On the existence and uniqueness of constant mean curvature hypersurfaces in hyperbolic space[1]

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1. Introduction.

Let Γ be an embedded codimension one submanifold of $\partial_\infty \mathbb{H}^{n+1}$ (the boundary at infinity of hyperbolic space). We study the problem of finding a constant mean curvature hypersurface of $\mathbb{H}^{n+1}$ with prescribed asymptotic boundary Γ. To state precisely our results, we must first give some definitions.

Consider the halfspace model for hyperbolic space, i.e.

$$\mathbb{H}^{n+1} = \{ (x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_{n+1} > 0 \}$$

with the *hyperbolic metric* $ds^2 = \sum_{i=1}^{n} \frac{dx_i^2}{x_{n+1}^2}$. In the halfspace model, we view $\Gamma = \partial\Omega$ as a codimension one submanifold of euclidean space $\mathbb{R}^n$, with Ω a bounded $C^{2,\alpha}$ domain. We denote by H_Γ the mean curvature of Γ with respect to the interior normal vector, computed in the Euclidean metric. We say that Γ is *mean convex* if $H_\Gamma > 0$ at each point of Γ. We remark that mean convexity is not an intrinsic hyperbolic notion.

We shall prove that any such mean convex Γ is the asymptotic boundary of a complete embedded hypersurface of $\mathbb{H}^{n+1}$ of constant mean curvature H, for each $H \in (0,1)$ (here and in the following the mean curvature is computed with respect to the upward normal vector). We construct the desired M as a limit of constant mean curvature graphs over a fixed compact domain in a horosphere, for constant boundary data. By graphs, we mean graphs in the system of coordinates defined as follows: at a point p on the horosphere $\{x_{n+1} = c\} = L(c)$ we associate a point on the geodesic passing by p and orthogonal to the horosphere (i.e. the vertical geodesic passing by p). Thus an important part of our study concerns the existence and uniqueness of constant mean curvature hypersurfaces which are graphs over

[1]Dedicated to Stefan Hildebrandt on his sixtieth birthday

a bounded domain in a horosphere, whose boundary is mean convex. For such graphs, we are able to prove existence and uniqueness for $H \in (0,1)$. This leads to the following

Theorem 1.1. *Let Ω be a bounded domain in $L(c)$, respectively $\partial_\infty \mathbb{H}^{n+1}$ such that $\Gamma = \partial\Omega$ is of class $C^{2,\alpha}$ and mean convex. Then for each $H \in (0,1)$ there exists a complete embedded hypersurface M of $\mathbb{H}^{n+1}$ of constant mean curvature H with $\partial M = \Gamma$, respectively $\partial_\infty M = \Gamma$. Moreover, M can be represented as a graph $x_{n+1} = u(x)$ over Ω with $u \in C^{2,\alpha}(\bar{\Omega})$ and there is a unique such graph.*

If Γ is mean convex and bounds a star-shaped domain Ω, we have a stronger uniqueness result. We say that Γ is the *asymptotic homological boundary* of a hypersurface M in $\mathbb{H}^{n+1}$ if, for each c sufficiently small, $M \cap L(c) = \Gamma(c)$, where $\Gamma(c) \to \Gamma$ as $c \to 0$ and $\Gamma(c)$ is homologous to 0 in M. We denote the asymptotic homological boundary of M by $\partial_\infty M$.

Theorem 1.2. *Let Ω be a bounded domain in $L(c)$, respectively $\partial_\infty \mathbb{H}^{n+1}$ such that $\Gamma = \partial\Omega$ is of class $C^{2,\alpha}$ and mean convex. Let M be an embedded hypersurface of constant mean curvature $H \in (0,1)$ such that $\partial M = \Gamma$, respectively $\partial_\infty M = \Gamma$ and such that the mean curvature vector at the highest point of M points upward. Then M is the unique graph constructed in Theorem* 1.1.

We remark that an embedded hypersurface of constant mean curvature bigger than one, with asymptotic homological boundary a codimension one embedded submanifold of $\partial_\infty \mathbb{H}^{n+1}$ does not exist; this follows easily by comparing such a hypersurface with horospheres.

The study of minimal hypersurfaces, $H = 0$, with prescribed asymptotic boundary was initiated by Anderson [A] using methods of geometric measure theory. The boundary regularity of these solutions was studied by Hardt and Lin [HL] and Lin [L]. The extension of these results to constant $H \in (0,1)$ is due to Tonegawa [T] who makes a detailed study of boundary regularity, using the methods of [L]. Our Theorem 1.2 says that the geometric measure theory solution studied by Tonegawa is actually a topological disk when Γ is mean convex and star-shaped.

In the case of intrinsic Gauss curvature between -1 and 0, Rosenberg and Spruck [RS] completely answered the question of existence and uniqueness of graph type solutions. They proved that any embedded codimension one submanifold of $\partial_\infty \mathbb{H}^{n+1}$ is the asymptotic homological boundary of a complete embedded hypersurface of constant intrinsic Gauss curvature K, for each $K \in (-1,0)$; furthermore they proved that in $\mathbb{H}^3$, there are exactly two such surfaces, each of one is a graph over one of the two components of $\partial_\infty \mathbb{H}^{n+1} \setminus \Gamma$. Our approach follows the spirit of [RS]. It is possible that the graphical solutions of Theorem 1.1 always exist for any smooth Ω without convexity condition (this is false for $H = 0$) but this is far from clear.

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2. Constant mean curvature graphs.

Theorem 2.1. *Let Ω be a $C^{2,\alpha}$ subdomain of a horosphere $L(c)$ such that $\Gamma = \partial\Omega$ is mean convex. Then, for each $H \in (0,1)$ there exists a $C^{2,\alpha}$ graph M over $\bar{\Omega}$ of constant mean curvature H such that $\partial M = \Gamma$. Furthermore the graph M is unique.*

We claim that if $u\colon \Omega \to \mathbb{R}$ is a $C^{2,\alpha}$ solution of the following Dirichlet problem

$$\begin{cases} F[u] = \operatorname{div}\left(\dfrac{\nabla u}{W_u}\right) - \dfrac{n}{u}\left(h - \dfrac{1}{W_u}\right) & \text{in } \Omega, \\ u = c \quad \text{on } \Gamma, \end{cases} \tag{A}$$

where $W_u = \sqrt{1+|\nabla u|^2}$, then the graph of u is a hypersurface of constant mean curvature H with boundary equal to Γ. In fact, the hyperbolic metric in the halfspace model is conformally equivalent to the Euclidean metric with coefficient of conformality x_{n+1}^{-2}, so the principal curvatures of M in $\mathbb{H}^{n+1}$ are given by

$$k = x_{n+1}k_e + n_{n+1},$$

where k_e is the Euclidean principal curvature and n_{n+1} is the last component of the unit (in the Euclidean metric) normal vector to M. So, if we denote by H_e the Euclidean mean curvature, we have

$$H = x_{n+1}H_e + n_{n+1}$$

The claim follows, if we substitute in the previous equality the well known formula for the Euclidean mean curvature of a graph

$$H_e = \frac{1}{n}\operatorname{div}\left(\frac{\nabla u}{W_u}\right).$$

Proof of Theorem 2.1. Set

$$S = \{\, t \in [0,1] \mid \exists\, u^t \text{ admissible solution of } (\mathrm{A}^t)\,\}.$$

By Remark 2.3, $0 \in S \neq \emptyset$ so if we prove that S is open and closed, we have $S = [0,1]$ and the admissible solution u_1 is a solution of the Dirichlet problem (A).

(i) First, we prove that we can solve (A^t) in a neighborhood of $t = 0$ in $[0,1]$ with the aid of the Implicit Function Theorem.

Consider the linear operator $\mathfrak{L}^t_u \colon C^{2,\alpha}(\bar\Omega) \to C^{0,\alpha}(\bar\Omega)$ defined by

$$\mathfrak{L}^t_u h = F^t_{ij}(x)h_{ij} + b^t_i(x)h_i + c^t(x)h$$

where

$$F^t_{ij}(x) = \frac{\partial F^t}{\partial u_{ij}}(x,u,\nabla u)$$
$$b^t_i(x) = \frac{\partial F^t}{\partial u_i}(x,u,\nabla u)$$
$$c^t(x) = \frac{\partial F^t}{\partial u}(x,u,\nabla u) = \frac{n}{u^2 W_t}(tHW_t - 1)$$

$\mathfrak{L}^0_{u^0}$ is invertible since $c^0(x) = -\frac{n}{(u^0)^2 W_{u^0}} < 0$ (cf. Theorem 6.14 [GT]).

Then, by the Implicit Function Theorem (cf. Theorem 17.6 [GT]), there exists $t_0 > 0$ such that for each $t \in [0, t_0)$ there exists a solution u^t of (A^t) and such that each u^t varies continuously in t in the norm $C^{2,\alpha}(\bar\Omega)$. In particular, there exists a positive constant M such that

$$|W_{u^t} - W_{u^0}| \leq M t_0$$

hence

$$tHW_{u^t} - 1 < t_0(W_{u^0} + Mt_0)H - 1$$

and the second term is negative if t_0 is small enough.

So, for each $t \in [0, t_0)$, we have found an admissible solution of (A^t).

To prove that S is open and closed, we can assume $t \geq t_0 > 0$.

(ii) S is open in $[0,1]$. Let $t_1 \in S$, $t_1 \geq t_0$, and let u^{t_1} be an admissible solution of (A^{t_1}). The linear operator $\mathfrak{L}^t_u$ defined in (i) is invertible in B^t as $c^t(x) \leq 0$ for each $u \in B^t$, for each $t \geq t_0$.

Then, by the Implicit Function Theorem, there exists ϵ, $0 < \epsilon < t_0$, such that for each $t \in (t_1 - \epsilon, t_1 + \epsilon)$ there exists a solution u^t of (A^t) and such that u^t varies continuously in t in the $C^{2,\alpha}(\bar{\Omega})$ norm. By the same argument used in (i), we obtain that $(t_1 - \epsilon, t_1 + \epsilon) \subset S$, hence S is open.

(iii) S is closed in $[0,1]$. Let $t \in \bar{S}$, $t \geq t_0$, and let $\{t_m\} \subset S$ be a sequence such that $t_m \geq t_0$ for each m, and $t_m \to t$. Let $\{u^m\}$ be the corresponding sequence of admissible solutions of (A^{t_m}).

For each m, $t_m W_{u^m} H - 1 < 0$, hence $W_{u^m} < (t_0 H)^{-1}$; so the set $\{u^m\}$ is $C^{2,\alpha}(\bar{\Omega})$ bounded by a constant not depending on m. Up to a subsequence there exists $u^t \in C^{2,\alpha}(\bar{\Omega})$ such that $u^m \to u^t$ in $C^{2,\alpha}(\bar{\Omega})$. By continuity u^t is a solution of (A^t) and $tW_{u^t}H - 1 \leq 0$.

To prove that u^t is an admissible solution, we have to show that $tW_{u^t}H - 1 < 0$.

First we prove that the maximum of $|\nabla u^t|$ is on the boundary Γ for each solution of (A^t) such that $tW_{u^t}H - 1 \leq 0$.

By differentiating equation $F^t[u] = 0$ with respect to x_k, $k \leq n$, we obtain that $v = u^t_k$ satisfies a linear differential equation of the form

$$a_{ij}(x)v_{ij} + b_i(x)v_i + c(x)v = 0, \tag{1}$$

where $c(x) = tHW_{u^t} - 1 \leq 0$. By the maximum principle, v attains its maximum at the boundary and hence

$$\sup_{\Omega} |\nabla u^t| = \sup_{\Gamma} |\nabla u^t|.$$

Now, we evaluate the maximum of W_{u^t} (i.e. $|\nabla u^t|$) on the boundary Γ.

Let $0 \in \Gamma$ be a point of maximum of $|\nabla u^t|$ and choose coordinates on $L(c)$ so that the positive x_n-axis is the interior normal to Γ at 0 (i.e. $u^t_n(0) = |\nabla u^t|(0)$ and $u_{nn}(0) \leq 0$). Near 0, we can represent Γ as a graph $x_n = \rho(x')$, where $x' = (x_1, \ldots, x_{n-1})$, $\rho(0) = \rho_\alpha(0) = 0$, $\alpha < n$.

Consider the constant function $\underline{u} = c$; as $F^t[\underline{u}] = -\frac{n}{c}(tH - 1) > 0$, $\underline{u}$ is a subsolution of (A^t), hence $u^t \geq c$ in Ω and $u^t_n > 0$ on Γ by the Hopf boundary point lemma.

Since $u^t(x', \rho(x')) = c$, differentiating with respect to $\alpha, \gamma < n$, we have

$$u^t_{\alpha\gamma}(0) = -u^t_n(0)\rho_{\alpha\gamma}(0).$$

Substituting in $F^t[u^t] = 0$ gives

$$u^t_{nn}(0) - W^2_{u^t} u^t_n(0) \sum_{\alpha<n} \rho_{\alpha\alpha}(0) - \frac{n}{u} W^2_u (tHW_{u^t} - 1) = 0.$$

Since $\sum_{\alpha<n} \rho_{\alpha\alpha} = (n-1)H_\Gamma$ and $u^t_{nn} \leq 0$, we obtain

$$u^t_n(0)(n-1)H_\Gamma + \frac{n}{c}(tHW_{u^t} - 1) \leq 0.$$

As $H_\Gamma > 0$ and $u^t_n > 0$ at each point of Γ, the last inequality implies at 0

$$tHW_{u^t} - 1 < 0$$

hence

$$\max_\Omega W_{u^t} \le \max_\Gamma W_{u^t} < \frac{1}{tH}.$$

Thus u^t is an admissible solution and S is closed.

We have proved the existence part of Theorem 2.1.

(iv) Uniqueness. Let u be an admissible solution of the Dirichlet problem (A) and v be an arbitrary solution; by Remark 2.2 $u \ge c$, $v \ge c$. Let $x^0 \in \Omega$ be such that the function $w = v - u$ takes on its maximum at x^0 (x^0 is interior); hence $\nabla w(x^0) = \nabla v(x^0) - \nabla u(x^0) = 0$. As u is admissible, in a neighborhood of x^0 we have $HW_v - 1 = HW_u - 1 < 0$. Thus w also satisfies a linear equation of the form (1) and so, by the maximum principle $w \le 0$. By reversing the role of u and v we find $w \equiv 0$.

Theorem 2.1 is proved. □

Theorem 2.5. *Let Ω be a $C^{2,\alpha}$ subdomain of $\partial_\infty \mathbb{H}^{n+1}$ such that $\Gamma = \partial\Omega$ is mean convex. Then for each constant $H \in (0,1)$ there exists a graph*

$$M = \{x_{n+1} = u(x),\ u \in C^\infty(\Omega) \cap C^{0,1}(\bar\Omega)\}$$

over Ω of constant mean curvature H such that $\partial_\infty M = \Gamma$. Furthermore the graph M is unique.

Proof. Let $\Gamma(c)$, $\Omega(c)$ be the vertical translations of Γ and Ω to $L(c)$. By Theorem 2.1, $\Gamma(c)$ is the boundary of a graph of a function u_c over $\Omega(c)$ of constant mean curvature H.

To prove the theorem we will pass to the limit as $c \to 0$ for the Dirichlet problems

$$\begin{cases} F[u] = 0 & \text{in } \Omega(c), \\ u = c & \text{on } \Gamma(c). \end{cases}$$

The sequence $\{u_c\}$ is decreasing with c. In fact, if $c' < c$

$$u_c|_{\Gamma(c)} - u_{c'}|_{\Gamma(c')} = c - c' > 0$$

hence, by the maximum principle ($HW-1<0$)

$$u_c - u_{c'} \geq 0 \quad \text{in } \Omega.$$

Furthermore, u_c has a positive lower bound independent of c in Ω. In fact equidistant spheres (i.e. the set of equidistant points from a hyperbolic hyperplane) with asymptotic boundary in Ω, of constant mean curvature H, whose mean curvature vector points upward are lower barriers for u_c for all c. Therefore, by Schauder estimates, we have uniform bounds for u_c in $C^{0,1}(\bar{\Omega}) \cap C^{2,\alpha}(\Omega')$ for any compact subdomain Ω' strictly contained in Ω, independent of c. So, we can pass to the limit (up to subsequence) for $c \to 0$ and obtain a solution $u \in C^{0,1}(\bar{\Omega}) \cap C^{2,\alpha}(\Omega)$ of the Dirichlet problem

$$\begin{cases} F[u] = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma. \end{cases} \tag{A$^\infty$}$$

By standard elliptic regularity $u \in C^\infty(\Omega)$.

Uniqueness is obtained as in (iv) of Theorem 2.1. □

3. Higher regularity.

The graph M of Theorem 2.5 was obtained by constructing a solution $u \in C^\infty(\Omega) \cap C^{0,1}(\bar{\Omega})$ of the degenerate Dirichlet problem

$$\begin{cases} F[u] = \operatorname{div}\left(\dfrac{\nabla u}{W_u}\right) - \dfrac{n}{u}\left(h - \dfrac{1}{W_u}\right) & \text{in } \Omega, \\ u = c \quad \text{on } \Gamma, \end{cases} \tag{A$^\infty$}$$

where $W_u = \sqrt{1+|\nabla u|^2} \leq H^{-1}$.

We will show in fact that $u \in C^{2,\alpha}(\bar{\Omega})$. Define $v(x) = \lambda d(x)$, where $\lambda = \frac{\sqrt{1-H^2}}{H}$ and $d(x)$ is the distance from x to $\partial\Omega$.

Lemma 3.1. *$u(x) < v(x)$ in Ω.*

Proof. Suppose $T = \sup_\Omega (u-v) > 0$ and $u(x_0) - v(x_0) = T$, $x_0 \in \Omega$. Let $y_0 \in \partial\Omega$ be a closest point to x_0 i.e. $|x_0 - y_0| = d(x_0)$, and choose coordinates $(y_1, \ldots, y_n)$ at y_0 such that e_n is the interior unit normal and $\partial\Omega$ is locally represented as a graph $y_n = \rho(y_1, \ldots, y_{n-1})$ with $\rho(0) = \rho_\alpha(0) = 0$ and $\rho_{\alpha\beta}(0) = \kappa_\alpha\delta_{\alpha\beta}$, $\alpha, \beta < n$, where κ_α are the principal curvatures of $\partial\Omega$ at y_0 with respect to the normal e_n. Then $u(x)-u(x_0) \leq v(x)-v(x_0) \leq \lambda|x-x_0|$, hence $|\nabla u(x_0)| \leq \lambda$. On the other hand,

$$u(y_0 + se_n) \leq v(y_0 + se_n) + T \leq \lambda s + T, \quad 0 \leq s \leq d(x_0)$$

and

$$u(y_0 + d(x_0)e_n) = u(x_0) = v(x_0) + T = \lambda d(x_0) + T.$$

Thus

$$|\nabla u(x_0)| = u_n(x_0) = \lambda, \quad u_{nn}(x_0) \leq 0.$$

Consider now the level set

$$\Gamma_0 = \{\, x \in \Omega \mid u(x) = \lambda d(x_0) + T \,\}$$

passing through x_0; since $|\nabla u(x_0)| = \lambda$, Γ_0 is smooth near x_0 and also $d(x) \geq d(x_0)$ on Γ_0. Hence we can find a small ball $B_\epsilon(z_0)$ (hence also $d(x) > d(x_0)$ on $B_\epsilon(z_0)$). According to the geometric meaning of $d(x)$, the ball of radius $d(x_0) + \epsilon$ centered at z_0 is contained in $\bar{\Omega}$ (for otherwise there exists $z \in B_\epsilon(z_0)$ with $d(z) < d(x_0)$; a contradiction). This implies that $z_0 = y_0 + (d(x_0) + \epsilon)e_n$ and moreover

$$1 - \kappa_i(y_0)d(x_0) \geq 1 - \frac{d(x_0)}{d(x_0) + \epsilon} = \frac{\epsilon}{d(x_0) + \epsilon} > 0, \quad i = 1, \ldots, n-1.$$

It follows that $d(x)$ is actually C^2 near x_0 and satisfies

$$|\nabla d| \equiv 1,$$
$$d_i d_j d_{ij} = 0$$

near x_0.

$$\Delta d(x_0) = -\sum_{i=1}^{n-1} \frac{\kappa_i(y_0)}{1 - \kappa_i(y_0)d(x_0)} \leq -(n-1)H_\Gamma(y_0) < 0.$$

Therefore $v - \lambda d(x)$ satisfies

$$\operatorname{div}\left(\frac{\nabla v}{W_v}\right) < 0, \quad HW_v - 1 = 0$$

at x_0.

Since $u - v$ has its maximum at x_0

$$\operatorname{div}\left(\frac{\nabla u}{W_u}\right) \leq \operatorname{div}\left(\frac{\nabla v}{W_v}\right) < 0, \quad HW_u - 1 = HW_v - 1 = 0$$

at x_0.

This gives $F[u] < 0$ at x_0; a contradiction. □

Remark 3.2. The above argument really shows that $v = \lambda d(x)$ is a strict viscosity supersolution of (A^∞).

Now fix δ_0 so small that each point $P \in \Gamma$ can be touched by an interior tangent ball B_{δ_0} of radius δ_0. Choosing P as origin and introducing coordinates $(x_1, \ldots, x_n)$ with e_n the interior normal to Γ at 0, there is an equidistant sphere solution $w(x) \leq u(x)$ of (A^∞) which is a graph over $B_{\delta_0}(\delta_0 e_n)$ given by

$$w(x) = -RH + \sqrt{R^2 - \sum_{\alpha<n} x_\alpha^2 - (x_n - \delta_0)^2}, \quad R = \frac{\delta_0}{\sqrt{1 - H^2}}.$$

Expanding w and $v = \lambda d(x)$ in a Taylor series about the origin, we find (with $\lambda = \frac{\sqrt{1-H^2}}{H}$)

$$w(x) = \lambda x_n + O(|x|^2),$$
$$v(x) = \lambda x_n + O(|x|^2).$$

Since $w \leq u \leq v$ in $B_{\delta_0}(\delta_0 e_n)$ this gives

Lemma 3.3. *Let $x_0 = \delta e_n$, $\delta \leq \delta_0$ and let $x \in B_{\frac{\delta}{2}}(x_0)$. Then*

$$|u(x) - \lambda x_n| \leq C\delta^2$$

with C independent of δ.

Now observe that $l(x) = \lambda x_n$ is also a solution of $F = 0$. Since homothety from $0 \in \Gamma$ is a hyperbolic isometry, the rescaled functions

$$u^\delta(x) = \frac{1}{\delta} u(\delta x), \quad l^\delta(x) = \frac{1}{\delta} l(\delta x) = l(x)$$

are solutions of $F = 0$ in $B_{\frac{1}{2}}(e_n)$. By standard interior estimates (since $|\nabla u^\delta| \leq \frac{1}{H}$, $\frac{\lambda}{10} \leq u^\delta \leq 10\lambda$ in $B_{\frac{1}{2}}(e_n)$) all derivatives of u^δ are uniformly bounded in $B_{\frac{1}{4}}(e_n)$. Therefore the difference $u^\delta - l^\delta$ satisfies a uniformly elliptic equation with nice coefficients. This implies that

$$\sup_{B_{\frac{1}{8}}(e_n)} |\nabla(u^\delta - l^\delta)| \leq C(H) \sup_{B_{\frac{1}{2}}(e_n)} |u^\delta - l^\delta| \leq C\delta,$$
$$\sup_{B_{\frac{1}{8}}(e_n)} |\nabla^2(u^\delta - l^\delta)| \leq C(H) \sup_{B_{\frac{1}{2}}(e_n)} |u^\delta - l^\delta| \leq C\delta$$

by Lemma 3.3. Returning to the original variables, this gives

$$|\nabla u(x_0) - \lambda e_n| \leq C\delta, \quad |\nabla^2 u(x_0)| \leq C$$

with C independent of δ. Thus we have proved

Theorem 3.4. *Let $u \in C^\infty(\Omega) \cap C^{0,1}(\bar{\Omega})$ be a solution of (A^∞). Then $u \in C^{1,1}(\bar{\Omega})$ and $W_u = H^{-1}$ on $\partial\Omega$.*

We will now utilize the work of Tonegawa (cf. [T]) to show that $u \in C^{2,\alpha}(\bar{\Omega})$. Let $0 \in \partial\Omega$ and represent $\partial\Omega$ near 0 as a graph $x_n = \rho(x)$, $\rho(0) = \rho_\alpha(0) = 0$, $\alpha < n$ with e_n the interior normal to $\partial\Omega$ at 0.

We flatten $\partial\Omega$ near 0 by the transformation

$$y = (x', u(x)).$$

Since $u_n(0) = \lambda > 0$, there is a local inverse map $x_n = \psi(y)$ (the zeroth order Legendre transform in terminology of [KNS]) defined on a small upper half-ball $B_\delta^+(0)$. Using the transformation rules

$$u_\alpha(x) = -\frac{\psi_\alpha(y)}{\psi_n(y)}, \quad \alpha < n,$$
$$u_n(x) = \frac{1}{\psi_n(y)},$$
$$\frac{\partial}{\partial x_\alpha} = \frac{\partial}{\partial y_\alpha} - \frac{\psi_\alpha}{\psi_n}\frac{\partial}{\partial y_n}, \quad \alpha < n,$$
$$\frac{\partial}{\partial x_n} = \frac{1}{\psi_n}\frac{\partial}{\partial y_n}$$

it follows that $\psi \in C^{1,1}(\bar{B}_\delta^+(0))$ satisfies

$$\begin{cases} \left(\delta_{ij} - \dfrac{\psi_i\psi_j}{W_\psi^2}\right)\psi_{ij} = \dfrac{n}{y_n}(\psi_n - HW_\psi) & \text{in } B_\delta^+(0), \\ \psi(y',0) = \rho(y') & \text{on } \{y_n = 0\}. \end{cases} \tag{$*$}$$

Geometrically, this is just the representations of the graph $x_{n+1} = u(x)$ over the vertical plane passing through the tangent plane to $\partial\Omega$ at the origin and $(*)$ is just the equation of constant H in these coordinates.

This is precisely the situation studied in [T]. Applying his Theorem 2.14, we obtain that $\psi \in C^{2,\alpha}(\bar{B}_\delta^+(0))$. This gives

Theorem 3.5. *Let $u \in C^\infty(\Omega) \cap C^{0,1}(\bar{\Omega})$ be a solution of* (A^∞). *Then* $u \in C^{2,\alpha}(\bar{\Omega})$.

For further regularity results see [T].

4. A uniqueness result.

Let M be an embedded hypersurface of $\mathbb{H}^{n+1}$ such that $\partial M = \Gamma \subset L(c)$, $c \geq 0$; let Ω be the compact domain in $L(c)$ such that $\partial\Omega = \Gamma$. By Remark 2.2, if M has constant mean curvature $H \in (0,1)$, then it lies above $L(c)$; hence M divides $\mathbb{H}^{n+1} \cap \{x_{n+1} \geq c\}$ in two connected components. We denote by $\mathfrak{B}$ the component of $(\mathbb{H}^{n+1} \setminus M) \cap \{x_{n+1} \geq c\}$ that does not contain Ω. As H is constant, the mean curvature vector of M points towards the same component at each point of M.

Theorem 4.1. *Let Ω be a star-shaped (with respect to some interior point) subdomain of a horosphere $L(c)$ (respectively $\partial_\infty\mathbb{H}^{n+1}$) such that $\Gamma = \partial\Omega$ is mean convex. Let M be an embedded hypersurface of constant mean curvature $H \in (0,1)$ such that $\partial M = \Gamma$ (respectively $\partial_\infty M = \Gamma$), with mean curvature vector that points towards $\mathfrak{B}$. Then M is a graph over Ω, so M is the unique disk given by Theorem* 2.1 *(respectively* 2.5*).*

Proof. We start by proving uniqueness for a hypersurface with boundary at infinity. By Theorem 2.5, there exists a graph S over Ω of constant mean curvature H, with asymptotic boundary Γ. Denote by 0 the point with respect to which Ω is star-shaped and consider the family of hyperbolic isometries $\{H_t\}_{t\in\mathbb{R}}$ generated by translations along the vertical geodesic passing by 0; each H_t is an Euclidean homothety and we can choose the parameter t such that $H_1 = Id$. The fact that Ω is star-shaped with respect to 0 guarantees that, if $t \neq 1$, then $H_t(\Gamma) \cap M = \emptyset$. For t big enough, $H_t(S) \cap M = \emptyset$ and $H_t(S)$ is above M. Then, decrease t until we have a first point of contact between $H_t(S)$ and M; by the maximum principle, the first point of contact must be on the boundary or $M = S$. Hence, S is above M or equal to it.

Now, let t be small enough to have $H_t(S) \cap M = \emptyset$ and $H_t(S)$ below M. By increasing t, we find a first point of contact, that cannot be interior by the maximum principle. Hence S is below M. So, $M = S$.

Now let M have compact boundary Γ on $L(c)$. Consider the family $\{h_t\}_{t\in\mathbb{R}}$ of horizontal homotheties about the point $0 \in \Omega$, such that $h_1 = Id$. As Ω is star-shaped with respect to 0, the family $\{h_t(\Gamma)\}_{t\in\mathbb{R}} = \{C_t\}_{t\in\mathbb{R}}$ is a foliation of $L(c)$ by mean convex codimension one submanifolds that are the boundaries of star-shaped domains Ω_t in $L(c)$.

By Theorem 2.1, for each t, there exists a unique graph N_t over Ω_t of constant mean curvature H, such that $\partial N_t = C_t$. We can choose $\tau, \sigma \in \mathbb{R}$, $\tau < 1$, $\sigma > 1$, such that N_τ is below M and N_σ is above M (Figure 1).

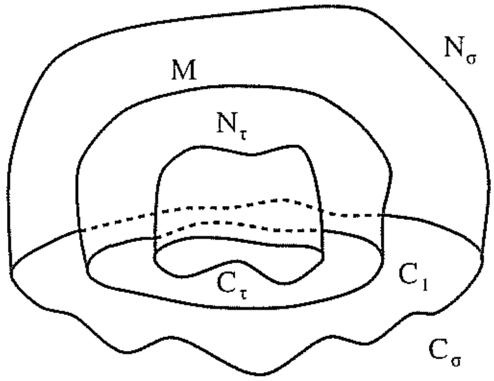

Figure 1.

In fact we can find an equidistant spheres S_1 of constant mean curvature H with constant mean curvature vector that points upward, lying below M and one such equidistant sphere S_2 lying above M. Let Γ_1 and Γ_2 be the codimension one spheres in $L(c)$ such that $\Gamma_1 = \partial S_1$ and $\Gamma_2 = \partial S_2$.

Choose τ and σ such that C_τ is interior to Γ_1 and C_σ is exterior to Γ_2. Then N_τ must lie below S_1, hence below M and N_σ must lie above S_2, hence above M (by the argument used at the beginning of the proof of this theorem).

We claim that the family $\mathfrak{F} = \{N_t\}$, $t \in [\tau, \sigma]$, is a foliation of the region (that contains M) bounded by $N_\tau \cup N_\sigma \cup [(\Omega_\sigma \setminus \Omega_\tau) \cap L(c)]$. In fact, by the maximum principle, two distinct N_t's cannot intersect, so we have only to prove that N_t varies continuously with t.

Fix $t_0 \in \mathbb{R}$; for ϵ small enough, for each $t \in [t_0 - \epsilon, t_0 + \epsilon]$, there exists a diffeomorphism $f_t \colon \Omega_{t_0} \to \Omega_t$ with the property $\|f_t\|_{C^2(\Omega_{t_0})} \leq \epsilon$.

Let F be defined as in Section 2 and for each $t \in [t_0 - \epsilon, t_o + \epsilon]$ consider the two families of equivalent Dirichlet problems

$$\begin{cases} F[u \circ f_t] = 0 & \text{in } \Omega_{t_0}, \\ u = c & \text{on } C_{t_0}, \end{cases} \tag{D^t}$$

$$\begin{cases} F[u] = 0 & \text{in } \Omega_t, \\ u = c & \text{on } C_t, \end{cases} \tag{E^t}$$

By Theorem 2.1, there exists a unique admissible solution u^{t_0} of (E^{t_0}); as u^{t_0} is admissible, $HW_{u^{t_0}} - 1 < 0$. Then, as $\|f_t\|_{C^2(\Omega_{t_0})} \leq \epsilon$, we have

$$HW_{u^{t_0} \circ f^{t_0}} - 1 < 0,$$

so the linearized operator associated to (D^{t_0}) as in Theorem 2.1, is invertible. Hence, by Implicit Function Theorem, there exists δ, $0 < \delta < \epsilon$, such that for each $t \in [t_0 - \delta, t_0 + \delta]$ there exists a solution v^t of (D^t) that depends continuously on t. Thus $u^t = v^t \circ (f^t)^{-1}$ is the unique solution of (E^t) and depends continuously on t and so $\mathfrak{F} = \{N_t\}$ is a foliation.

Now, using the foliation $\mathfrak{F}$, we prove that $M = N_1$, hence it is a graph and it is unique by Theorem 2.1.

For $t < 1$ no N_t intersect M. Otherwise, for the smallest such t, N_t is on one side of M at an intersection point (necessarily interior) and their mean curvature vectors both point towards $\mathfrak{B}$, so N_t would be equal to M by the maximum principle. This is impossible as $\partial N_t \neq \partial M$ for $t < 1$. Thus M is above N_1 (or equal to it). We repeat the same argument starting with N_σ and decreasing to N_1 and, as before, we conclude that M is below N_1. So $M = N_1$ and the Theorem 3.1 is proved. □

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