(p,R)\cap M) \tag{2}$$

Using the same techniques as in the proof of SSY-Theorem, Do Carmo and Peng [8] showed that a minimal, stable, complete hypersurface is a hyperplane if

$$\lim_{R\longrightarrow\infty} \frac{\int_{B_R}|A|^2}{R^{2+2q}} = 0, \quad q < \sqrt{\frac{2}{n}}.$$

Cao, Shen and Zhu [5] proved that a minimal, stable, complete hypersurface in the Euclidean space must have only one end. This result has recently been extended to any ambient space with positive sectional curvature by Li and Wang [12]. Q. Chen [4] showed that, if the number of connected components of the intersection between a minimal, stable, complete hypersurface $M$ and any ball of $\mathbb{R}^{n+1}$, $n = 3, 4$, is bounded by some constant, then $M$ is a hyperplane. D. Zhou and X. Chen have told us that they have recently proved the following: if the $L^p$ norm of the second fundamental form of a minimal, stable, complete hypersurface $M$ in $\mathbb{R}^{n+1}$, $n \leq 4$, $p \geq n$, is bounded, then $M$ is a hyperplane (personal communication).

# 3 New Results

We have been exploring the problem of existence of non planar, complete, minimal, stable hypersurfaces embedded in $\mathbb{R}^{n+1}, 3 \leq n \leq 7$ and we have obtained some new results. In this section we state them and we describe briefly the techniques of the proofs. The proofs are essentially contained in [14], [15].

First, we are able to reduce the problem to the bounded curvature case. Let $M$ be a submanifold of $\mathbb{R}^{n+1}$ and $|A|$ the length of the second fundamental form of $M$. We say that $M$ has *uniformly bounded curvature* if there is a positive constant $C$ such that $|A|(x) \leq C$, $\forall x \in M$.

**Theorem 1** *If there exists a non flat, complete, minimal, stable hypersurface, properly immersed in* $\mathbb{R}^{n+1}$, *then there exists such a hypersurface with uniformly* *bounded curvature.*

Theorem 1 is an easy consequence of the following curvature estimate (where, we do not assume that $M$ has codimension one).

**Theorem 2** *Assume that a complete minimal, stable submanifold of* $\mathbb{R}^{n+1}$ *with uniformly bounded curvature is flat. Let* $M$ *be a properly immersed minimal, stable submanifold of* $\mathbb{R}^{n+1}$. *We have*

*(i) if* $\partial M \neq \emptyset$, *then for any point* $p \in M$

$$|A|(p) \leq \frac{C}{\inf_{q\in\partial M} |p-q]} \tag{3}$$

*where* $C$ *is some universal constant;*

*(ii) if* $M$ *is complete without boundary, then* $M$ *is flat.*

In the proof of Theorem 2, we use a well known rescaling technique. We assume by contradiction that the universal constant of inequality (3) does not exist. This yields a sequence of minimal hypersurfaces, contradicting inequality (3). We rescale each of them by a homothety whose ratio is essentially the maximum of the norm of its second fundamental form. The rescaled hypersurfaces have uniformly bounded curvature. Then, we can extract a subsequence converging to a minimal stable hypersurface with uniformly bounded curvature and with curvature one at one point. Such hypersurface can not be flat and this gives a contradiction to the assumption (see [15] for the proof).

Next result answers Bernstein's parametric problem in a particular case.

**Theorem 3** *Let* $M$ *be a complete minimal stable hypersurface immersed in* $\mathbb{R}^{n+1}$, $n < 5$. *If there exist* $\epsilon_0 > 0$ *and* $N \in \mathbb{N}$ *such that, for any* $p \in M$ *and any* $\epsilon \leq \epsilon_0$, *the number of connected components of* $M \cap B(p, \epsilon)$ *is bounded by* $N$, *then* $M$ *is a hyperplane.*

Here is an idea of the proof of Theorem 3. By Theorem 1, we can restrict to the case of uniformly bounded curvature. Then, $M$ is locally a graph and by hypothesis, in each extrinsic ball, of radius smaller thaw $\epsilon_0$ there are a finite number of such graphs. Hence, the area of $M$ in each ball of radius smaller thaw $\epsilon_0$ is uniformly bounded. Now, the result follows easily from the Schoen-Simon-Yau's curvature estimate contained in the inequality (2).

We recall the results of [14], that are more significative in this context. We notice that, by Theorem 1, the assumption of uniformly bounded curvature for the hypersurface $M$ is not a strong hypothesis.

Let $r : M \longrightarrow \mathbb{R}$ be a $C^2$ function defined on the hypersurface $M$ and let $N$ be a unit normal vector field to $M$. We call *Tube* of radius $r$ around $M$ the set

$$T(M,r) = \{x \in \mathbb{R}^{n+1} \mid \exists p \in M,\ x = p + tN(p),\ t \leq r(p)\}.$$

In our opinion, strong evidence for the non existence of a non planar, embedded, minimal stable hypersurface is given by the following Theorem.

**Theorem 4** *Let $M$ be a non planar, stable, minimal hypersurface embedded in $\mathbb{R}^{n+1}$, $n \leq 5$, with uniformly bounded curvature. Fix a point $\sigma$ in $M$ and denote by $d(p,\sigma)$ the intrinsic distance between $\sigma$ and any point $p \in M$. Let $0 < c_1 \leq 1$, $c_2 > 0$, $\delta \geq 1$ and consider any $C^2$ function $r$ on $M$ such that $r(p) \geq \inf\{c_1|A(p)|^{-1}, c_2 d(p,\sigma)^\delta\}$. Then the tube $T(M,r)$, is not embedded.*

Theorem 4 means that the subfocal tube of a non planar minimal, stable hypersurface embedded in $\mathbb{R}^{n+1}, n \leq 5$, cannot be embedded. More precisely: if $n \leq 4$, such hypersurfaces admit no embedded tube of constant radius, whatever small the radius is .

However, the following Theorem shows that, assuming a further hypothesis on the embedding, such hypersurfaces admit an embedded tube whose radius decays sufficiently fast.

**Theorem 5** *Let $M$ be a minimal, non planar hypersurface embedded in $\mathbb{R}^{n+1}$, $n \leq 5$, with uniformly bounded curvature.*

*(i) If $M$ is stable and there is an Euclidean ball $B(p)$ in $\mathbb{R}^{n+1}$ centered at a point $p \in M$ such that $B(p) \cap M$ consists of a finite number of connected components, then there exists an embedded tube around $M$.*

*(ii) If $M$ is not stable, then either $M$ is proper, or $M$ is properly embedded in a open set bounded by a complete minimal stable hypersurface (possibly multiply-connected).*

Let us say some words about the proof of Theorem 4 (see Theorem 1 in [14]). We call $T(R, r)$ the tube around a ball of radius $R$ of $M$. If such a tube were embedded in $\mathbb{R}^{n+1}$, the order of its volume in terms of $R$ should be at most $n + 1$. We then compute the volume of $T(R, r)$ in terms of the integral of the norm of the second fundamental form of $M$. By a subtle application of inequality (2), one obtains that the order of the volume of $T(R, r)$ is larger than $n + 1$, hence $T(R, r)$ cannot be embedded.

For the proof of (i) of Theorem 5 we use a purely topological argument, while (ii) is a generalization to higher dimension of a result about laminations, contained in [13]. In fact, the result in [13] is much stronger than ours, because there, the authors can use the parametric form of Bernstein's Theorem in dimension two (see Theorem 2 in [14] for details).

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