

Stably embedded minimal hypersurfaces

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Abstract We use Schoen's curvature estimates to prove that the subfocal tubular neighborhood of a nonplanar minimal hypersurface with bounded second fundamental form, stably embedded in \mathbb{R}^{n+1} , $n < 5$, whose radius decays sufficiently slowly cannot be embedded. In particular such hypersurfaces admit no embedded tubular neighborhoods of constant radius, whatever small the radius. However, assuming a further hypothesis on the embedding, we prove that such hypersurfaces admit an embedded tube whose radius decays sufficiently fast.

Keywords Minimal · Hypersurface · Embedding · Stable

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1 Introduction

While a complete stable minimal surface in \mathbb{R}^3 is known to be a plane [5], there are examples of stable nonplanar minimal hypersurfaces embedded in \mathbb{R}^{n+1} for $n \geq 7$ [1]. Whether the only minimal, stable, complete hypersurfaces in \mathbb{R}^{n+1} , $3 \leq n \leq 6$, are hyperplanes is an open problem. Exploring it yields new results about embedded stable minimal hypersurfaces. Let us recall some of them.

In [6] it is proved that a minimal, stable, complete hypersurface is a hyperplane as soon as

$$\lim_{R \rightarrow \infty} \frac{\int_{B_R} |K|}{R^{2+2q}} = 0, \quad q < \sqrt{2/n}.$$

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In [2], it is proved that a minimal, stable, complete hypersurface must have only one end. This result has been extended recently to any ambient space with positive sectional curvature [11]. Chen [3] has shown that if the number of connected components of the intersection between a minimal, stable, complete hypersurface M and any ball of \mathbb{R}^{n+1} , $3 \leq n \leq 4$, is bounded by some constant, then M is a hyperplane. D. Zhou and X. Chen proved that, if the L^p norm of the second fundamental form of a minimal, stable, complete hypersurface M in \mathbb{R}^{n+1} , $n \leq 4, p \geq n$, is bounded, then M is a hyperplane (Personal communication).

In the following we always assume $n \geq 3$. The problem then is to know whether there are non-flat minimal, stable, complete minimal hypersurfaces in \mathbb{R}^{n+1} for $n \leq 5$ (as it is the case when $n = 2$). A strong evidence for this non-existence result is given by the following Theorem.

Theorem 1 *Let M be a nonplanar, stable, minimal hypersurface in \mathbb{R}^{n+1} , $n \leq 5$, with bounded second fundamental form. Fix a point σ in M and denote by $d(p, \sigma)$ the intrinsic distance between σ and any point $p \in M$. Let $0 < c_1 \leq 1$, $c_2 > 0$, $\delta \geq 1$ and consider any function on M such that $r(p) \geq \inf\{c_1|A(p)|^{-1}, c_2d(p, \sigma)^\delta\}$. Then the tube $T(M, r(p))$ is not embedded.*

Theorem 1 means that the subfocal tubular neighborhood of a nonplanar minimal, stable hypersurface embedded in \mathbb{R}^{n+1} , $n \leq 5$, cannot be embedded. More precisely: such hypersurfaces admit no embedded tubular neighborhoods of constant radius, whatever small the radius. However, the following Theorem shows that assuming a further hypothesis on the embedding, such hypersurfaces admit an embedded tube whose radius decays sufficiently fast.

Theorem 2 *Let M be a minimal, nonplanar hypersurface embedded in \mathbb{R}^{n+1} , $n \leq 5$, with uniformly bounded second fundamental form.*

- (i) *If M is stable and there is an Euclidean ball $B^E(p)$ in \mathbb{R}^{n+1} centered at a point $p \in M$ such that $B^E(p) \cap M$ consists of a finite number of connected components, then there exists an embedded tube around M .*
- (ii) *If M is not stable, then either M is proper, or M is properly embedded in a open set bounded by a complete minimal stable hypersurface (possibly multi-connected).*

The paper is organized as follows.

In Sect. 2 we prove Theorem 1. From Theorem 1 we know that there many tubes of a non flat minimal hypersurface M that intersect other pieces of M . These pieces define graphs in the normal bundle to M .

In Sect. 3, we study such graphs. More precisely we compute the equation that the distance function ϕ from points of a minimal piece of a minimal hypersurface to another minimal piece, satisfies. When M has bounded curvature, ϕ satisfies a linear elliptic inequality, namely $F(\phi) \leq 0$, where F is a linear elliptic operator.

In Sect. 4, we give a lower estimate for positive eigenfunctions of $F(\phi) \leq 0$. We use this fact to show that, roughly speaking, the distance function ϕ is bounded from below at a point $q \in M$ in terms of the value of ϕ at a point p at distance R from q and of the distance R (the method used in this section is similar to [17]). This gives a lower estimate for the (non-constant) radius of tubes that separate the pieces. This result gives a measure of the radius of a tube that separate M (when considering it as the zero section of its normal bundle) from any other piece of M . Our result does not

prevent from having an infinite number of connected components of M cutting the tube around M .

In Sect. 5 we deal with the infinite number of components case (see Theorem 2). The proof Theorem 2 is topological in nature and it is proved without using the estimate on the radius of the tube found in Sect. 4.

2 Volume of tubes

We will give a geometric application of an estimate found in [15]. Let M be a hypersurface in \mathbb{R}^{n+1} with an orientation \mathbf{N} and let $r : M \rightarrow \mathbb{R}$ be a smooth positive function. Fix a point σ of M and let B_R be the ball of M centered at σ of radius R in the metric induced by the immersion. Denote by $|B_R|$ the volume of such a ball. Consider a bundle $\pi : U \rightarrow M$ with U an open neighborhood of M in \mathbb{R}^{n+1} , for which the zero section $s : M \rightarrow U$ is the inclusion of M in U . Such a bundle is called a *Tubular neighborhood* of M in \mathbb{R}^{n+1} and we denote $s(M) = M_0$. It is clear that one can choose a tubular neighborhood that is locally equivalent to a domain of the normal bundle; hence the exponential map $\exp : NM \rightarrow \mathbb{R}^{n+1}$ is a local diffeomorphism between U and a neighborhood of the zero section of the normal bundle. We will often write our definitions up to the local diffeomorphism \exp .

Definition 1 We call *Tube of radius r around M* the set

$$T(M, r) = \{x \in \mathbb{R}^{n+1} \mid \exists p \in M, x = p + t\mathbf{N}(p), t \leq r(p)\}.$$

Let A be the second fundamental form of M . For simplicity, we assume that at any point of M , $|A| \neq 0$.

Definition 2 Assume that M has bounded second fundamental form. The tube

$$T(M, c|A|^{-1})$$

is called a *Subfocal Tube* for any $c \leq 1$.

Denote by $T(R, r)$ the tube of radius r about $B_R \subset M$, by dT its volume element and let

$$V(R, r) = \int_{T(R,r)} dT.$$

The following Proposition is a weaker form of Theorem 1 and contains almost all the ideas of the proof

Proposition 1 Let M be a nonplanar stable minimal hypersurface in $\mathbb{R}^{n+1}, n \leq 5$, with bounded second fundamental form. Then, for R sufficiently large, the tube $T(R, r), r(p) = c|A(p)|^{-1}, c \leq 1$, is not embedded. In particular the subfocal Tube $T(M, c|A(p)|^{-1})$ is not embedded.

In order to prove Proposition 1, we need some preliminaries. Let

$$M_t = \{x \in \mathbb{R}^{n+1} \mid \exists p \in M, x = p + t\mathbf{N}(p)\}.$$

Note that

$$M_t = \{x \in \mathbb{R}^{n+1} \mid d(x, M) = t\}$$

and that $dT = dM_t dt$, where dM_t is the volume element of M_t . It is not hard to see that

$$dM_t = \det(1 - tA)dM,$$

where dM is the volume element of M (cf. [9] for details). By a straightforward computation we have:

$$\det(1 - tA) = \prod_{h=1}^n (1 - t\kappa_h) = 1 + \sum_{h=1}^n (-1)^h t^h S_h,$$

where κ_h are the principal curvatures of M and S_h are the elementary symmetric functions of $\kappa_1, \dots, \kappa_n$ and $S_0 = 1$. For $r(p) \leq c|A(p)|^{-1}$, $c \leq 1$, one has:

$$\begin{aligned} V(R, r) &= \int_{T(R,r)} dM_t dt = \int_{T(R,r)} \left(1 + \sum_{h=1}^n (-1)^h t^h S_h \right) dM dt \\ &= \int_{B_R} dM \int_0^{r(p)} \left(1 + \sum_{h=1}^n (-1)^h t^h S_h \right) dt \end{aligned}$$

From now on, we assume that the hypersurface M is minimal, hence $S_1 = 0$ and

$$V(R, r) = \int_{B_R} r(p) dM + \sum_{h=2}^n \frac{(-1)^h}{h+1} \int_{B_R} r(p)^{h+1} S_h dM.$$

We recall a fundamental result of [15] that provides estimates on the L^p norm of the second fundamental form of M .

SSY-Theorem *For any $n \in \mathbb{N}$ and $p \in \left[0, 4 + \sqrt{\frac{8}{n}}\right]$, there exists a constant $\beta(p, n)$ satisfying the following condition: if M is a stable minimal hypersurface of \mathbb{R}^{n+1} then, for any $R > 0$*

$$\int_{B_R} |A|^p \leq \beta(p, n) R^{-p} |B_R|. \tag{1}$$

Using [SSY]-Theorem, we can easily prove the following Lemma.

Lemma 1 *Let $n \leq 7$. There exists a constant $\gamma(n)$ satisfying the following conditions:*

- (i) *if M is a nonplanar stable minimal hypersurface in \mathbb{R}^{n+1} , $0 < \varepsilon < -\frac{1}{2} + \sqrt{\frac{2}{n}}$ and R is sufficiently large, then*

$$|B_R| > \gamma(n) R^{5+\varepsilon} \tag{2}$$

- (ii) *if M is a nonplanar stable minimal hypersurface in \mathbb{R}^{n+1} , for each $\alpha > 1$, and ε as above, there exists a sufficiently large $\tilde{R} > R$ such that*

$$|B_{\tilde{R}}| - |B_{\alpha^{-1}\tilde{R}}| > \gamma(n) R^{5+\frac{\varepsilon}{2}}. \tag{3}$$

Proof (i) As $n \leq 7$, in (1) we can choose $p = 5 + 2\varepsilon$ and we have

$$\int_{B_R} |A|^{5+2\varepsilon} \leq \beta(5 + 2\varepsilon, n)R^{-5-2\varepsilon}|B_R|$$

provided $\varepsilon < -\frac{1}{2} + \sqrt{\frac{2}{n}}$. Suppose, by contradiction, that for any positive constant $\gamma(n)$ there exists some R arbitrarily large such that

$$|B_R| \leq \gamma(n)R^{5+\varepsilon}$$

It follows that

$$\int_{B_R} |A|^{5+2\varepsilon} \leq \beta(5 + 2\varepsilon, n)\gamma(n)R^{-5-2\varepsilon}R^{5+\varepsilon} = \frac{\beta(5 + 2\varepsilon, n)\gamma(n)}{R^\varepsilon}.$$

Taking the limit $R \rightarrow \infty$ we deduce that $|A| \equiv 0$ on M , hence M is a hyperplane and this is a contradiction.

(ii) Let $R > 1$ be such that $|B_R| > \gamma(n)R^{5+\varepsilon}$ with ε as in (i). Let $\alpha > 1$ and consider the following sequence:

$$R_1 = R, R_2 = \alpha R_1, R_3 = \alpha^2 R_1, \dots, R_k = \alpha^{k-1} R, \dots$$

One can write

$$|B_{R_k}| = (|B_{R_k}| - |B_{R_{k-1}}|) + (|B_{R_{k-1}}| - |B_{R_{k-2}}|) + \dots + (|B_{R_2}| - |B_{R_1}|) + |B_{R_1}|.$$

As $R_k \geq R$, one has $|B_{R_k}| > \gamma(n)R_k^{5+\varepsilon}$, hence there exists $1 \leq j_k \leq k$ such that

$$|B_{R_{j_k}}| - |B_{R_{j_k-1}}| > \frac{\gamma(n)R_k^{5+\varepsilon}}{k}. \tag{4}$$

We notice that it cannot happen that

$$|B_{R_1}| > \frac{\gamma(n)R_k^{5+\varepsilon}}{k} = \frac{\gamma(n)(\alpha^{k-1}R_1)^{5+\varepsilon}}{k}$$

because R_1 is fixed and the right hand side of the previous inequality tends to infinity as $k \rightarrow \infty$.

For k sufficiently large, one can write (4) in the following way:

$$\begin{aligned} |B_{R_{j_k}}| - |B_{R_{j_k-1}}| &> \gamma(n)(\alpha^{j_k-1}R)^{5+\frac{\varepsilon}{2}}(\alpha^{k-j_k})^{5+\frac{\varepsilon}{2}}\frac{(\alpha^{k-1})^{\frac{\varepsilon}{2}}}{k}R^{\frac{\varepsilon}{2}} \\ &\geq \gamma(n)(R_{j_k})^{5+\frac{\varepsilon}{2}}. \end{aligned}$$

Denoting R_{j_k} with \tilde{R} one obtains (3). □

Proof of Proposition 1 For $h = 0, \dots, n$, let

$$V_h(R, r) = \frac{(-1)^h}{h+1} \int_{B_R} \frac{r^{h+1}}{|A|^{h+1}} S_h dM$$

be the term of order h of the volume of the tube $T(R, r)$. Denote by $S_{h,i}$ the h -th elementary symmetric function of the $n - 1$ variables $\kappa_j, j \in \{1, \dots, n\} \setminus i$. One has (cf. [12]):

$$hS_h = \sum_{i=1}^n S_{h-1,i}\kappa_i.$$

Using previous equality, we obtain easily:

$$S_2 = -\frac{|A|^2}{2}, \quad S_3 = \frac{1}{3} \sum_{i=1}^n \kappa_i^3,$$

$$S_4 = \frac{1}{8}|A|^4 - \frac{1}{4} \sum_{i=1}^n \kappa_i^4, \quad S_5 = -\frac{1}{6}|A|^2 \sum_{i=1}^n \kappa_i^3 + \frac{1}{5} \sum_{i=1}^n \kappa_i^5.$$

As $trA = \sum_{i=1}^n \kappa_i = 0$, hence for any $j, |\kappa_j| \leq \sum_{i \neq j, i=1}^n |\kappa_i|$. From Cauchy–Schwarz formula:

$$\kappa_j^2 \leq (n - 1) \sum_{i \neq j, i=1}^n |\kappa_i|^2.$$

A fortiori:

$$\kappa_j^2 \leq \frac{n - 1}{n} |A|^2.$$

Then, it is not hard to see that for every $0 \leq h \leq n \leq 5$, there exist a positive constant $c(h, n)$ such that

$$(-1)^{h+1} S_h \leq c(h, n) |A|^h, \tag{5}$$

where we can choose $c(0, n) = 1, c(1, n) = 0, c(2, n) = \frac{1}{2}$ and $0 < \sum_{h=3}^n \frac{c(h, n)}{h+1} < \frac{5}{6}$.

Then

$$V_0(R, r) = \int_{B_R} \frac{1}{|A|} dM, \quad V_1(R, r) = 0, \quad V_2(R, r) = -\frac{1}{6} \int_{B_R} \frac{1}{|A|} dM.$$

Hence

$$V(R, r) \geq \sum_{h=0}^n \int_{B_R} \frac{c(h, n)}{|A|} dM = \left(\frac{5}{6} - \sum_{h=3}^n c(h, n) \right) \int_{B_R} \frac{1}{|A|} dM. \tag{6}$$

Let us estimate $\int_{B_R} \frac{1}{|A|} dM$ from below. By Cauchy–Schwarz inequality:

$$|B_R| \leq \left(\int_{B_R} \frac{1}{|A|} \right)^{\frac{1}{2}} \left(\int_{B_R} |A| \right)^{\frac{1}{2}} \leq \beta(1, n)^{\frac{1}{2}} \left(\int_{B_R} \frac{1}{|A|} \right)^{\frac{1}{2}} |B_R|^{\frac{1}{2}} R^{-\frac{1}{2}},$$

where last inequality follows from (1), with $p = 1$. Hence

$$\int_{B_R} \frac{1}{|A|} \geq \beta (1, n)^{-1} |B_R| R.$$

Replacing in (6) one has, for R sufficiently large

$$V(R, r) \geq \left(\frac{5}{6} - \sum_{h=3}^n c(h, n) \right) |B_R| \beta (1, n)^{-1} R \geq C(n) R^{6+\epsilon},$$

where $C(n) = \gamma(n) \beta (1, n)^{-1} \left(\frac{5}{6} - \sum_{h=3}^n c(h, n) \right)$ and last inequalities is by (i) of Lemma 1.

Let us prove that the inequality

$$V(R, r) \geq C(n) R^{6+\epsilon} \tag{7}$$

implies that the tube $T(R, r)$ is not embedded.

Denote by $B^E(\sigma, R)$ the Euclidean ball in \mathbb{R}^{n+1} of radius R , centered at a point $\sigma \in M$. As for any point $p \in M$ the Euclidean distance between σ and p is less or equal to the intrinsic distance, one has

$$B_R \subset B^E(\sigma, R).$$

Hence, letting $a = \inf_{B_R} \frac{1}{|A|}$ one has

$$T(R, r) \subset B^E(\sigma, R + 2a).$$

Assume by contradiction that the tube $T(R, r)$ is embedded, then

$$V(R, r) \leq \text{vol}(B^E(\sigma, R + 2a)) \leq \omega_{n+1} (R + 2a)^{n+1},$$

where ω_{n+1} is the volume of the unit ball in \mathbb{R}^{n+1} . This is in contradiction with (7) for $R \rightarrow \infty$, as $n \leq 5$.

If M is contained in a Euclidean ball of finite radius R , centered at σ i.e. M is not properly embedded, then $T(M, r)$ would be contained in $B^E(\sigma, R + a)$. So, if $T(M, r)$ would be embedded, it should have finite volume, and this is again a contradiction by (7). □

Theorem 1 *Let M be a nonplanar stable, minimal hypersurface in \mathbb{R}^{n+1} , $n \leq 5$, with bounded second fundamental form. Fix a point σ in M and denote by $d(p, \sigma)$ the intrinsic distance between σ and any point $p \in M$. Let $0 < c_1 \leq 1, c_2 > 0, \delta \geq 1$ and consider any function on M such that $r(p) \geq \inf\{c_1|A(p)|^{-1}, c_2d(p, \sigma)^\delta\}$. Then the tube $T(M, r(p))$ is not embedded.*

Proof In order to simplify computation, we show the details of the proof for $n = 3$ and $c_1 = c_2 = 1$. In this case, one has

$$V(R, r) = \int_{B_R} r(p) dM + \frac{1}{3} \int_{B_R} r(p)^3 S_2 - \frac{1}{4} \int_{B_R} r(p)^4 S_3 dM.$$

Denote by B_R^+ the subset of B_R where $r(p) = |A(p)|^{-1}$ and let $B_R^- = B_R \setminus B_R^+$. Then

$$\begin{aligned}
 V(R, r) &= \int_{B_R^+} |A(p)|^{-1} dM - \frac{1}{6} \int_{B_R^+} |A(p)|^{-3} S_2 dM - \frac{1}{4} \int_{B_R^+} |A(p)|^{-4} S_3 dM \\
 &\quad + \int_{B_R^-} d(p, \sigma)^\delta dM - \frac{1}{6} \int_{B_R^-} d(p, \sigma)^{3\delta} S_2 dM - \frac{1}{4} \int_{B_R^-} d(p, \sigma)^{4\delta} S_3 dM.
 \end{aligned}$$

For the two integrals over B_R^+ we proceed as in the proof of Proposition 1. Let us estimate the integrals over B_R^- (actually we only write the job for S_3 , as it is analogous for S_2). Inequality (5) implies that

$$- \int_{B_R^-} d(p, \sigma)^{4\delta} S_3 dM \geq -c(3, 3) \int_{B_R^-} d(p, \sigma)^\delta dM.$$

Hence:

$$\begin{aligned}
 &\int_{B_R^-} d(p, \sigma)^\delta dM - \frac{1}{4} \int_{B_R^-} d(p, \sigma)^{4\delta} S_3 dM \\
 &\geq \left(1 - \frac{c(3, 3)}{4}\right) \int_{B_R^-} d(p, \sigma)^\delta dM
 \end{aligned}$$

On the other side

$$\begin{aligned}
 \int_{B_R^-} d(p, \sigma)^\delta dM &= \int_{B_R^- \setminus B_{\alpha^{-1}R}^-} d(p, \sigma)^\delta dM + \int_{B_{\alpha^{-1}R}^-} d(p, \sigma)^\delta dM \\
 &\geq \int_{B_R^- \setminus B_{\alpha^{-1}R}^-} d(p, \sigma)^\delta dM \geq \left(\frac{R}{\alpha}\right)^\delta (|B_R^-| - |B_{\alpha^{-1}R}^-|).
 \end{aligned}$$

As in the proof of Proposition 1 and using the previous inequality, one has

$$\begin{aligned}
 V(R, r) &\geq \left(\frac{5}{6} - c(3, 3)\right) (|B_R^+| - |B_{\alpha^{-1}R}^+| + |B_{\alpha^{-1}R}^+|) R \\
 &\quad + \left(1 - \frac{c(3, 3)}{4}\right) \left(\frac{R}{\alpha}\right)^\delta (|B_R^-| - |B_{\alpha^{-1}R}^-|).
 \end{aligned}$$

So, choosing $C = \min\left(\frac{5}{6} - c(3, 3), 1 - \frac{c(3, 3)}{4}\right)$, one has

$$V(R, r) \geq \min\left(C, \frac{1}{\alpha^\delta}\right) (|B_R| - |B_{\alpha^{-1}R}|) R^{\min(\delta, 1)}.$$

By (ii) of Lemma 1, there exists $\tilde{R} > R$ such that

$$\begin{aligned}
 V(\tilde{R}, r) &\geq \min\left(C, \frac{1}{\alpha^\delta}\right) \tilde{R}^{\min(\delta, 1)} \gamma(n) \tilde{R}^{5+\frac{\epsilon}{2}} \\
 V(\tilde{R}, r) &\geq \gamma(n) \min\left(C, \frac{1}{\alpha^\delta}\right) \tilde{R}^{6+\frac{\epsilon}{2}}
 \end{aligned}$$

as $\delta \geq 1$

Hence the tube $T(\tilde{R}, r)$ is not embedded. In particular the tube $T(M, r)$ is not embedded. □

We observe that for $n = 3, 4$ the hypothesis $\delta > 0$ is enough to obtain the result of Theorem 1.

Remark 1 Let $a = \inf_M \frac{1}{|A|} > 0$, then, in the hypotheses of Theorem 1, the tube $T(M, \frac{a}{2})$ is not embedded. So, the inverse image restricted to $T(M, a)$ of $M \subset M_0$ by the exponential map

$$\exp : NM \longrightarrow \mathbb{R}^{n+1}$$

is not empty and $T(M, a)$ contains pieces of M that are graphs over a piece of the zero section M_0 .

3 Minimal distance equation

3.1 The minimal distance equation in the general case

Let M and M' be two disjoint minimal hypersurfaces in \mathbb{R}^{n+1} . Fix a point q in M' and let $\phi(q)$ be the Euclidean distance from q to M . Let p be a point of M such that $\phi(q) = d(p, q)$ and let U be a neighborhood of p in M . Denote by $X : U \rightarrow \mathbb{R}^{n+1}$ the position vector of M and by N a unit normal vector field to M . We define the position vector of M' by $\tilde{X} = X + \phi N$, $\tilde{X} : U \rightarrow \mathbb{R}^{n+1}$. Let \tilde{N} be the unit normal vector field to M' . Let $\mathcal{B} = \{e_1, \dots, e_n\}$ be a principal orthonormal frame at p ie such that $\nabla_{e_i} e_j = \delta_{ij} \lambda_i N$, where $\lambda_i, i = 1, \dots, n$ are the principal curvatures at p . Extend the basis \mathcal{B} in the neighborhood of p by parallel transport. Finally denote by ∇ ($\tilde{\nabla}$) the connection on M (M' respectively).

Lemma 2 *The following equalities hold:*

$$\nabla_{e_i} \nabla_{e_i} X(p) = -\lambda_i N(p), \quad \nabla_{e_i} \nabla_{e_i} N(p) = \sum_{j=1}^n e_j(\lambda_i) e_j - \lambda_i^2 N(p). \tag{8}$$

Proof The former formula is straightforward. Let us prove the latter.

$$\nabla_{e_i} \nabla_{e_i} N = \nabla_{e_i} \left(\sum_{j=1}^n \langle \nabla_{e_i} N, e_j \rangle e_j \right) = \sum_{j=1}^n (e_i(\langle \nabla_{e_i} N, e_j \rangle) e_j + \lambda_i \nabla_{e_i} e_i).$$

From Codazzi equation for all $i, j = 1, \dots, d$:

$$e_i(\langle \nabla_{e_i} N, e_j \rangle) = e_j(\langle \nabla_{e_i} N, e_i \rangle).$$

Hence

$$\nabla_{e_i} \nabla_{e_i} N = \sum_{j=1}^n e_j (\langle \nabla_{e_i} N, e_i \rangle) e_j + \lambda_i \nabla_{e_i} e_i = \sum_{j=1}^n e_j (\lambda_i) e_j - \lambda_i^2 N.$$

□

The vector fields $\{X, \bar{X}, N, \bar{N}\}$ are defined on U . However we can extend them (as well as any vector field Y defined on M) to a neighborhood of M by parallel transport in \mathbb{R}^{n+1} . That is:

$$\tilde{Y}(\exp_p(sN(p))) = Y(p), \quad s \in \mathbb{R}.$$

In particular we can compute the covariant derivative ∇ of the extended vector field at the point $q \in M'$ at distance $\phi(q) = s$ from the point p ($q = \exp_p(sN)$):

$$\nabla_{e_i} \tilde{Y}(q) = \frac{1}{(1 + \lambda_i s)} \nabla_{e_i} Y(p), \quad \partial_N \tilde{Y} = 0. \tag{9}$$

As M' is minimal, one has

$$0 = \bar{\Delta}(\bar{X}) = \bar{\Delta}(X + \phi N) = \bar{\Delta}X + (\bar{\Delta}\phi)N + \phi \bar{\Delta}N + 2\bar{\nabla}_{\bar{\nabla}\phi} N,$$

and taking the scalar product with N in \mathbb{R}^{n+1}

$$\bar{\Delta}\phi + 2 \langle \bar{\nabla}_{\bar{\nabla}\phi} N, N \rangle + \langle \bar{\Delta}N, N \rangle + \phi \langle \bar{\Delta}X, N \rangle = 0. \tag{10}$$

Keeping (9) in mind, we will develop (10).

Lemma 3 *Let M be a minimal hypersurface in \mathbb{R}^{n+1} and let N be a unit normal vector field of M . Then, for any C^2 function $f : M \rightarrow \mathbb{R}$ at any point $p \in M$,*

$$\Delta_{\mathbb{R}^{n+1}} f(p) = \Delta_M f(p) + \partial_{NN}^2 f(p)$$

at any point $p \in M$.

Proof Let $\{e_1, \dots, e_n\}$ be an orthonormal basis of $T_p M$, then $\{e_1, \dots, e_n, N\}$ is a basis of $T_p \mathbb{R}^{n+1}$. Extend such basis to a neighborhood of p in M by parallel transport. Let X be a vector field in M and denote

$$X^T = \sum_{i=1}^n \langle X, e_i \rangle e_i, \quad X^\perp = X - X^T.$$

$$\operatorname{div}_{\mathbb{R}^{n+1}} X = \sum_{i=1}^n \langle \nabla_{e_i} X, e_i \rangle + \langle \nabla_N X, N \rangle. \tag{11}$$

As M is minimal

$$\operatorname{div}_M X = \operatorname{div}_M X^T.$$

Furthermore by a straightforward computation

$$\langle \nabla_N X, N \rangle = \nabla_N \langle X, N \rangle.$$

Replacing in (11) one has

$$\operatorname{div}_{\mathbb{R}^{n+1}} X = \operatorname{div}_M X^T + \nabla_N \langle X, N \rangle.$$

Now let $X = \nabla f$ and replace in the previous equation:

$$\begin{aligned} \Delta_{\mathbb{R}^{n+1}}f(p) &= \operatorname{div}_{\mathbb{R}^{n+1}} \nabla f(p) \\ &= \operatorname{div}_M \nabla_M f(p) + \nabla_N \langle \nabla f(p), N \rangle \\ &= \Delta f(p) + \partial_N \partial_N f(p). \end{aligned}$$

□

(i) *Computation of $\langle \bar{\Delta} \tilde{X}, N \rangle$.* From Lemma 3:

$$\langle \bar{\Delta} \tilde{X}, N \rangle = \langle \Delta_{\mathbb{R}^{n+1}} \tilde{X}, N \rangle - \langle \partial_{\bar{N}} \partial_{\bar{N}} \tilde{X}, N \rangle.$$

We first compute $\Delta_{\mathbb{R}^{n+1}} \tilde{X}$. For $i = 1, \dots, n$, using (9):

$$\begin{aligned} \nabla_{e_i} \nabla_{e_i} \tilde{X} &= \frac{1}{(1 + \lambda_i \phi)} \nabla_{e_i} (\widetilde{\nabla_{e_i} X}) + \nabla_{e_i} \left(\frac{1}{1 + \lambda_i \phi} \right) \nabla_{e_i} X \\ &= \left(\frac{1}{1 + \lambda_i \phi} \right)^2 \nabla_{e_i} \nabla_{e_i} X - \frac{\lambda_i \nabla_{e_i} \phi + \phi \nabla_{e_i} \lambda_i}{(1 + \lambda_i \phi)^2} e_i. \end{aligned}$$

Then, from the first equality of (8):

$$\langle \Delta_{\mathbb{R}^{n+1}} X, N \rangle = - \sum_{i=1}^n \frac{\lambda_i}{(1 + \lambda_i \phi)^2}. \tag{12}$$

We now compute $\partial_{\bar{N}} \partial_{\bar{N}} \tilde{X}$

$$\partial_{\bar{N}} \tilde{X} = \partial_{\sum_{i=1}^n \langle \bar{N}, e_i \rangle e_i} \tilde{X} = \sum_{i=1}^n \frac{1}{1 + \lambda_i \phi} \langle \bar{N}, e_i \rangle e_i.$$

Hence:

$$\langle \partial_{\bar{N}} \tilde{X}, N \rangle = 0,$$

and consequently

$$\langle \partial_{\bar{N}} \partial_{\bar{N}} \tilde{X}, N \rangle = - \langle \partial_{\bar{N}} \tilde{X}, \partial_{\bar{N}} N \rangle.$$

It remains to compute $\partial_{\bar{N}} \tilde{N}$. As before:

$$\partial_{\bar{N}} \tilde{N} = \sum_{i=1}^n \frac{1}{1 + \lambda_i \phi} \langle \bar{N}, e_i \rangle \nabla_{e_i} N = \sum_{i=1}^n \frac{\lambda_i}{1 + \lambda_i \phi} \langle \bar{N}, e_i \rangle e_i.$$

So:

$$\langle \partial_{\bar{N}} \tilde{X}, \partial_{\bar{N}} \tilde{N} \rangle = \sum_{i=1}^n \frac{\lambda_i}{(1 + \lambda_i \phi)^2} \langle \bar{N}, e_i \rangle^2. \tag{13}$$

Finally, from (12) and (13)

$$\langle \bar{\Delta} \tilde{X}, N \rangle = - \sum_{i=1}^n \frac{\lambda_i}{(1 + \lambda_i \phi)^2} + \sum_{i=1}^n \frac{\lambda_i}{(1 + \lambda_i \phi)^2} \langle \bar{N}, e_i \rangle^2. \tag{14}$$

(ii) *Computation of $\langle \bar{\Delta}\tilde{N}, N \rangle$.* By (9) one has:

$$\nabla_{e_i}\tilde{N} = \frac{1}{1 + \lambda_i\phi} \nabla_{e_i}N,$$

hence:

$$\begin{aligned} \Delta_{\mathbb{R}^{n+1}}\tilde{N} &= \nabla_{e_i}(\nabla_{e_i}\tilde{N}) = \left(\frac{1}{1 + \lambda_i\phi}\right)^2 \nabla_{e_i}\nabla_{e_i}N + \nabla_{e_i}\left(\frac{1}{1 + \lambda_i\phi}\right) \nabla_{e_i}N \\ &= \left(\frac{1}{1 + \lambda_i\phi}\right)^2 \nabla_{e_i}\nabla_{e_i}N - \frac{(\lambda_i\nabla_{e_i}\phi + \phi\nabla_{e_i}\lambda_i)\lambda_i}{(1 + \lambda_i\phi)^2} e_i. \end{aligned}$$

Then, using the second equality of (8):

$$\langle \Delta_{\mathbb{R}^{n+1}}\tilde{N}, N \rangle = - \sum \frac{\lambda_i^2}{(1 + \lambda_i\phi)^2}.$$

Furthermore:

$$\langle \partial_{\tilde{N}}\partial_{\tilde{N}}\tilde{N}, N \rangle = - \sum \frac{\lambda_i^2 \langle \tilde{N}, e_i \rangle^2}{(1 + \lambda_i\phi)^2},$$

hence, using Lemma 3:

$$\langle \bar{\Delta}\tilde{N}, N \rangle = \sum \frac{\lambda_i^2}{(1 + \lambda_i\phi)^2} (\langle \tilde{N}, e_i \rangle^2 - 1). \tag{15}$$

(iii) *Computation of $\langle \bar{\nabla}_{\bar{\nabla}\phi}N, N \rangle$.* Denote by $(\cdot)^{\bar{T}}$ the projection on T_qM' :

$$\bar{\nabla}_{\bar{\nabla}\phi}\tilde{N} = (\nabla_{\bar{\nabla}\phi}\tilde{N})^{\bar{T}} = \sum_{i=1}^n \langle \bar{\nabla}\phi, e_i \rangle \frac{\lambda_i}{1 + \lambda_i\phi} e_i^{\bar{T}}.$$

Hence

$$\langle \bar{\nabla}_{\bar{\nabla}\phi}N, N \rangle = \sum_{i=1}^n \frac{\lambda_i}{1 + \lambda_i\phi} \langle e_i^{\bar{T}}, N \rangle \langle \bar{\nabla}\phi, e_i \rangle. \tag{16}$$

We plug (14), (15), (16) into (10) and we obtain the *minimal surface equation*:

$$\bar{\Delta}\phi + \sum_{i=1}^n \frac{\lambda_i}{1 + \lambda_i\phi} a_i = 0, \tag{17}$$

where $a_i = 2 \langle \bar{\nabla}\phi, e_i \rangle \langle e_i^{\bar{T}}, N \rangle + \langle \tilde{N}, e_i \rangle^2 - 1$.

Remark 2 The coefficients a_i satisfy $-1 \leq a_i \leq 1, i = 1, \dots, n$. In fact, for $i = 1, \dots, n$, let $\alpha_i, \beta_i \in [0, \pi[$ such that $\cos \alpha = \langle \tilde{N}, e_i \rangle, \cos \beta = \langle N, \tilde{N} \rangle$. By a straightforward computation

$$a_i = \cos 2\alpha_i + \cos^2 \alpha_i \cos 2\beta_i.$$

We then check that

$$-1 \leq a_i \leq 1.$$

3.2 The minimal distance equation in the bounded geometry case

We want to prove that when M and M' have uniformly bounded second fundamental form, then at points where the distance ϕ is small, the zero order term of (17) has positive coefficient and the first order terms of equation (17) is uniformly bounded.

First we write the $a_i, i = 1, \dots, n$, in terms of $\bar{\nabla}\phi$.

One has:

$$\bar{N} = \sum_{i=1}^n (\nabla_{e_i}\phi)e_i - \frac{1}{\sqrt{1+|\nabla\phi|^2}}N.$$

Thus:

$$\begin{cases} \langle \bar{N}, e_i \rangle = \frac{\langle \nabla\phi, e_i \rangle}{\sqrt{1+|\nabla\phi|^2}} \\ \langle \bar{N}, N \rangle = -\frac{1}{\sqrt{1+|\nabla\phi|^2}} \end{cases} \tag{18}$$

Furthermore:

$$\langle \bar{\nabla}\phi, e_i \rangle = \frac{\langle \nabla\phi, e_i \rangle}{1+|\nabla\phi|^2}, \quad \langle \nabla\phi, e_i \rangle = \frac{\langle \bar{\nabla}\phi, e_i \rangle}{1-|\bar{\nabla}\phi|^2},$$

and

$$|\bar{\nabla}\phi|^2 = \frac{|\nabla\phi|^2}{1+|\nabla\phi|^2}.$$

Replacing in (18) one has:

$$\begin{cases} \langle \bar{N}, e_i \rangle = \frac{\langle \bar{\nabla}\phi, e_i \rangle}{\sqrt{1-|\bar{\nabla}\phi|^2}} \\ \langle \bar{N}, N \rangle = -\sqrt{1-|\bar{\nabla}\phi|^2} \end{cases} \tag{19}$$

Then from (19)

$$\langle e_i^{\bar{T}}, N \rangle = -\langle e_i, \bar{N} \rangle \langle \bar{N}, N \rangle = \langle \bar{\nabla}\phi, e_i \rangle. \tag{20}$$

Using (19) and (20), one has

$$a_i = 2 \langle \bar{\nabla}\phi, e_i \rangle^2 + \frac{\langle \bar{\nabla}\phi, e_i \rangle^2}{1-|\bar{\nabla}\phi|^2} - 1.$$

Now we we write differently $\sum_{i=1}^n \frac{\lambda_i}{1+\lambda_i\phi}$. Notice that:

$$\prod_1^n (1 + \lambda_i\phi) = 1 + S_1\phi + \dots + S_n\phi^n$$

As M is minimal, $S_1 = 0$ and

$$\begin{aligned} \sum_{i=1}^n \frac{\lambda_i}{1 + \lambda_i \phi} &= \frac{1}{\prod_{i=1}^n (1 + \lambda_i \phi)} (2S_2 \phi + \dots + nS_n \phi^{n-1}) \\ &= \phi \frac{(2S_2 + \dots + nS_n \phi^{n-2})}{1 + \phi^2(S_2 + S_3 \phi + \dots + S_{n-2} \phi^{n-2})} \\ &= \phi \left(\frac{-|A|^2 + \sum_{k=3}^n kS_k \phi^{k-2}}{1 + \sum_{k=2}^n \phi^k S_k} \right). \end{aligned}$$

Replacing in (17), one has:

$$\bar{\Delta} \phi + \sum_{i=1}^n \frac{\lambda_i}{1 + \lambda_i \phi} < \bar{\nabla} \phi, e_i >^2 \left(2 + \frac{1}{1 - |\bar{\nabla} \phi|^2} \right) + b \phi = 0, \tag{21}$$

where

$$b = \frac{|A|^2 - \sum_{k=3}^n kS_k \phi^{k-2}}{1 + \sum_{k=2}^n S_k \phi^k}.$$

If M' is in a sufficiently small tubular neighborhood of M , i.e. if ϕ is small enough, then $b \geq 0$. More precisely, we have the following estimate.

Lemma 4 *There exists a constant $0 < \epsilon(n) \leq 1$ depending only on n such that, if $|A|\phi \leq \epsilon(n)$, then*

$$b \geq |A|^2 \frac{1 - \alpha(n)\epsilon(n)}{1 + \alpha(n)\epsilon(n)^2},$$

where $\alpha(n) = \frac{(n-1)(n+2)}{2} \sup_{k \leq n} c(k, n)$.

Proof Using inequality (5) and the hypothesis one has

$$\begin{aligned} \left| \sum_{k=3}^n kS_k \phi^{k-2} \right| &\leq \sum_{k=3}^n kc(k, n)|A|^k \phi^{k-2} \leq |A|^2 \sup_k c(k, n) \sum_{k=3}^n k\epsilon^{k-2} \\ &\leq \frac{(n+3)(n-2)}{2} \epsilon(n) |A|^2 \sup_k c(k, n) \\ 1 + \sum_{k=2}^n S_k \phi^k &\leq 1 + \sum_{k=2}^n \phi^k c(k, n) |A|^k \\ &\leq \sum_{k=2}^n c(k, n) \epsilon^k \leq \frac{(n-1)(n+2)}{2} \epsilon(n)^2 \sup_k c(k, n). \end{aligned}$$

Hence, taking $\alpha(n) = \frac{1}{2}(n-1)(n+2) \sup_k c(n, k)$:

$$b \geq |A|^2 \frac{1 - \alpha(n)\epsilon(n)}{1 + \alpha(n)\epsilon(n)^2}.$$

If we choose $\epsilon(n) \leq \frac{1}{\alpha(n)}$, then $b \geq 0$. □

As the second fundamental forms of M and M' are uniformly bounded, C^0 estimates on ϕ , imply C^1 estimates on ϕ , then one has $|\bar{\nabla}\phi| \leq C\phi$ for a positive constant C (see [18] for the case $n = 2$). Hence the first order term of (21) satisfies

$$\left| \sum_{i=1}^n \frac{\lambda_i}{1 + \lambda_i\phi} \langle \bar{\nabla}\phi, e_i \rangle^2 \left(2 + \frac{1}{1 - |\bar{\nabla}\phi|^2} \right) \right| \leq 3bC^2\phi^3$$

and it is uniformly bounded.

Furthermore, one has the following differential inequality on ϕ .

Corollary 1 *With previous notations, if $|A|\phi \leq \epsilon(n)$ and $\phi \leq \frac{1}{6C^2}$ then*

$$\bar{\Delta}\phi + \delta|A|^2\phi \leq 0,$$

where $0 < \delta \leq \frac{1 - \alpha(n)\epsilon(n)}{2(1 + \alpha(n)\epsilon(n)^2)}$.

4 An estimate for positive eigenfunctions of $\Delta\phi + \alpha|A|^2\phi \leq 0$

We consider a positive function u satisfying the following inequality

$$\Delta u + Vu \leq 0 \tag{22}$$

on a hypersurface M in \mathbb{R}^{n+1} . Assume $V \geq \alpha|A|^2$, where $|A|$ is the norm of the second fundamental form of M and $\alpha < 1$ is a constant to be determined later. We apply an ingenious method of Fischer-Colbrie in order to estimate the function u (see [7, 16, 17]). Roughly speaking, we will estimate the function u from below at a point $q \in M$ in terms of the value of u at a point p at distance R from q and of the distance R .

The key step in the estimate is contained in the following proposition.

Proposition 2 *Let u be a positive solution of $\Delta u + Vu \leq 0$, $V \geq \alpha|A|^2$, $0 < \alpha < 1$, on a complete hypersurface $M \subset \mathbb{R}^{n+1}$, $n \leq 5$. Then for any two point $p, q \in M$, there is a path γ joining q to p such that*

$$c(n) \int_{\gamma} \left(\frac{d\varphi}{ds} \right)^2 ds \geq \int_{\gamma} \varphi^2 \left(\frac{d \ln(u)}{ds} \right)^2 ds, \tag{23}$$

where $c(n)$ is a constant depending only on n , s is the arc length on γ , and φ is any smooth compact support function on γ .

Proof Here, we proceed as Fischer-Colbrie in [7].

For k positive, define

$$d\bar{s}^2 = v^2 ds^2, \quad v = u^k$$

Let γ be a minimizing geodesic of the metric $d\bar{s}^2$ joining a fixed point $\sigma \in M$ to a point $p \in M$.

Choose a basis $\bar{\mathcal{B}} = \{\bar{e}_1, \bar{e}_2, \dots, \bar{e}_n\}$ of $T_{\sigma}M$, orthonormal for the metric $d\bar{s}^2$, such that \bar{e}_1 is the unit tangent vector to γ at the point σ . Extend the basis $\bar{\mathcal{B}}$ by parallel transport along γ . Let $\mathcal{B} = \{e_1, e_2, \dots, e_n\}$ be a basis of $T_{\sigma}M$ orthonormal for the metric ds^2 , such that e_1 is parallel to \bar{e}_1 and $A(e_i, e_j) = 0, i \neq j$, where A is the second fundamental form of M at σ .

As γ is a minimizing geodesic, the second variation of length is positive, for any normal deformation Y with compact support along γ , i.e.,

$$\int_{\gamma} \left(\frac{d|Y|}{d\bar{s}} \right)^2 d\bar{s} \geq \int_{\gamma} \langle \langle \bar{R}(\bar{e}_1, Y)\bar{e}_1, Y \rangle \rangle d\bar{s}, \tag{24}$$

where \bar{R} is the curvature tensor of M and $\langle \langle \cdot, \cdot \rangle \rangle$ is the scalar product, both for the metric $d\bar{s}$.

Let φ be any smooth function on γ with compact support and consider the normal deformations $Y_i = \varphi \bar{e}_i$ to γ for $i = 2, \dots, n$. Applying (24) to each Y_i and summing up for $i = 2, \dots, n$, one has:

$$(n - 1) \int_{\gamma} \left(\frac{d\varphi}{d\bar{s}} \right)^2 d\bar{s} \geq \int_{\gamma} \varphi^2 \bar{R}_{\bar{1}\bar{1}} d\bar{s}, \tag{25}$$

where $\bar{R}_{\bar{1}\bar{1}}$ is the Ricci curvature in the e_1 direction i.e.

$$\bar{R}_{\bar{1}\bar{1}} = \sum_2^n \langle \langle \bar{R}(\bar{e}_1, \bar{e}_i)\bar{e}_1, \bar{e}_i \rangle \rangle .$$

The relation between the Ricci curvature for the metric $d\bar{s}^2$ in the \bar{e}_1 direction and the Ricci curvature for the metric ds^2 in the e_1 direction is given by:

$$\bar{R}_{\bar{1}\bar{1}} = \frac{1}{v^2} (R_{11} - (n - 2)(\ln v)_{11} - \Delta(\ln v)) \tag{26}$$

(cf. [10,17])

Let us compute R_{11} in terms of the second fundamental form of M . By Gauss Equation, for $i = 1, \dots, n$:

$$R_{11} = \sum_{i=2}^n \langle R(e_1, e_i)e_1, e_i \rangle = \sum_{i=2}^n (\langle A(e_1, e_1), A(e_i, e_i) \rangle - |A(e_1, e_i)|^2).$$

As M is minimal, $\sum_{i=2}^n A(e_i, e_i) = -A(e_1, e_1)$, hence

$$R_{11} = -A_{11}^2. \tag{27}$$

Plug (27) in (26):

$$\bar{R}_{\bar{1}\bar{1}} = \frac{1}{v^2} (-A_{11}^2 - (n - 2)(\ln v)_{11} - \Delta(\ln v)). \tag{28}$$

By definition of v :

$$\Delta(\ln v) = k \frac{\Delta u}{u} - \frac{k}{u^2} |\nabla u|^2.$$

Using equation (22), one has

$$\Delta(\ln v) \leq -kV - k^{-1} |\nabla \ln v|^2.$$

If we choose $kV \geq A_{11}^2$, previous inequality yields

$$\Delta(\ln v) + A_{11}^2 \leq -k^{-1} |\nabla \ln v|^2,$$

and replacing in (28)

$$\bar{R}_{\bar{1}\bar{1}} \geq \frac{1}{v^2} (-(n-2)(\ln v)_{11} + k^{-1} |\nabla \ln v|^2).$$

As in the proof of Proposition 1

$$|A|^2 \geq \frac{n}{n-1} A_{11}^2. \tag{29}$$

As $V \geq \alpha |A|^2$, the desired inequality $kV \geq A_{11}^2$ is satisfied as soon as

$$k \geq \frac{n-1}{\alpha n}. \tag{30}$$

Now inequality (25) becomes:

$$(n-1) \int_{\gamma} (\varphi)_s^2 \frac{ds}{v} \geq -(n-2) \int_{\gamma} \varphi^2 (\ln v)_{ss} \frac{ds}{v} + \frac{1}{k} \int_{\gamma} \varphi^2 |\nabla \ln v|^2 \frac{ds}{v}, \tag{31}$$

where $(\cdot)_s, (\cdot)_{ss}$ denote the first and the second derivative with respect to s respectively.

In (31) we get rid of v in the denominator, by substituting $\varphi \sqrt{v}$ to φ .

After this substitution (31) is transformed into:

$$\begin{aligned} (n-1) \int_{\gamma} (\varphi)_s^2 ds &\geq -(n-2) \int_{\gamma} \varphi^2 (\ln v)_{ss} ds \\ &+ \left(\frac{1}{k} - \frac{(n-1)}{4} \right) \int_{\gamma} \varphi^2 (\ln v)_s^2 ds - (n-1) \int_{\gamma} \phi \phi_s (\ln v)_s ds. \end{aligned}$$

Replacing $u = v^k$ and integrating by parts the term $\int_{\gamma} \varphi^2 (\ln u)_{ss} ds$ yields

$$(n-1) \int_{\gamma} (\varphi)_s^2 ds \geq k \left(1 - \frac{k(n-1)}{4} \right) \int_{\gamma} \varphi^2 (\ln u)_s^2 ds - k(n-3) \int_{\gamma} \phi \phi_s (\ln u)_s ds. \tag{32}$$

The case $n = 3$ is readily solved because one of the coefficient in (32) is zero and one has

$$4 \int_{\gamma} (\varphi)_s^2 ds \geq k(2-k) \int_{\gamma} \varphi^2 (\ln u)_s^2 ds.$$

The best choice is $k = 1$ and keeping in mind (30) one should take $\alpha \geq \frac{2}{3}$. Then

$$4 \int_{\gamma} (\varphi)_s^2 ds \geq \int_{\gamma} \varphi^2 (\ln u)_s^2 ds.$$

Now consider $4 \leq n \leq 5$. Cauchy-Schwarz inequality yields for $\epsilon > 0$

$$-2 \int_{\gamma} \varphi \phi_s (\ln u)_s ds \leq \frac{1}{\epsilon} \int_{\gamma} \varphi^2 (\ln u)_s^2 ds + \epsilon \int_{\gamma} \phi_s^2 ds. \tag{33}$$

Replacing (33) in (32) one has

$$\left(n - 1 + \frac{k(n - 3)}{2\epsilon}\right) \int_{\gamma} \varphi_s^2 ds \geq \frac{k}{4}(4 - k(n - 1) - 2\epsilon(n - 3)) \int_{\gamma} \varphi^2 (\ln u)_s^2 ds. \tag{34}$$

For $n = 4$ (choosing $k = 1, \epsilon = \frac{1}{3}, \alpha \geq \frac{3}{4}$) (34) gives:

$$54 \int_{\gamma} \varphi_s^2 ds \geq \int_{\gamma} \varphi^2 (\ln u)_s^2 ds.$$

For $n = 5$ (choosing $k = \frac{9}{10}, \epsilon = \frac{1}{20}, \alpha \geq \frac{8}{9}$) (34) gives

$$49 \int_{\gamma} \varphi_s^2 ds \geq \int_{\gamma} \varphi^2 (\ln u)_s^2 ds.$$

□

Proposition 3 *Let u be a positive function defined on a curve γ of a hypersurface M in $\mathbb{R}^{n+1}, n \leq 5$ such that*

$$c(n) \int_{\gamma} (\varphi)_s^2 ds \geq \int_{\gamma} \varphi^2 (\ln u)_s^2, \tag{35}$$

for any $\varphi \in C_0^\infty(\gamma)$. Let p, q two points of γ such that $p = \gamma(1)$ and $q = \gamma(eR)$.

Then, there exist a point $\tilde{p} \in \gamma$, in a neighborhood of radius e^2 of p such that

$$u(q) \geq \frac{u(\tilde{p})}{R^{\beta(n)}}, \tag{36}$$

where $\beta(n) = \sqrt{\frac{(e-1)(4-e)c(n)}{(3-e)}}$.

Proof Let φ be the test-function defined on γ as follows:

$$\varphi(s) = \begin{cases} s/R & \forall s \in [0, R], \\ 1 & \forall s \in [R, eR] \\ \frac{s-3R}{(e-3)R} & \forall s \in [eR, 3R], \end{cases}$$

where s is the arc length of γ . Plug φ into (35):

$$\frac{c(n)(4 - e)}{(3 - e)R} \geq \int_0^{3R} \chi^2 (\ln u)_s^2 ds, \tag{37}$$

where

$$\chi(s) = (s/R)\delta_{[0,R]} + \delta_{[R,eR]} + \left(\frac{3}{(3 - e)} - \frac{s}{(3 - e)R}\right) \delta_{[eR,3R]}$$

Apply Cauchy–Schwarz inequality as follows:

$$\left| \int_R^{eR} \chi (\ln u)_s ds \right| \leq \int_R^{eR} |\chi (\ln u)_s| ds \leq \sqrt{\int_R^{eR} (\chi (\ln u)_s)^2 ds} \sqrt{(e - 1)R}.$$

Then, using (37):

$$\left| \int_R^{eR} (\ln u)_s ds \right| \leq \beta(n), \tag{38}$$

where $\beta(n) = \sqrt{\frac{(4-e)(e-1)c(n)}{(3-e)}}$. Integrate the first term of (38)

$$\left| \ln \frac{u(eR)}{u(R)} \right| \leq \beta(n). \tag{39}$$

From (39) it follows that

$$u(eR) \geq u(R)e^{-\beta(n)}. \tag{40}$$

Let $\delta = \ln(eR)$ and let $\tilde{p} = \gamma(e^{\delta-[\delta]+1})$. Iterating $[\delta] - 1$ times (40) one has:

$$u(q) \geq \frac{u(\tilde{p})}{R^{\beta(n)}}.$$

□

5 Proper minimal hypersurfaces

Theorem 2 *Let M be a minimal, nonplanar hypersurface embedded in \mathbb{R}^{n+1} , $n \leq 5$, with uniformly bounded second fundamental form.*

- (i) *If M is stable and there is an Euclidean ball $B^E(p)$ in \mathbb{R}^{n+1} centered at a point $p \in M$ such that $B^E(p) \cap M$ consists of a finite number of connected components, then there exists an embedded tube around M .*
- (ii) *If M is not stable, then either M is proper, or M is properly embedded in a open set bounded by a complete minimal stable hypersurface (possibly multi-connected).*

Proof (i) As M has uniformly bounded curvature, there exists $a > 0$ such that

$$a = \inf_{q \in M} \frac{1}{|A(q)|} > 0.$$

By Remark 1, we may assume that the subfocal tube $T(M, \frac{a}{2})$ is not embedded. Then $\exp^{-1}(M)$ consists in the zero section M_0 of the normal bundle (identified with M by the exponential diffeomorphism) together with pieces of minimal hypersurfaces that intersect a tube of radius $\frac{a}{2}$ around M_0 ($\exp^{-1}(T(M, a)) \setminus M_0 \neq \emptyset$); in particular there is a piece of M in $T(M_0, a)$, that defines graphs of positive functions defined on subdomains of M_0 . The case of one piece, i.e. when there is only one graph defined on a subdomain of M , has been treated in Sect. 4. Here, we may have an infinite number of components that enters the tube $T(M_0, a)$. In order to deal with this case, we use a purely topological argument, that goes as follows. Let R_0 be the radius of the Euclidean ball around $p \in M$ that contains a finite number of components of M . Eventually decreasing R_0 , one can assume that there is only one component. Then, there is a $R_1 > R_0$ such that $T(R_1, a) \setminus M_0$ contains a finite number of connected pieces of M , say M_{11}, \dots, M_{1k} . These pieces are graphs of positive functions u_i defined on disjoint sub-domains of $M_0 : D_{11}, \dots, D_{1k}$. We claim that the tubes $T(D_{1i}, u_{1i}), i = 1, \dots, k$, are disjoint and embedded.

Indeed, suppose that, for some k , $T(D_{1k}, u_{1k})$ is not embedded, then there is a piece N of M that is a graph over D_{1k} and lies between D_{1k} and M_{1k} up to the boundary of D_{1k} , since M is embedded. In particular N cuts $T(R_1, a) \setminus M_0$ and hence N must be one of the M_{1i} .

Then consider an increasing sequence $\{R_j\}$ going to infinity and apply the same reasoning to each $T(R_j, a)$. Finally the following tube around M :

$$\left[T(M, a) \setminus \left(\bigcup_{j,i_j} T(D_{j,i_j}, a) \right) \right] \cup \left[\bigcup_{j,i_j} T(D_{j,i_j}, u_{j,i_j}) \right].$$

is embedded.

(ii) The closure \bar{M} of M is a lamination of \mathbb{R}^{n+1} by minimal smooth embedded hypersurfaces. In the case $n = 2$, this fact is proved in [13]. The proof is analogous here and we give it, for the reader’s convenience.

As the curvature of M is uniformly bounded, then there is a positive δ such that for each $p \in M$, M is locally a graph over the disk $D_\delta(p)$ in T_pM centered at p of radius δ . Now, assume that M is not proper and let $x \in \mathbb{R}^{n+1}$ be an accumulation point of M . Let x_n a sequence of points in M converging to x . We can assume that the tangent planes $T_{x_n}M$ converge to some plane P through x . Indeed the unit normal vectors at x_n form a subset of the unit sphere; hence there is an accumulation point, and we can choose a subsequence x_n such that $T_{x_n}M$ converge to P . Since M is a graph G_n over each $D_\delta(x_n) \subset T_{x_n}M$, then, for n sufficiently large, G_n is a graph over $D_{\frac{\delta}{2}}(x) \subset P$. As G_n has bounded curvature, G_n is a graph of bounded slope of a C^2 -function ϕ_n defined on $D_{\frac{\delta}{2}}(x)$. ϕ_n satisfies the minimal graph equation which is a uniform quasi-linear elliptic partial differential equation. Hence, by classical PDE theory (cf. [8]), there is a subsequence of $\{\phi_n\}$ converging in the C^2 norm to a C^2 -function defined over $D_{\frac{\delta}{2}}(x)$ whose graph is a minimal surface with bounded curvature. In other terms, a subsequence of G_n converges in the C^2 norm to a minimal graph G_∞ over $D_{\frac{\delta}{2}}(x)$, $x \in G_\infty$. Then, we observe that G_∞ does not depend on the choice of the sequence x_n converging to x and that G_∞ is disjoint from each G_n . Thus one has a local lamination of \bar{M} near the point x .

The graph G_∞ is contained in \bar{M} , hence for any point $y \in \partial G_\infty$ there is a limit graph $G_\infty(y)$ over a disk of radius $\frac{\delta}{2}$ centered at y and by uniqueness of the limit $G_\infty = G_\infty(y)$ where they intersect. This means that G_∞ extends to a complete minimal hypersurface in \bar{M} .

Now, let L one of the limit leaf: we prove that \tilde{L} is stable.

Where \tilde{L} is the universal covering space of L . On the normal bundle over \tilde{L} we consider the flat metric given by its immersion in \mathbb{R}^{n+1} .

Let \tilde{D} be a compact simply connected domain of \tilde{L} projecting on a domain D in L . Each point of D has a neighborhood that is a uniform limit of pairwise disjoint local graphs in M . One can lift each of these local graphs in the normal bundle over \tilde{L} along the lifting paths in D . Using holonomy, one obtains that \tilde{D} is the uniform limit of pairwise disjoint embedded minimal hypersurfaces F_n in the normal bundle of \tilde{L} .

\tilde{D} is stable, because it is the uniform limit of disjoint minimal domains F_n . We prove it by contradiction. If \tilde{D} were unstable, then the first eigenvalue λ_1 of the stability operator $L = \Delta + |A|^2$ would be negative. Let f be the eigenfunction of λ_1 , then $L(f) + \lambda_1 f = 0, f > 0$ in \tilde{D} and $f|_{\partial\tilde{D}} = 0$.

Consider the variation of \tilde{D} given by

$$\mathbf{X}_t(x) = x + tf(x)\mathbf{N}(x),$$

where $x \in \tilde{D}$ and \mathbf{N} is the unit normal vector field along \tilde{D} in the normal bundle.

We have the following formula for the first variation of the mean curvature of \mathbf{X}_t at $t = 0$:

$$\dot{H}(0) = L(f) = -\lambda_1 f.$$

As $\lambda_1 < 0$, $f(x) > 0$ for $x \in \tilde{D}$, then $\dot{H}(0) > 0$, i.e. the mean curvature vector of \mathbf{X}_t for t small points away from \tilde{D} .

Now, for t small, choose n_0 large enough such that F_{n_0} intersects $\mathbf{X}_{t_0}(\tilde{D})$. Decreasing t from t_0 to 0 there is a smallest positive t such that $\mathbf{X}_t(\tilde{D})$ has a non empty intersection with F_{n_0} . At a point $x \in F_{n_0} \cap \mathbf{X}_t(\tilde{D})$, F_{n_0} lies in the mean convex side of $\mathbf{X}_t(\tilde{D})$ and this is impossible because F_{n_0} is minimal.

Denote by \mathcal{L} the union of the limit leaves. As M is not stable, $M \cap \mathcal{L} = \emptyset$ and M is proper in $\mathbb{R}^{n+1} \setminus \mathcal{L}$. \square

Remark 3 We observe that (i) implies that for any point of $q \in M$ there exists an Euclidean ball around q , $B^E(q)$, such that $B^E(q) \cap M$ consists of a finite number of connected components.

Remark 4 In (ii), if $\bar{M} = \mathbb{R}^{n+1}$ i.e. \bar{M} is a foliation of \mathbb{R}^{n+1} , then, applying the flux formula to the vector field normal to the foliation, we can easily deduce that M is area minimizing. In particular the volume of a ball of M of intrinsic radius R in M has order at most R^n (cf. [SSY]). As $n \leq 5$, then Lemma 1 implies that M is a hyperplane.

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