

# Stably embedded minimal hypersurfaces

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Received: 9 February 2004 / Accepted: 3 May 2005 /  
Published online: 26 July 2006  
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**Abstract** We use Schoen's curvature estimates to prove that the subfocal tubular neighborhood of a nonplanar minimal hypersurface with bounded second fundamental form, stably embedded in  $\mathbb{R}^{n+1}$ ,  $n < 5$ , whose radius decays sufficiently slowly cannot be embedded. In particular such hypersurfaces admit no embedded tubular neighborhoods of constant radius, whatever small the radius. However, assuming a further hypothesis on the embedding, we prove that such hypersurfaces admit an embedded tube whose radius decays sufficiently fast.

**Keywords** Minimal · Hypersurface · Embedding · Stable

**Mathematics Subject Classification (2000)** (53 A 10)

## 1 Introduction

While a complete stable minimal surface in  $\mathbb{R}^3$  is known to be a plane [5], there are examples of stable nonplanar minimal hypersurfaces embedded in  $\mathbb{R}^{n+1}$  for  $n \geq 7$  [1]. Whether the only minimal, stable, complete hypersurfaces in  $\mathbb{R}^{n+1}$ ,  $3 \leq n \leq 6$ , are hyperplanes is an open problem. Exploring it yields new results about embedded stable minimal hypersurfaces. Let us recall some of them.

In [6] it is proved that a minimal, stable, complete hypersurface is a hyperplane as soon as

$$\lim_{R \rightarrow \infty} \frac{\int_{B_R} |K|}{R^{2+2q}} = 0, \quad q < \sqrt{2/n}.$$

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In [2], it is proved that a minimal, stable, complete hypersurface must have only one end. This result has been extended recently to any ambient space with positive sectional curvature [11]. Chen [3] has shown that if the number of connected components of the intersection between a minimal, stable, complete hypersurface  $M$  and any ball of  $\mathbb{R}^{n+1}$ ,  $3 \leq n \leq 4$ , is bounded by some constant, then  $M$  is a hyperplane. D. Zhou and X. Chen proved that, if the  $L^p$  norm of the second fundamental form of a minimal, stable, complete hypersurface  $M$  in  $\mathbb{R}^{n+1}$ ,  $n \leq 4, p \geq n$ , is bounded, then  $M$  is a hyperplane (Personal communication).

In the following we always assume  $n \geq 3$ . The problem then is to know whether there are non-flat minimal, stable, complete minimal hypersurfaces in  $\mathbb{R}^{n+1}$  for  $n \leq 5$  (as it is the case when  $n = 2$ ). A strong evidence for this non-existence result is given by the following Theorem.

**Theorem 1** *Let  $M$  be a nonplanar, stable, minimal hypersurface in  $\mathbb{R}^{n+1}$ ,  $n \leq 5$ , with bounded second fundamental form. Fix a point  $\sigma$  in  $M$  and denote by  $d(p, \sigma)$  the intrinsic distance between  $\sigma$  and any point  $p \in M$ . Let  $0 < c_1 \leq 1$ ,  $c_2 > 0$ ,  $\delta \geq 1$  and consider any function on  $M$  such that  $r(p) \geq \inf\{c_1|A(p)|^{-1}, c_2d(p, \sigma)^\delta\}$ . Then the tube  $T(M, r(p))$  is not embedded.*

Theorem 1 means that the subfocal tubular neighborhood of a nonplanar minimal, stable hypersurface embedded in  $\mathbb{R}^{n+1}$ ,  $n \leq 5$ , cannot be embedded. More precisely: such hypersurfaces admit no embedded tubular neighborhoods of constant radius, whatever small the radius. However, the following Theorem shows that assuming a further hypothesis on the embedding, such hypersurfaces admit an embedded tube whose radius decays sufficiently fast.

**Theorem 2** *Let  $M$  be a minimal, nonplanar hypersurface embedded in  $\mathbb{R}^{n+1}$ ,  $n \leq 5$ , with uniformly bounded second fundamental form.*

- (i) *If  $M$  is stable and there is an Euclidean ball  $B^E(p)$  in  $\mathbb{R}^{n+1}$  centered at a point  $p \in M$  such that  $B^E(p) \cap M$  consists of a finite number of connected components, then there exists an embedded tube around  $M$ .*
- (ii) *If  $M$  is not stable, then either  $M$  is proper, or  $M$  is properly embedded in a open set bounded by a complete minimal stable hypersurface (possibly multi-connected).*

The paper is organized as follows.

In Sect. 2 we prove Theorem 1. From Theorem 1 we know that there many tubes of a non flat minimal hypersurface  $M$  that intersect other pieces of  $M$ . These pieces define graphs in the normal bundle to  $M$ .

In Sect. 3, we study such graphs. More precisely we compute the equation that the distance function  $\phi$  from points of a minimal piece of a minimal hypersurface to another minimal piece, satisfies. When  $M$  has bounded curvature,  $\phi$  satisfies a linear elliptic inequality, namely  $F(\phi) \leq 0$ , where  $F$  is a linear elliptic operator.

In Sect. 4, we give a lower estimate for positive eigenfunctions of  $F(\phi) \leq 0$ . We use this fact to show that, roughly speaking, the distance function  $\phi$  is bounded from below at a point  $q \in M$  in terms of the value of  $\phi$  at a point  $p$  at distance  $R$  from  $q$  and of the distance  $R$  (the method used in this section is similar to [17]). This gives a lower estimate for the (non-constant) radius of tubes that separate the pieces. This result gives a measure of the radius of a tube that separate  $M$  (when considering it as the zero section of its normal bundle) from any other piece of  $M$ . Our result does not

prevent from having an infinite number of connected components of  $M$  cutting the tube around  $M$ .

In Sect. 5 we deal with the infinite number of components case (see Theorem 2). The proof Theorem 2 is topological in nature and it is proved without using the estimate on the radius of the tube found in Sect. 4.

## 2 Volume of tubes

We will give a geometric application of an estimate found in [15]. Let  $M$  be a hypersurface in  $\mathbb{R}^{n+1}$  with an orientation  $\mathbf{N}$  and let  $r : M \rightarrow \mathbb{R}$  be a smooth positive function. Fix a point  $\sigma$  of  $M$  and let  $B_R$  be the ball of  $M$  centered at  $\sigma$  of radius  $R$  in the metric induced by the immersion. Denote by  $|B_R|$  the volume of such a ball. Consider a bundle  $\pi : U \rightarrow M$  with  $U$  an open neighborhood of  $M$  in  $\mathbb{R}^{n+1}$ , for which the zero section  $s : M \rightarrow U$  is the inclusion of  $M$  in  $U$ . Such a bundle is called a *Tubular neighborhood* of  $M$  in  $\mathbb{R}^{n+1}$  and we denote  $s(M) = M_0$ . It is clear that one can choose a tubular neighborhood that is locally equivalent to a domain of the normal bundle; hence the exponential map  $\exp : NM \rightarrow \mathbb{R}^{n+1}$  is a local diffeomorphism between  $U$  and a neighborhood of the zero section of the normal bundle. We will often write our definitions up to the local diffeomorphism  $\exp$ .

**Definition 1** We call *Tube of radius  $r$  around  $M$*  the set

$$T(M, r) = \{x \in \mathbb{R}^{n+1} \mid \exists p \in M, x = p + t\mathbf{N}(p), t \leq r(p)\}.$$

Let  $A$  be the second fundamental form of  $M$ . For simplicity, we assume that at any point of  $M$ ,  $|A| \neq 0$ .

**Definition 2** Assume that  $M$  has bounded second fundamental form. The tube

$$T(M, c|A|^{-1})$$

is called a *Subfocal Tube* for any  $c \leq 1$ .

Denote by  $T(R, r)$  the tube of radius  $r$  about  $B_R \subset M$ , by  $dT$  its volume element and let

$$V(R, r) = \int_{T(R,r)} dT.$$

The following Proposition is a weaker form of Theorem 1 and contains almost all the ideas of the proof

**Proposition 1** Let  $M$  be a nonplanar stable minimal hypersurface in  $\mathbb{R}^{n+1}, n \leq 5$ , with bounded second fundamental form. Then, for  $R$  sufficiently large, the tube  $T(R, r), r(p) = c|A(p)|^{-1}, c \leq 1$ , is not embedded. In particular the subfocal Tube  $T(M, c|A(p)|^{-1})$  is not embedded.

In order to prove Proposition 1, we need some preliminaries. Let

$$M_t = \{x \in \mathbb{R}^{n+1} \mid \exists p \in M, x = p + t\mathbf{N}(p)\}.$$

Note that

$$M_t = \{x \in \mathbb{R}^{n+1} \mid d(x, M) = t\}$$

and that  $dT = dM_t dt$ , where  $dM_t$  is the volume element of  $M_t$ . It is not hard to see that

$$dM_t = \det(1 - tA)dM,$$

where  $dM$  is the volume element of  $M$  (cf. [9] for details). By a straightforward computation we have:

$$\det(1 - tA) = \prod_{h=1}^n (1 - t\kappa_h) = 1 + \sum_{h=1}^n (-1)^h t^h S_h,$$

where  $\kappa_h$  are the principal curvatures of  $M$  and  $S_h$  are the elementary symmetric functions of  $\kappa_1, \dots, \kappa_n$  and  $S_0 = 1$ . For  $r(p) \leq c|A(p)|^{-1}$ ,  $c \leq 1$ , one has:

$$\begin{aligned} V(R, r) &= \int_{T(R,r)} dM_t dt = \int_{T(R,r)} \left( 1 + \sum_{h=1}^n (-1)^h t^h S_h \right) dM dt \\ &= \int_{B_R} dM \int_0^{r(p)} \left( 1 + \sum_{h=1}^n (-1)^h t^h S_h \right) dt \end{aligned}$$

From now on, we assume that the hypersurface  $M$  is minimal, hence  $S_1 = 0$  and

$$V(R, r) = \int_{B_R} r(p) dM + \sum_{h=2}^n \frac{(-1)^h}{h+1} \int_{B_R} r(p)^{h+1} S_h dM.$$

We recall a fundamental result of [15] that provides estimates on the  $L^p$  norm of the second fundamental form of  $M$ .

**SSY-Theorem** For any  $n \in \mathbb{N}$  and  $p \in \left[0, 4 + \sqrt{\frac{8}{n}}\right]$ , there exists a constant  $\beta(p, n)$  satisfying the following condition: if  $M$  is a stable minimal hypersurface of  $\mathbb{R}^{n+1}$  then, for any  $R > 0$

$$\int_{B_R} |A|^p \leq \beta(p, n) R^{-p} |B_R|. \tag{1}$$

Using [SSY]-Theorem, we can easily prove the following Lemma.

**Lemma 1** Let  $n \leq 7$ . There exists a constant  $\gamma(n)$  satisfying the following conditions:

- (i) if  $M$  is a nonplanar stable minimal hypersurface in  $\mathbb{R}^{n+1}$ ,  $0 < \varepsilon < -\frac{1}{2} + \sqrt{\frac{2}{n}}$  and  $R$  is sufficiently large, then

$$|B_R| > \gamma(n) R^{5+\varepsilon} \tag{2}$$

- (ii) if  $M$  is a nonplanar stable minimal hypersurface in  $\mathbb{R}^{n+1}$ , for each  $\alpha > 1$ , and  $\varepsilon$  as above, there exists a sufficiently large  $\tilde{R} > R$  such that

$$|B_{\tilde{R}}| - |B_{\alpha^{-1}\tilde{R}}| > \gamma(n) R^{5+\frac{\varepsilon}{2}}. \tag{3}$$

*Proof* (i) As  $n \leq 7$ , in (1) we can choose  $p = 5 + 2\varepsilon$  and we have

$$\int_{B_R} |A|^{5+2\varepsilon} \leq \beta(5 + 2\varepsilon, n)R^{-5-2\varepsilon}|B_R|$$

provided  $\varepsilon < -\frac{1}{2} + \sqrt{\frac{2}{n}}$ . Suppose, by contradiction, that for any positive constant  $\gamma(n)$  there exists some  $R$  arbitrarily large such that

$$|B_R| \leq \gamma(n)R^{5+\varepsilon}$$

It follows that

$$\int_{B_R} |A|^{5+2\varepsilon} \leq \beta(5 + 2\varepsilon, n)\gamma(n)R^{-5-2\varepsilon}R^{5+\varepsilon} = \frac{\beta(5 + 2\varepsilon, n)\gamma(n)}{R^\varepsilon}.$$

Taking the limit  $R \rightarrow \infty$  we deduce that  $|A| \equiv 0$  on  $M$ , hence  $M$  is a hyperplane and this is a contradiction.

(ii) Let  $R > 1$  be such that  $|B_R| > \gamma(n)R^{5+\varepsilon}$  with  $\varepsilon$  as in (i). Let  $\alpha > 1$  and consider the following sequence:

$$R_1 = R, R_2 = \alpha R_1, R_3 = \alpha^2 R_1, \dots, R_k = \alpha^{k-1} R, \dots$$

One can write

$$|B_{R_k}| = (|B_{R_k}| - |B_{R_{k-1}}|) + (|B_{R_{k-1}}| - |B_{R_{k-2}}|) + \dots + (|B_{R_2}| - |B_{R_1}|) + |B_{R_1}|.$$

As  $R_k \geq R$ , one has  $|B_{R_k}| > \gamma(n)R_k^{5+\varepsilon}$ , hence there exists  $1 \leq j_k \leq k$  such that

$$|B_{R_{j_k}}| - |B_{R_{j_k-1}}| > \frac{\gamma(n)R_k^{5+\varepsilon}}{k}. \tag{4}$$

We notice that it cannot happen that

$$|B_{R_1}| > \frac{\gamma(n)R_k^{5+\varepsilon}}{k} = \frac{\gamma(n)(\alpha^{k-1}R_1)^{5+\varepsilon}}{k}$$

because  $R_1$  is fixed and the right hand side of the previous inequality tends to infinity as  $k \rightarrow \infty$ .

For  $k$  sufficiently large, one can write (4) in the following way:

$$\begin{aligned} |B_{R_{j_k}}| - |B_{R_{j_k-1}}| &> \gamma(n)(\alpha^{j_k-1}R)^{5+\frac{\varepsilon}{2}}(\alpha^{k-j_k})^{5+\frac{\varepsilon}{2}}\frac{(\alpha^{k-1})^{\frac{\varepsilon}{2}}}{k}R^{\frac{\varepsilon}{2}} \\ &\geq \gamma(n)(R_{j_k})^{5+\frac{\varepsilon}{2}}. \end{aligned}$$

Denoting  $R_{j_k}$  with  $\tilde{R}$  one obtains (3). □

*Proof of Proposition 1* For  $h = 0, \dots, n$ , let

$$V_h(R, r) = \frac{(-1)^h}{h+1} \int_{B_R} \frac{r^{h+1}}{|A|^{h+1}} S_h dM$$

be the term of order  $h$  of the volume of the tube  $T(R, r)$ . Denote by  $S_{h,i}$  the  $h$ -th elementary symmetric function of the  $n - 1$  variables  $\kappa_j, j \in \{1, \dots, n\} \setminus i$ . One has (cf. [12]):

$$hS_h = \sum_{i=1}^n S_{h-1,i}\kappa_i.$$

Using previous equality, we obtain easily:

$$S_2 = -\frac{|A|^2}{2}, \quad S_3 = \frac{1}{3} \sum_{i=1}^n \kappa_i^3,$$

$$S_4 = \frac{1}{8}|A|^4 - \frac{1}{4} \sum_{i=1}^n \kappa_i^4, \quad S_5 = -\frac{1}{6}|A|^2 \sum_{i=1}^n \kappa_i^3 + \frac{1}{5} \sum_{i=1}^n \kappa_i^5.$$

As  $trA = \sum_{i=1}^n \kappa_i = 0$ , hence for any  $j, |\kappa_j| \leq \sum_{i \neq j, i=1}^n |\kappa_i|$ . From Cauchy–Schwarz formula:

$$\kappa_j^2 \leq (n - 1) \sum_{i \neq j, i=1}^n |\kappa_i|^2.$$

A fortiori:

$$\kappa_j^2 \leq \frac{n - 1}{n} |A|^2.$$

Then, it is not hard to see that for every  $0 \leq h \leq n \leq 5$ , there exist a positive constant  $c(h, n)$  such that

$$(-1)^{h+1} S_h \leq c(h, n) |A|^h, \tag{5}$$

where we can choose  $c(0, n) = 1, c(1, n) = 0, c(2, n) = \frac{1}{2}$  and  $0 < \sum_{h=3}^n \frac{c(h, n)}{h+1} < \frac{5}{6}$ .

Then

$$V_0(R, r) = \int_{B_R} \frac{1}{|A|} dM, \quad V_1(R, r) = 0, \quad V_2(R, r) = -\frac{1}{6} \int_{B_R} \frac{1}{|A|} dM.$$

Hence

$$V(R, r) \geq \sum_{h=0}^n \int_{B_R} \frac{c(h, n)}{|A|} dM = \left( \frac{5}{6} - \sum_{h=3}^n c(h, n) \right) \int_{B_R} \frac{1}{|A|} dM. \tag{6}$$

Let us estimate  $\int_{B_R} \frac{1}{|A|} dM$  from below. By Cauchy–Schwarz inequality:

$$|B_R| \leq \left( \int_{B_R} \frac{1}{|A|} \right)^{\frac{1}{2}} \left( \int_{B_R} |A| \right)^{\frac{1}{2}} \leq \beta(1, n)^{\frac{1}{2}} \left( \int_{B_R} \frac{1}{|A|} \right)^{\frac{1}{2}} |B_R|^{\frac{1}{2}} R^{-\frac{1}{2}},$$

where last inequality follows from (1), with  $p = 1$ . Hence

$$\int_{B_R} \frac{1}{|A|} \geq \beta (1, n)^{-1} |B_R| R.$$

Replacing in (6) one has, for  $R$  sufficiently large

$$V(R, r) \geq \left( \frac{5}{6} - \sum_{h=3}^n c(h, n) \right) |B_R| \beta (1, n)^{-1} R \geq C(n) R^{6+\epsilon},$$

where  $C(n) = \gamma(n) \beta (1, n)^{-1} \left( \frac{5}{6} - \sum_{h=3}^n c(h, n) \right)$  and last inequalities is by (i) of Lemma 1.

Let us prove that the inequality

$$V(R, r) \geq C(n) R^{6+\epsilon} \tag{7}$$

implies that the tube  $T(R, r)$  is not embedded.

Denote by  $B^E(\sigma, R)$  the Euclidean ball in  $\mathbb{R}^{n+1}$  of radius  $R$ , centered at a point  $\sigma \in M$ . As for any point  $p \in M$  the Euclidean distance between  $\sigma$  and  $p$  is less or equal to the intrinsic distance, one has

$$B_R \subset B^E(\sigma, R).$$

Hence, letting  $a = \inf_{B_R} \frac{1}{|A|}$  one has

$$T(R, r) \subset B^E(\sigma, R + 2a).$$

Assume by contradiction that the tube  $T(R, r)$  is embedded, then

$$V(R, r) \leq \text{vol}(B^E(\sigma, R + 2a)) \leq \omega_{n+1} (R + 2a)^{n+1},$$

where  $\omega_{n+1}$  is the volume of the unit ball in  $\mathbb{R}^{n+1}$ . This is in contradiction with (7) for  $R \rightarrow \infty$ , as  $n \leq 5$ .

If  $M$  is contained in a Euclidean ball of finite radius  $R$ , centered at  $\sigma$  i.e.  $M$  is not properly embedded, then  $T(M, r)$  would be contained in  $B^E(\sigma, R + a)$ . So, if  $T(M, r)$  would be embedded, it should have finite volume, and this is again a contradiction by (7). □

**Theorem 1** *Let  $M$  be a nonplanar stable, minimal hypersurface in  $\mathbb{R}^{n+1}$ ,  $n \leq 5$ , with bounded second fundamental form. Fix a point  $\sigma$  in  $M$  and denote by  $d(p, \sigma)$  the intrinsic distance between  $\sigma$  and any point  $p \in M$ . Let  $0 < c_1 \leq 1, c_2 > 0, \delta \geq 1$  and consider any function on  $M$  such that  $r(p) \geq \inf\{c_1|A(p)|^{-1}, c_2d(p, \sigma)^\delta\}$ . Then the tube  $T(M, r(p))$  is not embedded.*

*Proof* In order to simplify computation, we show the details of the proof for  $n = 3$  and  $c_1 = c_2 = 1$ . In this case, one has

$$V(R, r) = \int_{B_R} r(p) dM + \frac{1}{3} \int_{B_R} r(p)^3 S_2 - \frac{1}{4} \int_{B_R} r(p)^4 S_3 dM.$$

Denote by  $B_R^+$  the subset of  $B_R$  where  $r(p) = |A(p)|^{-1}$  and let  $B_R^- = B_R \setminus B_R^+$ . Then

$$\begin{aligned}
 V(R, r) = & \int_{B_R^+} |A(p)|^{-1} dM - \frac{1}{6} \int_{B_R^+} |A(p)|^{-3} S_2 dM - \frac{1}{4} \int_{B_R^+} |A(p)|^{-4} S_3 dM \\
 & + \int_{B_R^-} d(p, \sigma)^\delta dM - \frac{1}{6} \int_{B_R^-} d(p, \sigma)^{3\delta} S_2 dM - \frac{1}{4} \int_{B_R^-} d(p, \sigma)^{4\delta} S_3 dM.
 \end{aligned}$$

For the two integrals over  $B_R^+$  we proceed as in the proof of Proposition 1. Let us estimate the integrals over  $B_R^-$  (actually we only write the job for  $S_3$ , as it is analogous for  $S_2$ ). Inequality (5) implies that

$$- \int_{B_R^-} d(p, \sigma)^{4\delta} S_3 dM \geq -c(3, 3) \int_{B_R^-} d(p, \sigma)^\delta dM.$$

Hence:

$$\begin{aligned}
 & \int_{B_R^-} d(p, \sigma)^\delta dM - \frac{1}{4} \int_{B_R^-} d(p, \sigma)^{4\delta} S_3 dM \\
 & \geq \left(1 - \frac{c(3, 3)}{4}\right) \int_{B_R^-} d(p, \sigma)^\delta dM
 \end{aligned}$$

On the other side

$$\begin{aligned}
 \int_{B_R^-} d(p, \sigma)^\delta dM &= \int_{B_R^- \setminus B_{\alpha^{-1}R}^-} d(p, \sigma)^\delta dM + \int_{B_{\alpha^{-1}R}^-} d(p, \sigma)^\delta dM \\
 &\geq \int_{B_R^- \setminus B_{\alpha^{-1}R}^-} d(p, \sigma)^\delta dM \geq \left(\frac{R}{\alpha}\right)^\delta (|B_R^-| - |B_{\alpha^{-1}R}^-|).
 \end{aligned}$$

As in the proof of Proposition 1 and using the previous inequality, one has

$$\begin{aligned}
 V(R, r) \geq & \left(\frac{5}{6} - c(3, 3)\right) (|B_R^+| - |B_{\alpha^{-1}R}^+| + |B_{\alpha^{-1}R}^+|) R \\
 & + \left(1 - \frac{c(3, 3)}{4}\right) \left(\frac{R}{\alpha}\right)^\delta (|B_R^-| - |B_{\alpha^{-1}R}^-|).
 \end{aligned}$$

So, choosing  $C = \min\left(\frac{5}{6} - c(3, 3), 1 - \frac{c(3, 3)}{4}\right)$ , one has

$$V(R, r) \geq \min\left(C, \frac{1}{\alpha^\delta}\right) (|B_R| - |B_{\alpha^{-1}R}|) R^{\min(\delta, 1)}.$$

By (ii) of Lemma 1, there exists  $\tilde{R} > R$  such that



$$\begin{aligned}
 V(\tilde{R}, r) &\geq \min\left(C, \frac{1}{\alpha^\delta}\right) \tilde{R}^{\min(\delta, 1)} \gamma(n) \tilde{R}^{5+\frac{\epsilon}{2}} \\
 V(\tilde{R}, r) &\geq \gamma(n) \min\left(C, \frac{1}{\alpha^\delta}\right) \tilde{R}^{6+\frac{\epsilon}{2}}
 \end{aligned}$$

as  $\delta \geq 1$

Hence the tube  $T(\tilde{R}, r)$  is not embedded. In particular the tube  $T(M, r)$  is not embedded. □

We observe that for  $n = 3, 4$  the hypothesis  $\delta > 0$  is enough to obtain the result of Theorem 1.

*Remark 1* Let  $a = \inf_M \frac{1}{|A|} > 0$ , then, in the hypotheses of Theorem 1, the tube  $T(M, \frac{a}{2})$  is not embedded. So, the inverse image restricted to  $T(M, a)$  of  $M \subset M_0$  by the exponential map

$$\exp : NM \longrightarrow \mathbb{R}^{n+1}$$

is not empty and  $T(M, a)$  contains pieces of  $M$  that are graphs over a piece of the zero section  $M_0$ .

### 3 Minimal distance equation

#### 3.1 The minimal distance equation in the general case

Let  $M$  and  $M'$  be two disjoint minimal hypersurfaces in  $\mathbb{R}^{n+1}$ . Fix a point  $q$  in  $M'$  and let  $\phi(q)$  be the Euclidean distance from  $q$  to  $M$ . Let  $p$  be a point of  $M$  such that  $\phi(q) = d(p, q)$  and let  $U$  be a neighborhood of  $p$  in  $M$ . Denote by  $X : U \rightarrow \mathbb{R}^{n+1}$  the position vector of  $M$  and by  $N$  a unit normal vector field to  $M$ . We define the position vector of  $M'$  by  $\tilde{X} = X + \phi N$ ,  $\tilde{X} : U \rightarrow \mathbb{R}^{n+1}$ . Let  $\tilde{N}$  be the unit normal vector field to  $M'$ . Let  $\mathcal{B} = \{e_1, \dots, e_n\}$  be a principal orthonormal frame at  $p$  ie such that  $\nabla_{e_i} e_j = \delta_{ij} \lambda_i N$ , where  $\lambda_i, i = 1, \dots, n$  are the principal curvatures at  $p$ . Extend the basis  $\mathcal{B}$  in the neighborhood of  $p$  by parallel transport. Finally denote by  $\nabla$  ( $\tilde{\nabla}$ ) the connection on  $M$  ( $M'$  respectively).

**Lemma 2** *The following equalities hold:*

$$\nabla_{e_i} \nabla_{e_i} X(p) = -\lambda_i N(p), \quad \nabla_{e_i} \nabla_{e_i} N(p) = \sum_{j=1}^n e_j(\lambda_i) e_j - \lambda_i^2 N(p). \tag{8}$$

*Proof* The former formula is straightforward. Let us prove the latter.

$$\nabla_{e_i} \nabla_{e_i} N = \nabla_{e_i} \left( \sum_{j=1}^n \langle \nabla_{e_i} N, e_j \rangle e_j \right) = \sum_{j=1}^n (e_i(\langle \nabla_{e_i} N, e_j \rangle) e_j + \lambda_i \nabla_{e_i} e_i).$$

From Codazzi equation for all  $i, j = 1, \dots, d$ :

$$e_i(\langle \nabla_{e_i} N, e_j \rangle) = e_j(\langle \nabla_{e_i} N, e_i \rangle).$$

Hence

$$\nabla_{e_i} \nabla_{e_i} N = \sum_{j=1}^n e_j (\langle \nabla_{e_i} N, e_i \rangle) e_j + \lambda_i \nabla_{e_i} e_i = \sum_{j=1}^n e_j (\lambda_i) e_j - \lambda_i^2 N.$$

□

The vector fields  $\{X, \bar{X}, N, \bar{N}\}$  are defined on  $U$ . However we can extend them (as well as any vector field  $Y$  defined on  $M$ ) to a neighborhood of  $M$  by parallel transport in  $\mathbb{R}^{n+1}$ . That is:

$$\tilde{Y}(\exp_p(sN(p))) = Y(p), \quad s \in \mathbb{R}.$$

In particular we can compute the covariant derivative  $\nabla$  of the extended vector field at the point  $q \in M'$  at distance  $\phi(q) = s$  from the point  $p$  ( $q = \exp_p(sN)$ ):

$$\nabla_{e_i} \tilde{Y}(q) = \frac{1}{(1 + \lambda_i s)} \nabla_{e_i} Y(p), \quad \partial_N \tilde{Y} = 0. \tag{9}$$

As  $M'$  is minimal, one has

$$0 = \bar{\Delta}(\bar{X}) = \bar{\Delta}(X + \phi N) = \bar{\Delta}X + (\bar{\Delta}\phi)N + \phi \bar{\Delta}N + 2\bar{\nabla}_{\bar{\nabla}\phi} N,$$

and taking the scalar product with  $N$  in  $\mathbb{R}^{n+1}$

$$\bar{\Delta}\phi + 2 \langle \bar{\nabla}_{\bar{\nabla}\phi} N, N \rangle + \langle \bar{\Delta}N, N \rangle + \phi \langle \bar{\Delta}X, N \rangle = 0. \tag{10}$$

Keeping (9) in mind, we will develop (10).

**Lemma 3** *Let  $M$  be a minimal hypersurface in  $\mathbb{R}^{n+1}$  and let  $N$  be a unit normal vector field of  $M$ . Then, for any  $C^2$  function  $f : M \rightarrow \mathbb{R}$  at any point  $p \in M$ ,*

$$\Delta_{\mathbb{R}^{n+1}} f(p) = \Delta_M f(p) + \partial_{NN}^2 f(p)$$

at any point  $p \in M$ .

*Proof* Let  $\{e_1, \dots, e_n\}$  be an orthonormal basis of  $T_p M$ , then  $\{e_1, \dots, e_n, N\}$  is a basis of  $T_p \mathbb{R}^{n+1}$ . Extend such basis to a neighborhood of  $p$  in  $M$  by parallel transport. Let  $X$  be a vector field in  $M$  and denote

$$X^T = \sum_{i=1}^n \langle X, e_i \rangle e_i, \quad X^\perp = X - X^T.$$

$$\operatorname{div}_{\mathbb{R}^{n+1}} X = \sum_{i=1}^n \langle \nabla_{e_i} X, e_i \rangle + \langle \nabla_N X, N \rangle. \tag{11}$$

As  $M$  is minimal

$$\operatorname{div}_M X = \operatorname{div}_M X^T.$$

Furthermore by a straightforward computation

$$\langle \nabla_N X, N \rangle = \nabla_N \langle X, N \rangle.$$

Replacing in (11) one has

$$\operatorname{div}_{\mathbb{R}^{n+1}} X = \operatorname{div}_M X^T + \nabla_N \langle X, N \rangle.$$

Now let  $X = \nabla f$  and replace in the previous equation:

$$\begin{aligned} \Delta_{\mathbb{R}^{n+1}}f(p) &= \operatorname{div}_{\mathbb{R}^{n+1}} \nabla f(p) \\ &= \operatorname{div}_M \nabla_M f(p) + \nabla_N \langle \nabla f(p), N \rangle \\ &= \Delta f(p) + \partial_N \partial_N f(p). \end{aligned}$$

□

(i) *Computation of  $\langle \bar{\Delta} \tilde{X}, N \rangle$ .* From Lemma 3:

$$\langle \bar{\Delta} \tilde{X}, N \rangle = \langle \Delta_{\mathbb{R}^{n+1}} \tilde{X}, N \rangle - \langle \partial_{\bar{N}} \partial_{\bar{N}} \tilde{X}, N \rangle.$$

We first compute  $\Delta_{\mathbb{R}^{n+1}} \tilde{X}$ . For  $i = 1, \dots, n$ , using (9):

$$\begin{aligned} \nabla_{e_i} \nabla_{e_i} \tilde{X} &= \frac{1}{(1 + \lambda_i \phi)} \nabla_{e_i} (\widetilde{\nabla_{e_i} X}) + \nabla_{e_i} \left( \frac{1}{1 + \lambda_i \phi} \right) \nabla_{e_i} X \\ &= \left( \frac{1}{1 + \lambda_i \phi} \right)^2 \nabla_{e_i} \nabla_{e_i} X - \frac{\lambda_i \nabla_{e_i} \phi + \phi \nabla_{e_i} \lambda_i}{(1 + \lambda_i \phi)^2} e_i. \end{aligned}$$

Then, from the first equality of (8):

$$\langle \Delta_{\mathbb{R}^{n+1}} X, N \rangle = - \sum_{i=1}^n \frac{\lambda_i}{(1 + \lambda_i \phi)^2}. \tag{12}$$

We now compute  $\partial_{\bar{N}} \partial_{\bar{N}} \tilde{X}$

$$\partial_{\bar{N}} \tilde{X} = \partial_{\sum_{i=1}^n \langle \bar{N}, e_i \rangle e_i} \tilde{X} = \sum_{i=1}^n \frac{1}{1 + \lambda_i \phi} \langle \bar{N}, e_i \rangle e_i.$$

Hence:

$$\langle \partial_{\bar{N}} \tilde{X}, N \rangle = 0,$$

and consequently

$$\langle \partial_{\bar{N}} \partial_{\bar{N}} \tilde{X}, N \rangle = - \langle \partial_{\bar{N}} \tilde{X}, \partial_{\bar{N}} N \rangle.$$

It remains to compute  $\partial_{\bar{N}} \tilde{N}$ . As before:

$$\partial_{\bar{N}} \tilde{N} = \sum_{i=1}^n \frac{1}{1 + \lambda_i \phi} \langle \bar{N}, e_i \rangle \nabla_{e_i} N = \sum_{i=1}^n \frac{\lambda_i}{1 + \lambda_i \phi} \langle \bar{N}, e_i \rangle e_i.$$

So:

$$\langle \partial_{\bar{N}} \tilde{X}, \partial_{\bar{N}} \tilde{N} \rangle = \sum_{i=1}^n \frac{\lambda_i}{(1 + \lambda_i \phi)^2} \langle \bar{N}, e_i \rangle^2. \tag{13}$$

Finally, from (12) and (13)

$$\langle \bar{\Delta} \tilde{X}, N \rangle = - \sum_{i=1}^n \frac{\lambda_i}{(1 + \lambda_i \phi)^2} + \sum_{i=1}^n \frac{\lambda_i}{(1 + \lambda_i \phi)^2} \langle \bar{N}, e_i \rangle^2. \tag{14}$$

(ii) *Computation of  $\langle \bar{\Delta}\tilde{N}, N \rangle$ .* By (9) one has:

$$\nabla_{e_i}\tilde{N} = \frac{1}{1 + \lambda_i\phi} \nabla_{e_i}N,$$

hence:

$$\begin{aligned} \Delta_{\mathbb{R}^{n+1}}\tilde{N} &= \nabla_{e_i}(\nabla_{e_i}\tilde{N}) = \left(\frac{1}{1 + \lambda_i\phi}\right)^2 \nabla_{e_i}\nabla_{e_i}N + \nabla_{e_i}\left(\frac{1}{1 + \lambda_i\phi}\right) \nabla_{e_i}N \\ &= \left(\frac{1}{1 + \lambda_i\phi}\right)^2 \nabla_{e_i}\nabla_{e_i}N - \frac{(\lambda_i\nabla_{e_i}\phi + \phi\nabla_{e_i}\lambda_i)\lambda_i}{(1 + \lambda_i\phi)^2} e_i. \end{aligned}$$

Then, using the second equality of (8):

$$\langle \Delta_{\mathbb{R}^{n+1}}\tilde{N}, N \rangle = - \sum \frac{\lambda_i^2}{(1 + \lambda_i\phi)^2}.$$

Furthermore:

$$\langle \partial_{\tilde{N}}\partial_{\tilde{N}}\tilde{N}, N \rangle = - \sum \frac{\lambda_i^2 \langle \tilde{N}, e_i \rangle^2}{(1 + \lambda_i\phi)^2},$$

hence, using Lemma 3:

$$\langle \bar{\Delta}\tilde{N}, N \rangle = \sum \frac{\lambda_i^2}{(1 + \lambda_i\phi)^2} (\langle \tilde{N}, e_i \rangle^2 - 1). \tag{15}$$

(iii) *Computation of  $\langle \bar{\nabla}_{\bar{\nabla}\phi}N, N \rangle$ .* Denote by  $(\cdot)^{\bar{T}}$  the projection on  $T_qM'$  :

$$\bar{\nabla}_{\bar{\nabla}\phi}\tilde{N} = (\nabla_{\bar{\nabla}\phi}\tilde{N})^{\bar{T}} = \sum_{i=1}^n \langle \bar{\nabla}\phi, e_i \rangle \frac{\lambda_i}{1 + \lambda_i\phi} e_i^{\bar{T}}.$$

Hence

$$\langle \bar{\nabla}_{\bar{\nabla}\phi}N, N \rangle = \sum_{i=1}^n \frac{\lambda_i}{1 + \lambda_i\phi} \langle e_i^{\bar{T}}, N \rangle \langle \bar{\nabla}\phi, e_i \rangle. \tag{16}$$

We plug (14), (15), (16) into (10) and we obtain the *minimal surface equation*:

$$\bar{\Delta}\phi + \sum_{i=1}^n \frac{\lambda_i}{1 + \lambda_i\phi} a_i = 0, \tag{17}$$

where  $a_i = 2 \langle \bar{\nabla}\phi, e_i \rangle \langle e_i^{\bar{T}}, N \rangle + \langle \tilde{N}, e_i \rangle^2 - 1$ .

*Remark 2* The coefficients  $a_i$  satisfy  $-1 \leq a_i \leq 1, i = 1, \dots, n$ . In fact, for  $i = 1, \dots, n$ , let  $\alpha_i, \beta_i \in [0, \pi[$  such that  $\cos \alpha = \langle \tilde{N}, e_i \rangle, \cos \beta = \langle N, \tilde{N} \rangle$ . By a straightforward computation

$$a_i = \cos 2\alpha_i + \cos^2 \alpha_i \cos 2\beta_i.$$

We then check that

$$-1 \leq a_i \leq 1.$$

### 3.2 The minimal distance equation in the bounded geometry case

We want to prove that when  $M$  and  $M'$  have uniformly bounded second fundamental form, then at points where the distance  $\phi$  is small, the zero order term of (17) has positive coefficient and the first order terms of equation (17) is uniformly bounded.

First we write the  $a_i, i = 1, \dots, n$ , in terms of  $\bar{\nabla}\phi$ .

One has:

$$\bar{N} = \sum_{i=1}^n (\nabla_{e_i}\phi)e_i - \frac{1}{\sqrt{1+|\nabla\phi|^2}}N.$$

Thus:

$$\begin{cases} \langle \bar{N}, e_i \rangle = \frac{\langle \nabla\phi, e_i \rangle}{\sqrt{1+|\nabla\phi|^2}} \\ \langle \bar{N}, N \rangle = -\frac{1}{\sqrt{1+|\nabla\phi|^2}} \end{cases} \tag{18}$$

Furthermore:

$$\langle \bar{\nabla}\phi, e_i \rangle = \frac{\langle \nabla\phi, e_i \rangle}{1+|\nabla\phi|^2}, \quad \langle \nabla\phi, e_i \rangle = \frac{\langle \bar{\nabla}\phi, e_i \rangle}{1-|\bar{\nabla}\phi|^2},$$

and

$$|\bar{\nabla}\phi|^2 = \frac{|\nabla\phi|^2}{1+|\nabla\phi|^2}.$$

Replacing in (18) one has:

$$\begin{cases} \langle \bar{N}, e_i \rangle = \frac{\langle \bar{\nabla}\phi, e_i \rangle}{\sqrt{1-|\bar{\nabla}\phi|^2}} \\ \langle \bar{N}, N \rangle = -\sqrt{1-|\bar{\nabla}\phi|^2} \end{cases} \tag{19}$$

Then from (19)

$$\langle e_i^{\bar{T}}, N \rangle = -\langle e_i, \bar{N} \rangle \langle \bar{N}, N \rangle = \langle \bar{\nabla}\phi, e_i \rangle. \tag{20}$$

Using (19) and (20), one has

$$a_i = 2 \langle \bar{\nabla}\phi, e_i \rangle^2 + \frac{\langle \bar{\nabla}\phi, e_i \rangle^2}{1-|\bar{\nabla}\phi|^2} - 1.$$

Now we we write differently  $\sum_{i=1}^n \frac{\lambda_i}{1+\lambda_i\phi}$ . Notice that:

$$\prod_1^n (1 + \lambda_i\phi) = 1 + S_1\phi + \dots + S_n\phi^n$$

As  $M$  is minimal,  $S_1 = 0$  and

$$\begin{aligned} \sum_{i=1}^n \frac{\lambda_i}{1 + \lambda_i \phi} &= \frac{1}{\prod_{i=1}^n (1 + \lambda_i \phi)} (2S_2 \phi + \dots + nS_n \phi^{n-1}) \\ &= \phi \frac{(2S_2 + \dots + nS_n \phi^{n-2})}{1 + \phi^2(S_2 + S_3 \phi + \dots + S_{n-2} \phi^{n-2})} \\ &= \phi \left( \frac{-|A|^2 + \sum_{k=3}^n kS_k \phi^{k-2}}{1 + \sum_{k=2}^n \phi^k S_k} \right). \end{aligned}$$

Replacing in (17), one has:

$$\bar{\Delta} \phi + \sum_{i=1}^n \frac{\lambda_i}{1 + \lambda_i \phi} < \bar{\nabla} \phi, e_i >^2 \left( 2 + \frac{1}{1 - |\bar{\nabla} \phi|^2} \right) + b \phi = 0, \tag{21}$$

where

$$b = \frac{|A|^2 - \sum_{k=3}^n kS_k \phi^{k-2}}{1 + \sum_{k=2}^n S_k \phi^k}.$$

If  $M'$  is in a sufficiently small tubular neighborhood of  $M$ , i.e. if  $\phi$  is small enough, then  $b \geq 0$ . More precisely, we have the following estimate.

**Lemma 4** *There exists a constant  $0 < \epsilon(n) \leq 1$  depending only on  $n$  such that, if  $|A|\phi \leq \epsilon(n)$ , then*

$$b \geq |A|^2 \frac{1 - \alpha(n)\epsilon(n)}{1 + \alpha(n)\epsilon(n)^2},$$

where  $\alpha(n) = \frac{(n-1)(n+2)}{2} \sup_{k \leq n} c(k, n)$ .

*Proof* Using inequality (5) and the hypothesis one has

$$\begin{aligned} \left| \sum_{k=3}^n kS_k \phi^{k-2} \right| &\leq \sum_{k=3}^n kc(k, n)|A|^k \phi^{k-2} \leq |A|^2 \sup_k c(k, n) \sum_{k=3}^n k\epsilon^{k-2} \\ &\leq \frac{(n+3)(n-2)}{2} \epsilon(n) |A|^2 \sup_k c(k, n) \\ 1 + \sum_{k=2}^n S_k \phi^k &\leq 1 + \sum_{k=2}^n \phi^k c(k, n) |A|^k \\ &\leq \sum_{k=2}^n c(k, n) \epsilon^k \leq \frac{(n-1)(n+2)}{2} \epsilon(n)^2 \sup_k c(k, n). \end{aligned}$$

Hence, taking  $\alpha(n) = \frac{1}{2}(n-1)(n+2) \sup_k c(n, k)$  :

$$b \geq |A|^2 \frac{1 - \alpha(n)\epsilon(n)}{1 + \alpha(n)\epsilon(n)^2}.$$

If we choose  $\epsilon(n) \leq \frac{1}{\alpha(n)}$ , then  $b \geq 0$ . □

As the second fundamental forms of  $M$  and  $M'$  are uniformly bounded,  $C^0$  estimates on  $\phi$ , imply  $C^1$  estimates on  $\phi$ , then one has  $|\bar{\nabla}\phi| \leq C\phi$  for a positive constant  $C$  (see [18] for the case  $n = 2$ ). Hence the first order term of (21) satisfies

$$\left| \sum_{i=1}^n \frac{\lambda_i}{1 + \lambda_i\phi} \langle \bar{\nabla}\phi, e_i \rangle^2 \left( 2 + \frac{1}{1 - |\bar{\nabla}\phi|^2} \right) \right| \leq 3bC^2\phi^3$$

and it is uniformly bounded.

Furthermore, one has the following differential inequality on  $\phi$ .

**Corollary 1** *With previous notations, if  $|A|\phi \leq \epsilon(n)$  and  $\phi \leq \frac{1}{6C^2}$  then*

$$\bar{\Delta}\phi + \delta|A|^2\phi \leq 0,$$

where  $0 < \delta \leq \frac{1 - \alpha(n)\epsilon(n)}{2(1 + \alpha(n)\epsilon(n)^2)}$ .

#### 4 An estimate for positive eigenfunctions of $\Delta\phi + \alpha|A|^2\phi \leq 0$

We consider a positive function  $u$  satisfying the following inequality

$$\Delta u + Vu \leq 0 \tag{22}$$

on a hypersurface  $M$  in  $\mathbb{R}^{n+1}$ . Assume  $V \geq \alpha|A|^2$ , where  $|A|$  is the norm of the second fundamental form of  $M$  and  $\alpha < 1$  is a constant to be determined later. We apply an ingenious method of Fischer-Colbrie in order to estimate the function  $u$  (see [7, 16, 17]). Roughly speaking, we will estimate the function  $u$  from below at a point  $q \in M$  in terms of the value of  $u$  at a point  $p$  at distance  $R$  from  $q$  and of the distance  $R$ .

The key step in the estimate is contained in the following proposition.

**Proposition 2** *Let  $u$  be a positive solution of  $\Delta u + Vu \leq 0$ ,  $V \geq \alpha|A|^2$ ,  $0 < \alpha < 1$ , on a complete hypersurface  $M \subset \mathbb{R}^{n+1}$ ,  $n \leq 5$ . Then for any two point  $p, q \in M$ , there is a path  $\gamma$  joining  $q$  to  $p$  such that*

$$c(n) \int_{\gamma} \left( \frac{d\varphi}{ds} \right)^2 ds \geq \int_{\gamma} \varphi^2 \left( \frac{d \ln(u)}{ds} \right)^2 ds, \tag{23}$$

where  $c(n)$  is a constant depending only on  $n$ ,  $s$  is the arc length on  $\gamma$ , and  $\varphi$  is any smooth compact support function on  $\gamma$ .

*Proof* Here, we proceed as Fischer-Colbrie in [7].

For  $k$  positive, define

$$d\bar{s}^2 = v^2 ds^2, \quad v = u^k$$

Let  $\gamma$  be a minimizing geodesic of the metric  $d\bar{s}^2$  joining a fixed point  $\sigma \in M$  to a point  $p \in M$ .

Choose a basis  $\bar{\mathcal{B}} = \{\bar{e}_1, \bar{e}_2, \dots, \bar{e}_n\}$  of  $T_{\sigma}M$ , orthonormal for the metric  $d\bar{s}^2$ , such that  $\bar{e}_1$  is the unit tangent vector to  $\gamma$  at the point  $\sigma$ . Extend the basis  $\bar{\mathcal{B}}$  by parallel transport along  $\gamma$ . Let  $\mathcal{B} = \{e_1, e_2, \dots, e_n\}$  be a basis of  $T_{\sigma}M$  orthonormal for the metric  $ds^2$ , such that  $e_1$  is parallel to  $\bar{e}_1$  and  $A(e_i, e_j) = 0, i \neq j$ , where  $A$  is the second fundamental form of  $M$  at  $\sigma$ .

As  $\gamma$  is a minimizing geodesic, the second variation of length is positive, for any normal deformation  $Y$  with compact support along  $\gamma$ , i.e.,

$$\int_{\gamma} \left( \frac{d|Y|}{d\bar{s}} \right)^2 d\bar{s} \geq \int_{\gamma} \langle \langle \bar{R}(\bar{e}_1, Y)\bar{e}_1, Y \rangle \rangle d\bar{s}, \tag{24}$$

where  $\bar{R}$  is the curvature tensor of  $M$  and  $\langle \langle \cdot, \cdot \rangle \rangle$  is the scalar product, both for the metric  $d\bar{s}$ .

Let  $\varphi$  be any smooth function on  $\gamma$  with compact support and consider the normal deformations  $Y_i = \varphi \bar{e}_i$  to  $\gamma$  for  $i = 2, \dots, n$ . Applying (24) to each  $Y_i$  and summing up for  $i = 2, \dots, n$ , one has:

$$(n - 1) \int_{\gamma} \left( \frac{d\varphi}{d\bar{s}} \right)^2 d\bar{s} \geq \int_{\gamma} \varphi^2 \bar{R}_{\bar{1}\bar{1}} d\bar{s}, \tag{25}$$

where  $\bar{R}_{\bar{1}\bar{1}}$  is the Ricci curvature in the  $e_1$  direction i.e.

$$\bar{R}_{\bar{1}\bar{1}} = \sum_2^n \langle \langle \bar{R}(\bar{e}_1, \bar{e}_i)\bar{e}_1, \bar{e}_i \rangle \rangle .$$

The relation between the Ricci curvature for the metric  $d\bar{s}^2$  in the  $\bar{e}_1$  direction and the Ricci curvature for the metric  $ds^2$  in the  $e_1$  direction is given by:

$$\bar{R}_{\bar{1}\bar{1}} = \frac{1}{v^2} (R_{11} - (n - 2)(\ln v)_{11} - \Delta(\ln v)) \tag{26}$$

(cf. [10,17])

Let us compute  $R_{11}$  in terms of the second fundamental form of  $M$ . By Gauss Equation, for  $i = 1, \dots, n$  :

$$R_{11} = \sum_{i=2}^n \langle R(e_1, e_i)e_1, e_i \rangle = \sum_{i=2}^n (\langle A(e_1, e_1), A(e_i, e_i) \rangle - |A(e_1, e_i)|^2).$$

As  $M$  is minimal,  $\sum_{i=2}^n A(e_i, e_i) = -A(e_1, e_1)$ , hence

$$R_{11} = -A_{11}^2. \tag{27}$$

Plug (27) in (26):

$$\bar{R}_{\bar{1}\bar{1}} = \frac{1}{v^2} (-A_{11}^2 - (n - 2)(\ln v)_{11} - \Delta(\ln v)). \tag{28}$$

By definition of  $v$  :

$$\Delta(\ln v) = k \frac{\Delta u}{u} - \frac{k}{u^2} |\nabla u|^2.$$

Using equation (22), one has

$$\Delta(\ln v) \leq -kV - k^{-1} |\nabla \ln v|^2.$$

If we choose  $kV \geq A_{11}^2$ , previous inequality yields

$$\Delta(\ln v) + A_{11}^2 \leq -k^{-1} |\nabla \ln v|^2,$$



and replacing in (28)

$$\bar{R}_{\bar{1}\bar{1}} \geq \frac{1}{v^2} (-(n-2)(\ln v)_{11} + k^{-1} |\nabla \ln v|^2).$$

As in the proof of Proposition 1

$$|A|^2 \geq \frac{n}{n-1} A_{11}^2. \tag{29}$$

As  $V \geq \alpha |A|^2$ , the desired inequality  $kV \geq A_{11}^2$  is satisfied as soon as

$$k \geq \frac{n-1}{\alpha n}. \tag{30}$$

Now inequality (25) becomes:

$$(n-1) \int_{\gamma} (\varphi)_s^2 \frac{ds}{v} \geq -(n-2) \int_{\gamma} \varphi^2 (\ln v)_{ss} \frac{ds}{v} + \frac{1}{k} \int_{\gamma} \varphi^2 |\nabla \ln v|^2 \frac{ds}{v}, \tag{31}$$

where  $(\cdot)_s, (\cdot)_{ss}$  denote the first and the second derivative with respect to  $s$  respectively.

In (31) we get rid of  $v$  in the denominator, by substituting  $\varphi \sqrt{v}$  to  $\varphi$ .

After this substitution (31) is transformed into:

$$\begin{aligned} (n-1) \int_{\gamma} (\varphi)_s^2 ds &\geq -(n-2) \int_{\gamma} \varphi^2 (\ln v)_{ss} ds \\ &+ \left( \frac{1}{k} - \frac{(n-1)}{4} \right) \int_{\gamma} \varphi^2 (\ln v)_s^2 ds - (n-1) \int_{\gamma} \phi \phi_s (\ln v)_s ds. \end{aligned}$$

Replacing  $u = v^k$  and integrating by parts the term  $\int_{\gamma} \varphi^2 (\ln u)_{ss} ds$  yields

$$(n-1) \int_{\gamma} (\varphi)_s^2 ds \geq k \left( 1 - \frac{k(n-1)}{4} \right) \int_{\gamma} \varphi^2 (\ln u)_s^2 ds - k(n-3) \int_{\gamma} \phi \phi_s (\ln u)_s ds. \tag{32}$$

The case  $n = 3$  is readily solved because one of the coefficient in (32) is zero and one has

$$4 \int_{\gamma} (\varphi)_s^2 ds \geq k(2-k) \int_{\gamma} \varphi^2 (\ln u)_s^2 ds.$$

The best choice is  $k = 1$  and keeping in mind (30) one should take  $\alpha \geq \frac{2}{3}$ . Then

$$4 \int_{\gamma} (\varphi)_s^2 ds \geq \int_{\gamma} \varphi^2 (\ln u)_s^2 ds.$$

Now consider  $4 \leq n \leq 5$ . Cauchy-Schwarz inequality yields for  $\epsilon > 0$

$$-2 \int_{\gamma} \varphi \phi_s (\ln u)_s ds \leq \frac{1}{\epsilon} \int_{\gamma} \varphi^2 (\ln u)_s^2 ds + \epsilon \int_{\gamma} \phi_s^2 ds. \tag{33}$$

Replacing (33) in (32) one has

$$\left(n - 1 + \frac{k(n - 3)}{2\epsilon}\right) \int_{\gamma} \varphi_s^2 ds \geq \frac{k}{4}(4 - k(n - 1) - 2\epsilon(n - 3)) \int_{\gamma} \varphi^2 (\ln u)_s^2 ds. \tag{34}$$

For  $n = 4$  (choosing  $k = 1, \epsilon = \frac{1}{3}, \alpha \geq \frac{3}{4}$ ) (34) gives:

$$54 \int_{\gamma} \varphi_s^2 ds \geq \int_{\gamma} \varphi^2 (\ln u)_s^2 ds.$$

For  $n = 5$  (choosing  $k = \frac{9}{10}, \epsilon = \frac{1}{20}, \alpha \geq \frac{8}{9}$ ) (34) gives

$$49 \int_{\gamma} \varphi_s^2 ds \geq \int_{\gamma} \varphi^2 (\ln u)_s^2 ds.$$

□

**Proposition 3** *Let  $u$  be a positive function defined on a curve  $\gamma$  of a hypersurface  $M$  in  $\mathbb{R}^{n+1}, n \leq 5$  such that*

$$c(n) \int_{\gamma} (\varphi)_s^2 ds \geq \int_{\gamma} \varphi^2 (\ln u)_s^2, \tag{35}$$

for any  $\varphi \in C_0^\infty(\gamma)$ . Let  $p, q$  two points of  $\gamma$  such that  $p = \gamma(1)$  and  $q = \gamma(eR)$ .

Then, there exist a point  $\tilde{p} \in \gamma$ , in a neighborhood of radius  $e^2$  of  $p$  such that

$$u(q) \geq \frac{u(\tilde{p})}{R^{\beta(n)}}, \tag{36}$$

where  $\beta(n) = \sqrt{\frac{(e-1)(4-e)c(n)}{(3-e)}}$ .

*Proof* Let  $\varphi$  be the test-function defined on  $\gamma$  as follows:

$$\varphi(s) = \begin{cases} s/R & \forall s \in [0, R], \\ 1 & \forall s \in [R, eR] \\ \frac{s-3R}{(e-3)R} & \forall s \in [eR, 3R], \end{cases}$$

where  $s$  is the arc length of  $\gamma$ . Plug  $\varphi$  into (35):

$$\frac{c(n)(4 - e)}{(3 - e)R} \geq \int_0^{3R} \chi^2 (\ln u)_s^2 ds, \tag{37}$$

where

$$\chi(s) = (s/R)\delta_{[0,R]} + \delta_{[R,eR]} + \left(\frac{3}{(3 - e)} - \frac{s}{(3 - e)R}\right) \delta_{[eR,3R]}$$

Apply Cauchy–Schwarz inequality as follows:

$$\left| \int_R^{eR} \chi (\ln u)_s ds \right| \leq \int_R^{eR} |\chi (\ln u)_s| ds \leq \sqrt{\int_R^{eR} (\chi (\ln u)_s)^2 ds} \sqrt{(e - 1)R}.$$

Then, using (37):

$$\left| \int_R^{eR} (\ln u)_s ds \right| \leq \beta(n), \tag{38}$$

where  $\beta(n) = \sqrt{\frac{(4-e)(e-1)c(n)}{(3-e)}}$ . Integrate the first term of (38)

$$\left| \ln \frac{u(eR)}{u(R)} \right| \leq \beta(n). \tag{39}$$

From (39) it follows that

$$u(eR) \geq u(R)e^{-\beta(n)}. \tag{40}$$

Let  $\delta = \ln(eR)$  and let  $\tilde{p} = \gamma(e^{\delta-[\delta]+1})$ . Iterating  $[\delta] - 1$  times (40) one has:

$$u(q) \geq \frac{u(\tilde{p})}{R^{\beta(n)}}.$$

□

### 5 Proper minimal hypersurfaces

**Theorem 2** *Let  $M$  be a minimal, nonplanar hypersurface embedded in  $\mathbb{R}^{n+1}$ ,  $n \leq 5$ , with uniformly bounded second fundamental form.*

- (i) *If  $M$  is stable and there is an Euclidean ball  $B^E(p)$  in  $\mathbb{R}^{n+1}$  centered at a point  $p \in M$  such that  $B^E(p) \cap M$  consists of a finite number of connected components, then there exists an embedded tube around  $M$ .*
- (ii) *If  $M$  is not stable, then either  $M$  is proper, or  $M$  is properly embedded in a open set bounded by a complete minimal stable hypersurface (possibly multi-connected).*

*Proof* (i) As  $M$  has uniformly bounded curvature, there exists  $a > 0$  such that

$$a = \inf_{q \in M} \frac{1}{|A(q)|} > 0.$$

By Remark 1, we may assume that the subfocal tube  $T(M, \frac{a}{2})$  is not embedded. Then  $\exp^{-1}(M)$  consists in the zero section  $M_0$  of the normal bundle (identified with  $M$  by the exponential diffeomorphism) together with pieces of minimal hypersurfaces that intersect a tube of radius  $\frac{a}{2}$  around  $M_0$  ( $\exp^{-1}(T(M, a)) \setminus M_0 \neq \emptyset$ ); in particular there is a piece of  $M$  in  $T(M_0, a)$ , that defines graphs of positive functions defined on subdomains of  $M_0$ . The case of one piece, i.e. when there is only one graph defined on a subdomain of  $M$ , has been treated in Sect. 4. Here, we may have an infinite number of components that enters the tube  $T(M_0, a)$ . In order to deal with this case, we use a purely topological argument, that goes as follows. Let  $R_0$  be the radius of the Euclidean ball around  $p \in M$  that contains a finite number of components of  $M$ . Eventually decreasing  $R_0$ , one can assume that there is only one component. Then, there is a  $R_1 > R_0$  such that  $T(R_1, a) \setminus M_0$  contains a finite number of connected pieces of  $M$ , say  $M_{11}, \dots, M_{1k}$ . These pieces are graphs of positive functions  $u_i$  defined on disjoint sub-domains of  $M_0 : D_{11}, \dots, D_{1k}$ . We claim that the tubes  $T(D_{1i}, u_{1i}), i = 1, \dots, k$ , are disjoint and embedded.

Indeed, suppose that, for some  $k$ ,  $T(D_{1k}, u_{1k})$  is not embedded, then there is a piece  $N$  of  $M$  that is a graph over  $D_{1k}$  and lies between  $D_{1k}$  and  $M_{1k}$  up to the boundary of  $D_{1k}$ , since  $M$  is embedded. In particular  $N$  cuts  $T(R_1, a) \setminus M_0$  and hence  $N$  must be one of the  $M_{1i}$ .

Then consider an increasing sequence  $\{R_j\}$  going to infinity and apply the same reasoning to each  $T(R_j, a)$ . Finally the following tube around  $M$ :

$$\left[ T(M, a) \setminus \left( \bigcup_{j,i_j} T(D_{j,i_j}, a) \right) \right] \cup \left[ \bigcup_{j,i_j} T(D_{j,i_j}, u_{j,i_j}) \right].$$

is embedded.

(ii) The closure  $\bar{M}$  of  $M$  is a lamination of  $\mathbb{R}^{n+1}$  by minimal smooth embedded hypersurfaces. In the case  $n = 2$ , this fact is proved in [13]. The proof is analogous here and we give it, for the reader's convenience.

As the curvature of  $M$  is uniformly bounded, then there is a positive  $\delta$  such that for each  $p \in M$ ,  $M$  is locally a graph over the disk  $D_\delta(p)$  in  $T_pM$  centered at  $p$  of radius  $\delta$ . Now, assume that  $M$  is not proper and let  $x \in \mathbb{R}^{n+1}$  be an accumulation point of  $M$ . Let  $x_n$  a sequence of points in  $M$  converging to  $x$ . We can assume that the tangent planes  $T_{x_n}M$  converge to some plane  $P$  through  $x$ . Indeed the unit normal vectors at  $x_n$  form a subset of the unit sphere; hence there is an accumulation point, and we can choose a subsequence  $x_n$  such that  $T_{x_n}M$  converge to  $P$ . Since  $M$  is a graph  $G_n$  over each  $D_\delta(x_n) \subset T_{x_n}M$ , then, for  $n$  sufficiently large,  $G_n$  is a graph over  $D_{\frac{\delta}{2}}(x) \subset P$ . As  $G_n$  has bounded curvature,  $G_n$  is a graph of bounded slope of a  $C^2$ -function  $\phi_n$  defined on  $D_{\frac{\delta}{2}}(x)$ .  $\phi_n$  satisfies the minimal graph equation which is a uniform quasi-linear elliptic partial differential equation. Hence, by classical PDE theory (cf. [8]), there is a subsequence of  $\{\phi_n\}$  converging in the  $C^2$  norm to a  $C^2$ -function defined over  $D_{\frac{\delta}{2}}(x)$  whose graph is a minimal surface with bounded curvature. In other terms, a subsequence of  $G_n$  converges in the  $C^2$  norm to a minimal graph  $G_\infty$  over  $D_{\frac{\delta}{2}}(x)$ ,  $x \in G_\infty$ . Then, we observe that  $G_\infty$  does not depend on the choice of the sequence  $x_n$  converging to  $x$  and that  $G_\infty$  is disjoint from each  $G_n$ . Thus one has a local lamination of  $\bar{M}$  near the point  $x$ .

The graph  $G_\infty$  is contained in  $\bar{M}$ , hence for any point  $y \in \partial G_\infty$  there is a limit graph  $G_\infty(y)$  over a disk of radius  $\frac{\delta}{2}$  centered at  $y$  and by uniqueness of the limit  $G_\infty = G_\infty(y)$  where they intersect. This means that  $G_\infty$  extends to a complete minimal hypersurface in  $\bar{M}$ .

Now, let  $L$  one of the limit leaf: we prove that  $\tilde{L}$  is stable.

Where  $\tilde{L}$  is the universal covering space of  $L$ . On the normal bundle over  $\tilde{L}$  we consider the flat metric given by its immersion in  $\mathbb{R}^{n+1}$ .

Let  $\tilde{D}$  be a compact simply connected domain of  $\tilde{L}$  projecting on a domain  $D$  in  $L$ . Each point of  $D$  has a neighborhood that is a uniform limit of pairwise disjoint local graphs in  $M$ . One can lift each of these local graphs in the normal bundle over  $\tilde{L}$  along the lifting paths in  $D$ . Using holonomy, one obtains that  $\tilde{D}$  is the uniform limit of pairwise disjoint embedded minimal hypersurfaces  $F_n$  in the normal bundle of  $\tilde{L}$ .

$\tilde{D}$  is stable, because it is the uniform limit of disjoint minimal domains  $F_n$ . We prove it by contradiction. If  $\tilde{D}$  were unstable, then the first eigenvalue  $\lambda_1$  of the stability operator  $L = \Delta + |A|^2$  would be negative. Let  $f$  be the eigenfunction of  $\lambda_1$ , then  $L(f) + \lambda_1 f = 0, f > 0$  in  $\tilde{D}$  and  $f|_{\partial\tilde{D}} = 0$ .

Consider the variation of  $\tilde{D}$  given by

$$\mathbf{X}_t(x) = x + tf(x)\mathbf{N}(x),$$

where  $x \in \tilde{D}$  and  $\mathbf{N}$  is the unit normal vector field along  $\tilde{D}$  in the normal bundle.

We have the following formula for the first variation of the mean curvature of  $\mathbf{X}_t$  at  $t = 0$  :

$$\dot{H}(0) = L(f) = -\lambda_1 f.$$

As  $\lambda_1 < 0$ ,  $f(x) > 0$  for  $x \in \tilde{D}$ , then  $\dot{H}(0) > 0$ , i.e. the mean curvature vector of  $\mathbf{X}_t$  for  $t$  small points away from  $\tilde{D}$ .

Now, for  $t$  small, choose  $n_0$  large enough such that  $F_{n_0}$  intersects  $\mathbf{X}_{t_0}(\tilde{D})$ . Decreasing  $t$  from  $t_0$  to 0 there is a smallest positive  $t$  such that  $\mathbf{X}_t(\tilde{D})$  has a non empty intersection with  $F_{n_0}$ . At a point  $x \in F_{n_0} \cap \mathbf{X}_t(\tilde{D})$ ,  $F_{n_0}$  lies in the mean convex side of  $\mathbf{X}_t(\tilde{D})$  and this is impossible because  $F_{n_0}$  is minimal.

Denote by  $\mathcal{L}$  the union of the limit leaves. As  $M$  is not stable,  $M \cap \mathcal{L} = \emptyset$  and  $M$  is proper in  $\mathbb{R}^{n+1} \setminus \mathcal{L}$ .  $\square$

*Remark 3* We observe that (i) implies that for any point of  $q \in M$  there exists an Euclidean ball around  $q$ ,  $B^E(q)$ , such that  $B^E(q) \cap M$  consists of a finite number of connected components.

*Remark 4* In (ii), if  $\bar{M} = \mathbb{R}^{n+1}$  i.e.  $\bar{M}$  is a foliation of  $\mathbb{R}^{n+1}$ , then, applying the flux formula to the vector field normal to the foliation, we can easily deduce that  $M$  is area minimizing. In particular the volume of a ball of  $M$  of intrinsic radius  $R$  in  $M$  has order at most  $R^n$  (cf. [SSY]). As  $n \leq 5$ , then Lemma 1 implies that  $M$  is a hyperplane.

**Acknowledgments** The first author would like to thank B. Fux-Svaiter for very useful advise during the revision of this paper.

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