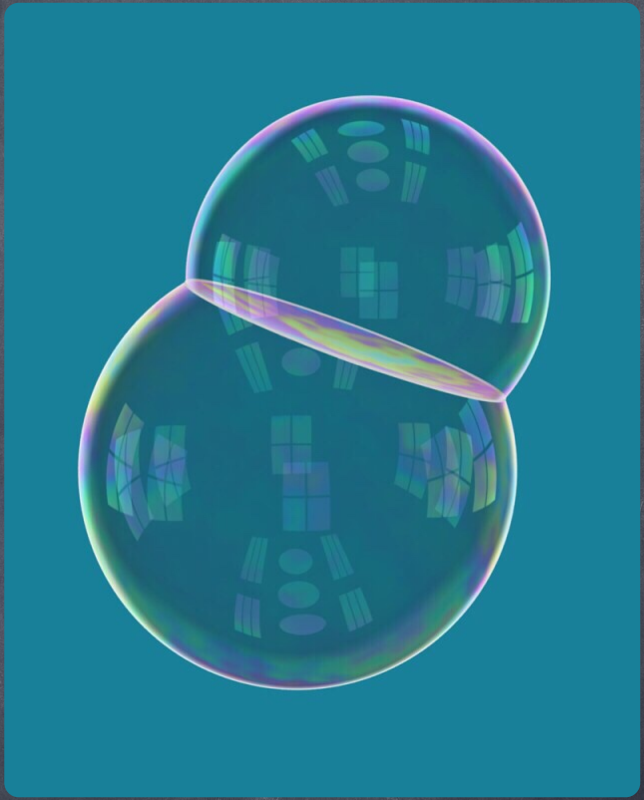
The Shape of Soap Films and Bubbles

Barbara Nelli
Università dell'Aquila

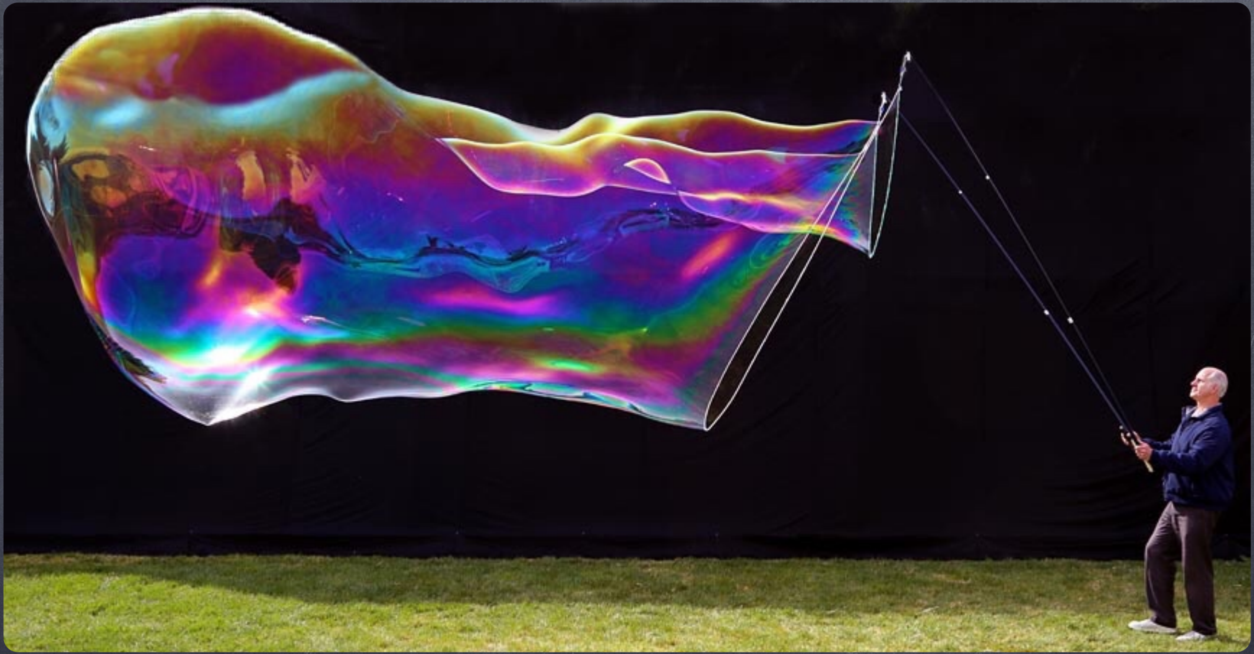
BEIRUT 15/02/2018

Soap Bubble

Soap bubble

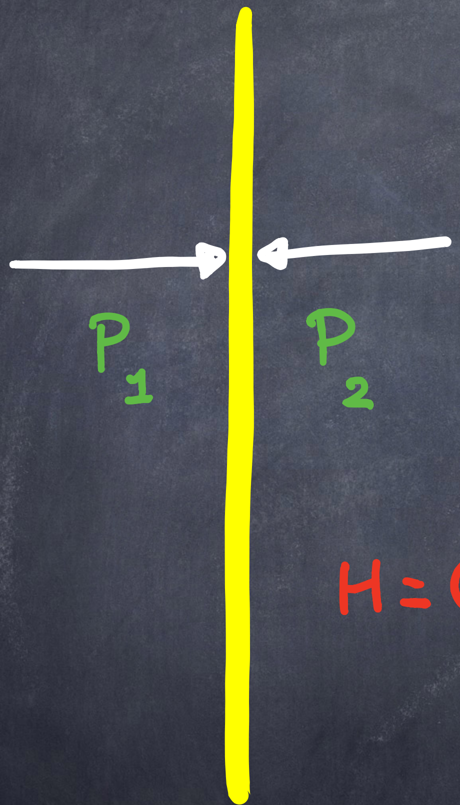


Double bubble



Large bubble

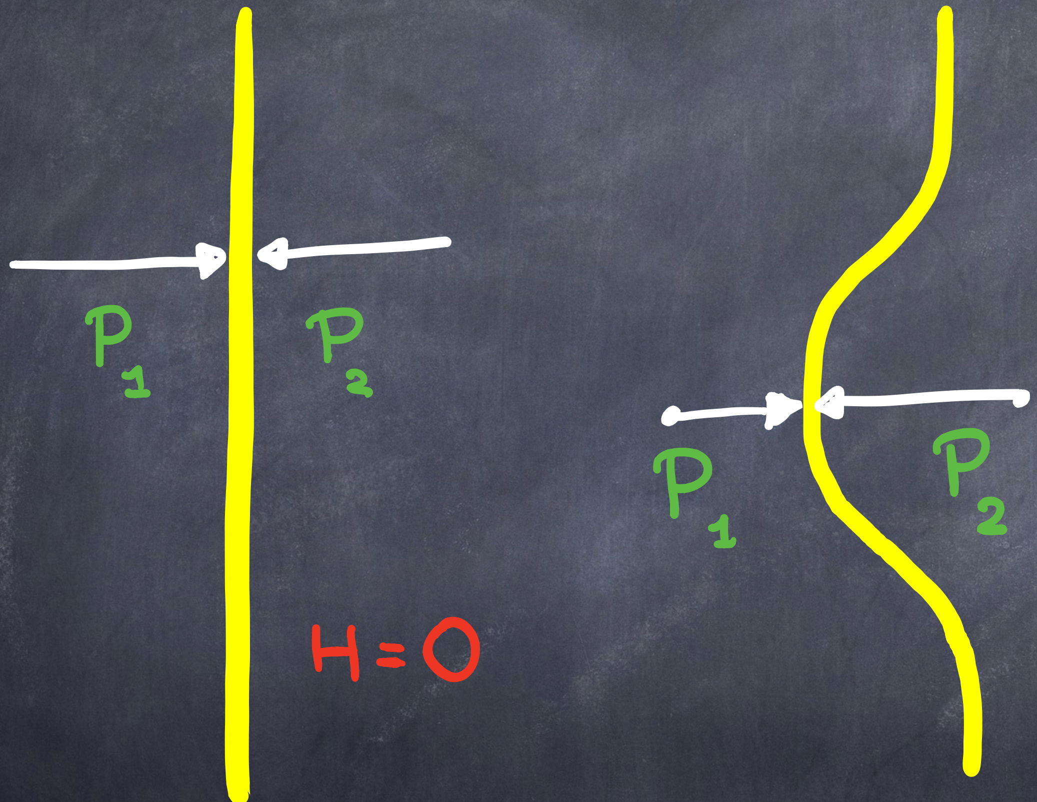
The mathematics of a soap bubble



$$H = 0$$

$$P_1 = P_2$$

The mathematics of a soap bubble



$H \propto P_2 - P_1$

$P_1 = P_2$ $\qquad$ $P_2 > P_1$

H is known as the mean curvature of a surface

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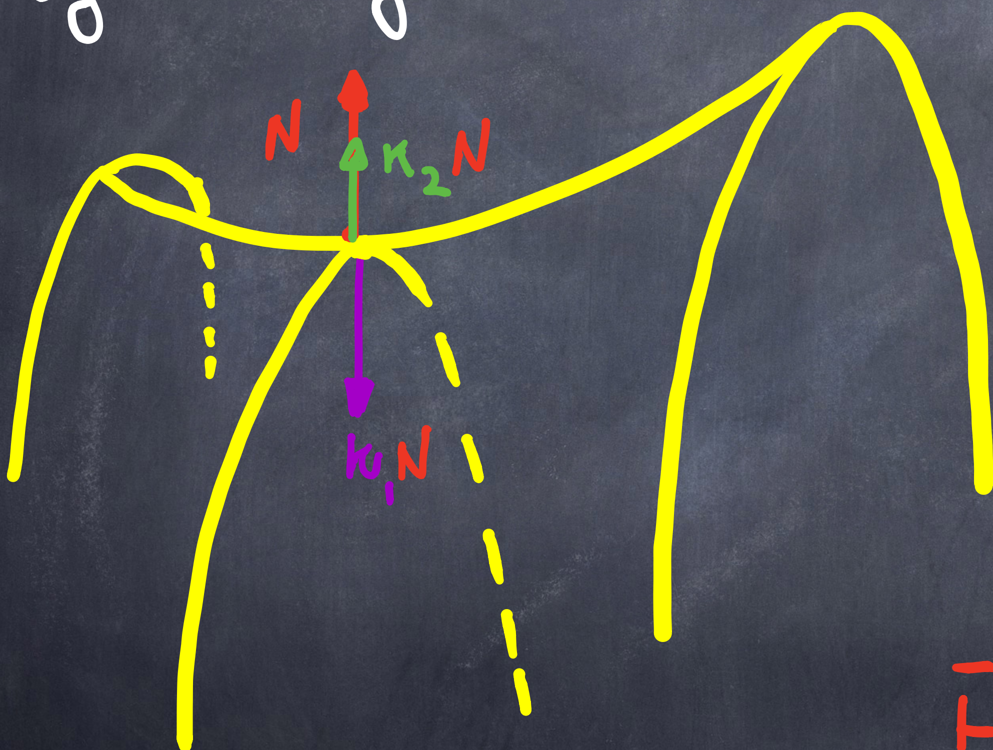


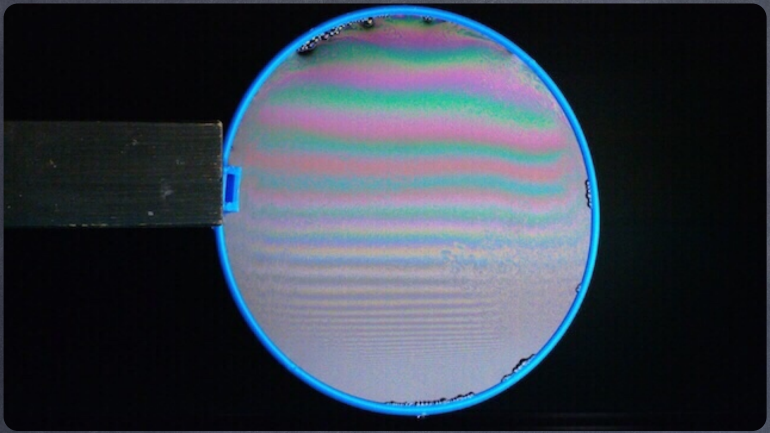
$$H = \frac{k_1 + k_2}{2}$$

k_1, k_2 principal curvatures

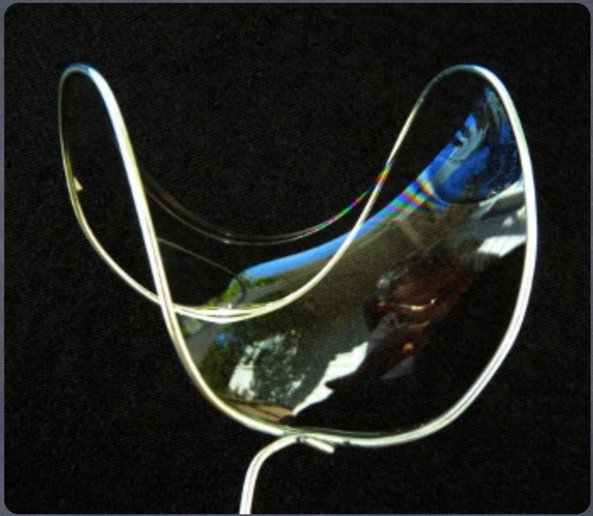
$$\vec{H} = HN$$

mean curvature vector

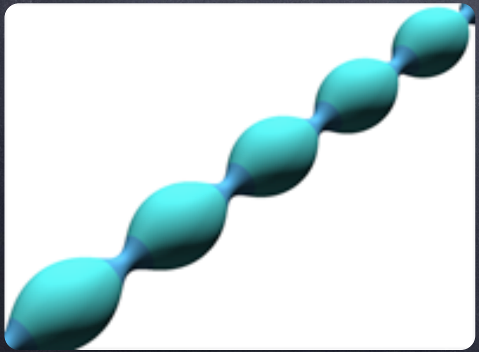
Examples



H=0, Flat wire frame



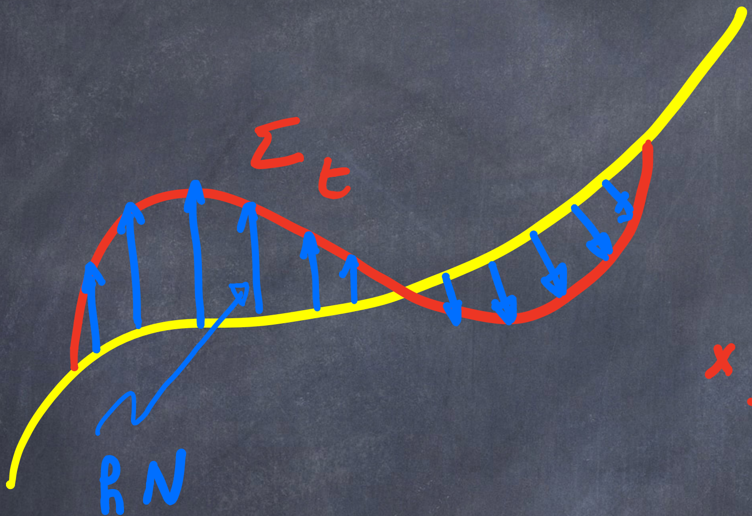
H=0, Saddle wire frame



H=c, Delaunay surface

H=1/R, bubble

Variational interpretation

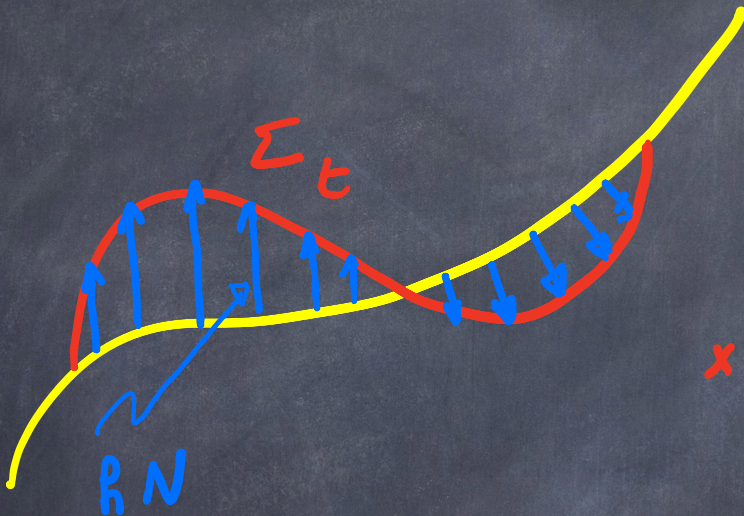


$$\Sigma \xrightarrow{x} \mathbb{R}^3$$

$$\Sigma_t \xrightarrow{x_t} \mathbb{R}^3$$

$$x_t = x + t\,h\,N$$

Variational interpretation



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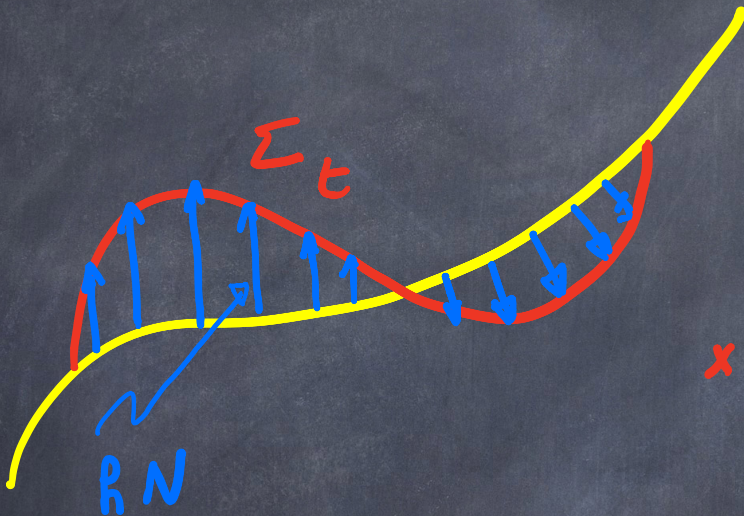

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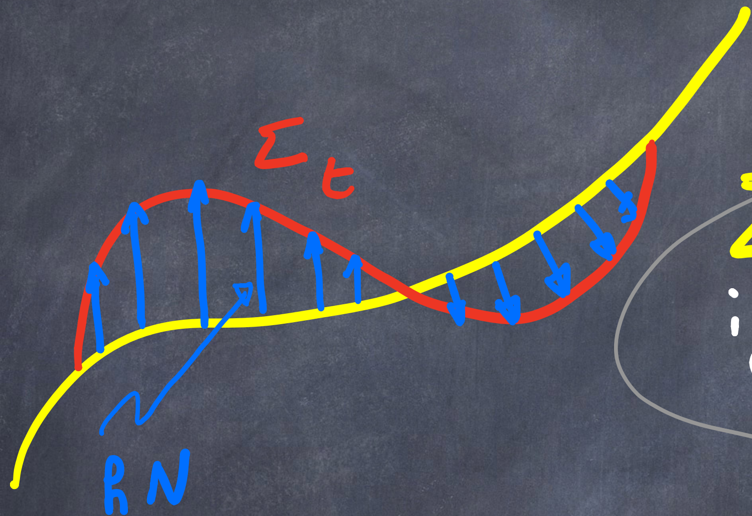

$\Sigma \xrightarrow{x} \mathbb{R}^3$

Σ is STABLE if it is a MINIMUM

If Σ has $H = 0$, it is a critical point for $\text{Area}(\Sigma_t)$.

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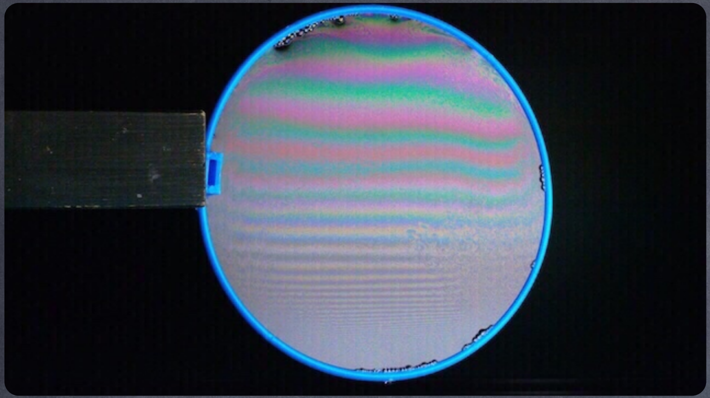
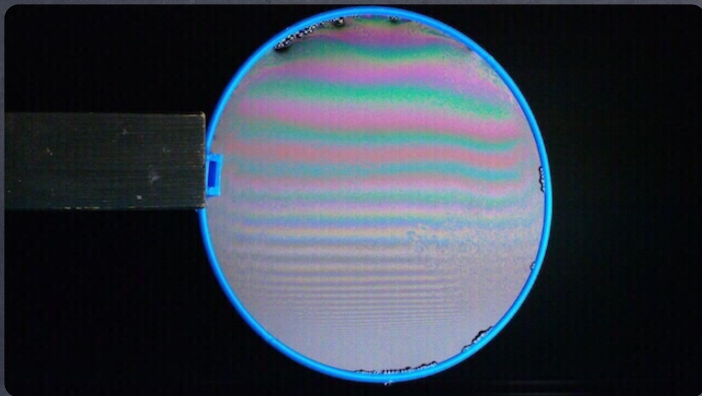


H=0

$H = 0$ is called minimal surface



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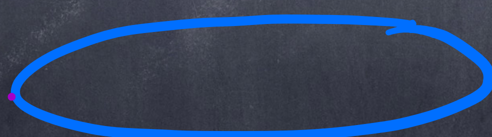
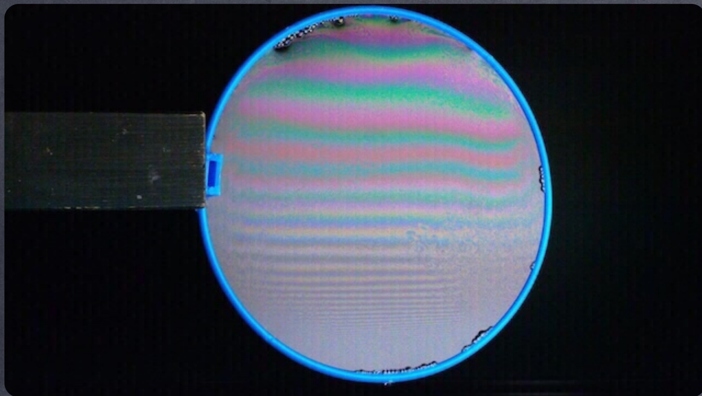


circle $= \partial S$



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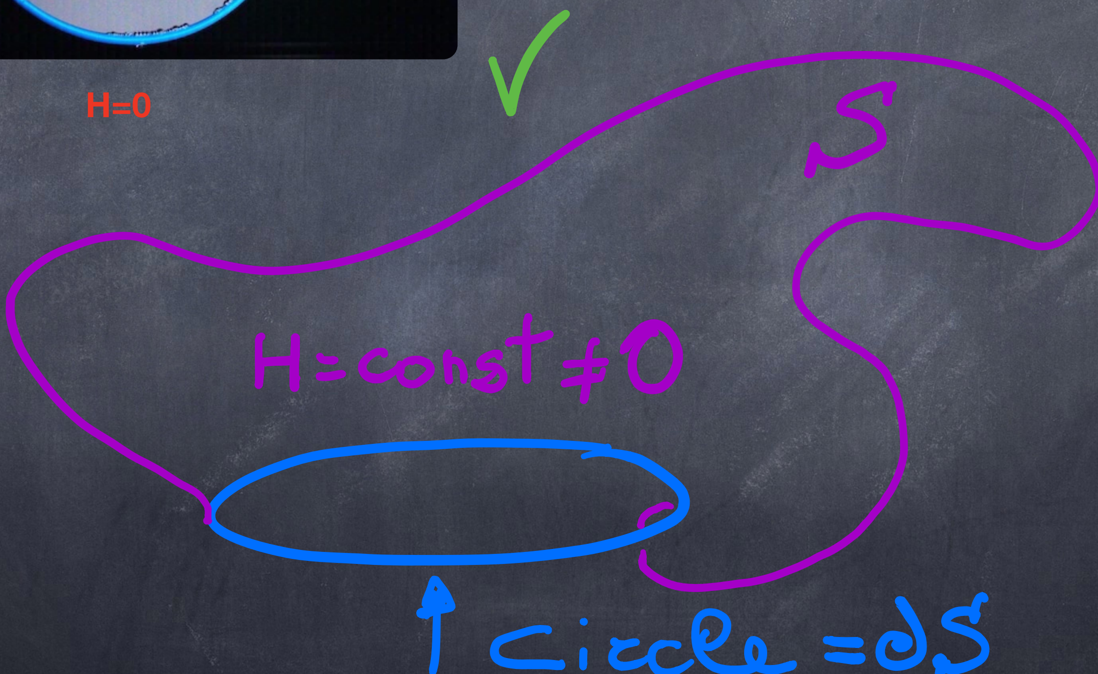
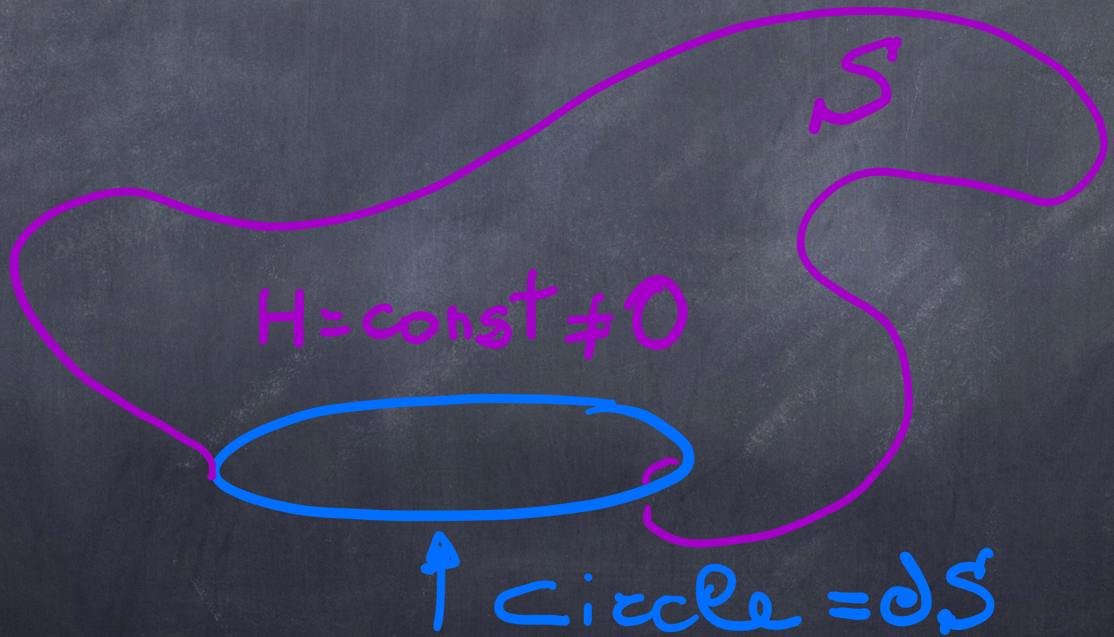


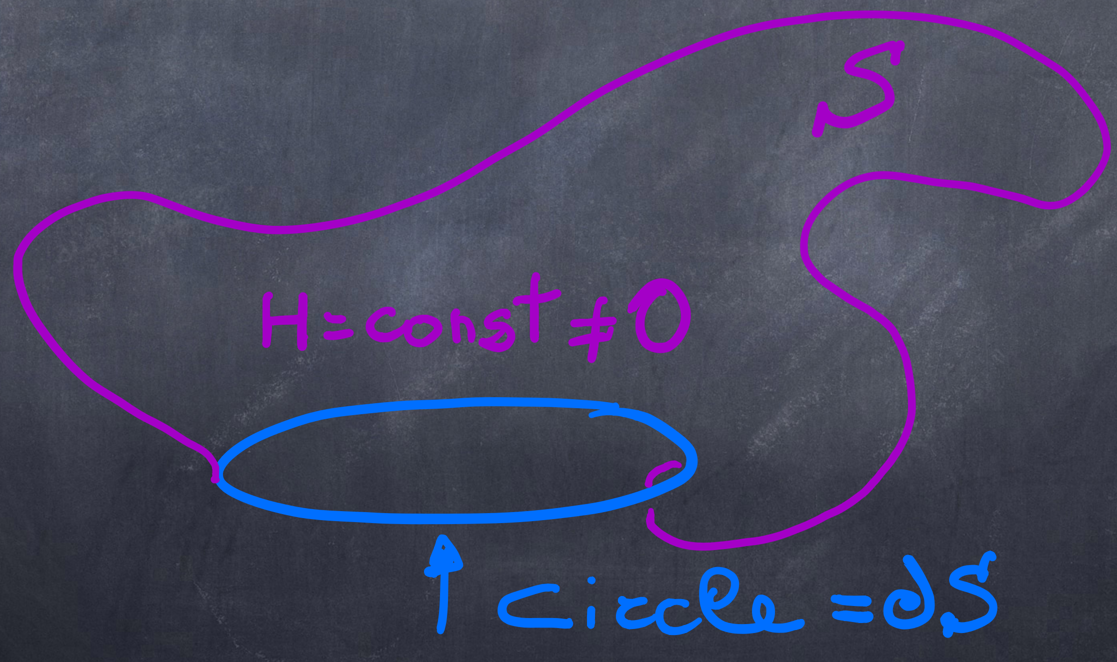
Which is the shape of S?

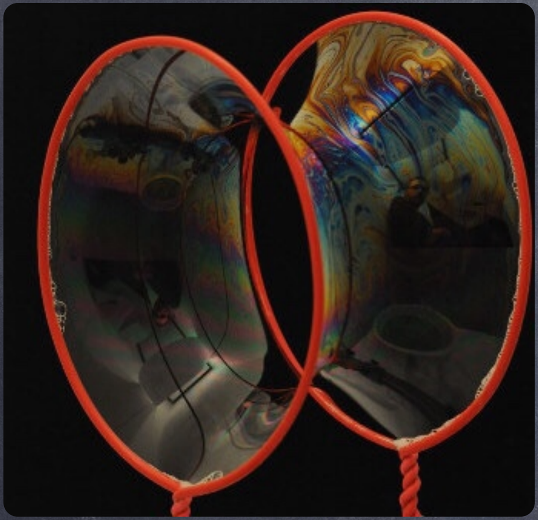
Is S a part of a round sphere?



Is S a part of a round sphere?

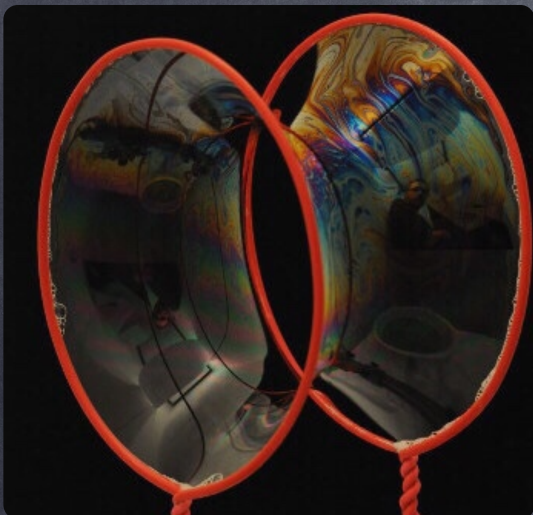
UNKNOWN





H=0 Catenoid

C_1, C_2 convex curves



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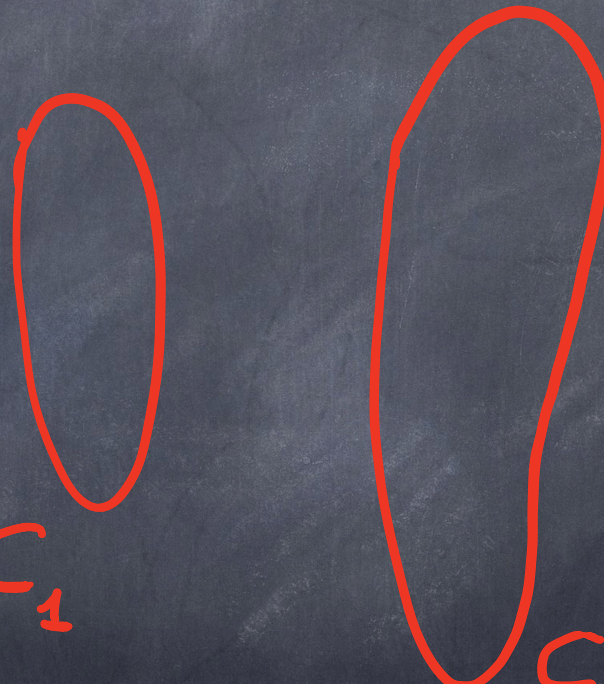

C_1 C_2

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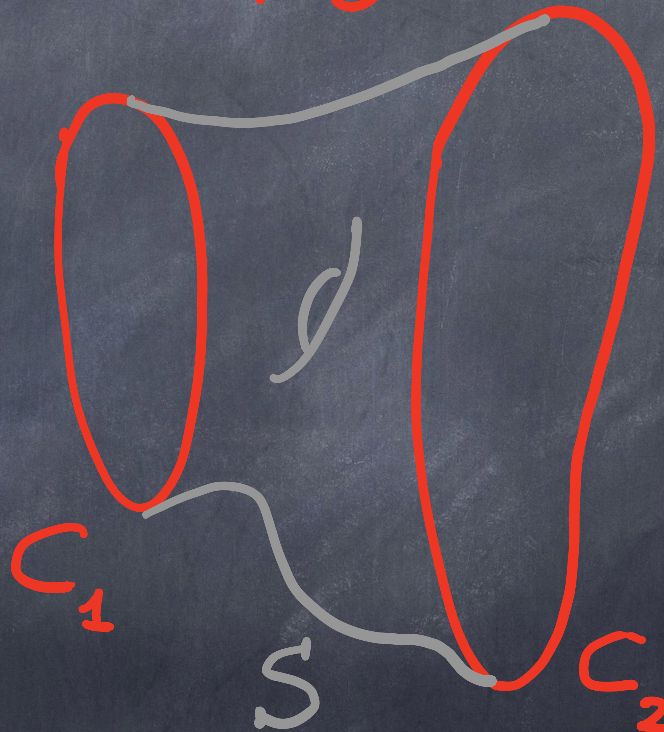
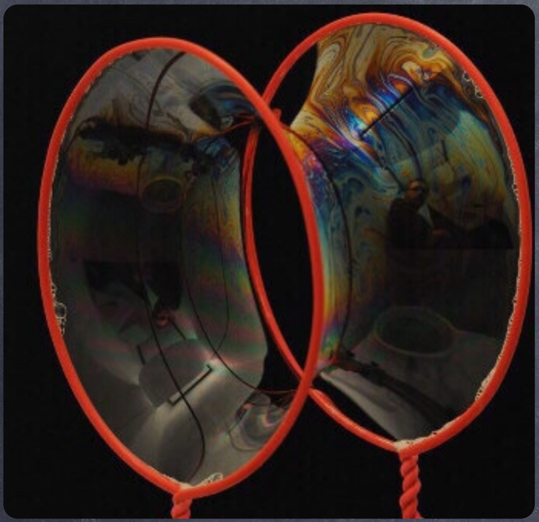
$H=0$



May S have genus?



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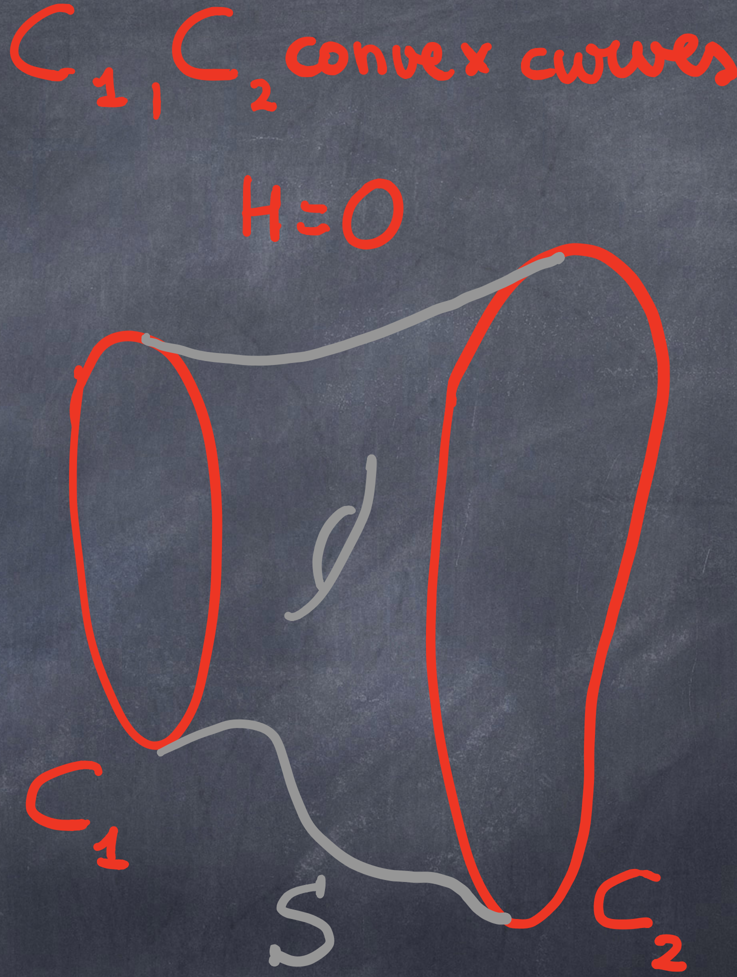

May S have genus? UNKNOWN

A contribution : [— 1998]

Comment. Math. Helv. 73 (1998) 298–305
0010-2571/98/020298-8 $ 1.50+0.20/0

Commentarii Mathematici Helvetici

An example of an immersed complete genus one minimal surface in $\mathbb{R}^3$ with two convex ends

Barbara Nelli

Abstract. We prove the existence of a compact genus one immersed minimal surface M, whose boundary is the union of two immersed locally convex curves lying in parallel planes. M is a part of a complete minimal surface with two finite total curvature ends.

Mathematics Subject Classification (1991). 53A10, 53C42.

Keywords. Minimal surface, convex boundary, Weierstrass representation, elliptic functions.

1. Introduction

In 1978 Meeks conjectured that a connected minimal surface bounded by two convex curves in two parallel planes is topologically an annulus; hence it has genus zero. The conjecture has never been proved and the most general result, due to Schoen, is the following.

Let $\Gamma = \Gamma_1 \cup \Gamma_2$ be any boundary consisting of two Jordan curves in parallel planes; assume that Γ is invariant by reflection through two planes P_1, P_2 orthogonal to the planes of the Γ_i and that both P_1 and P_2 divide Γ into pieces which are graphs with locally bounded slope over the dividing plane. Then any minimal surface spanning Γ is topologically an annulus and is an embedded surface meeting each parallel plane between the planes of the Γ_i in smooth Jordan curves.

In particular, if Γ_1 and Γ_2 are circles such that the line joining their centers is perpendicular to the planes in which they lie, then M is a catenoid (cf. [Sc]).

In 1991, Meeks and White studied the space of minimal annuli bounded by convex curves in parallel planes (cf. [MW]).

In this paper we prove the existence of a compact genus one immersed minimal surface M, whose boundary is the union of two immersed locally convex curves lying in parallel planes. In fact M is a part of a complete minimal surface with two finite total curvature ends.

The method we use to construct our surface is the following.

It is well known that a minimal surface of genus g and k ends can be described

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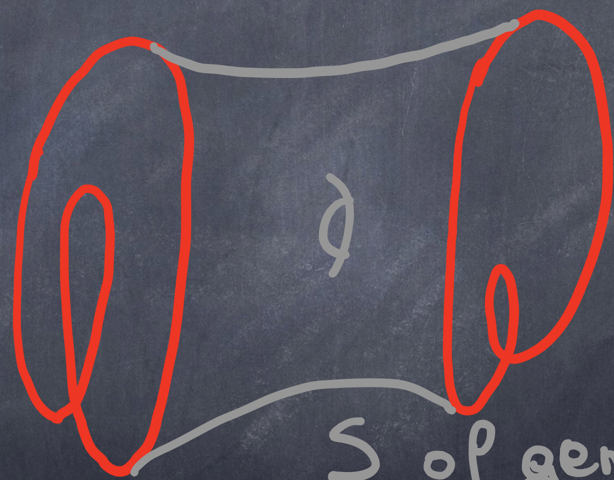

S of genus 1

C_1, C_2 locally convex

A question one is able to answer

Alexandrov Theorem. A compact surface with constant mean curvature $H \neq 0$, embedded in $\mathbb{R}^3$ is a round sphere.

[Alexandrov '56]

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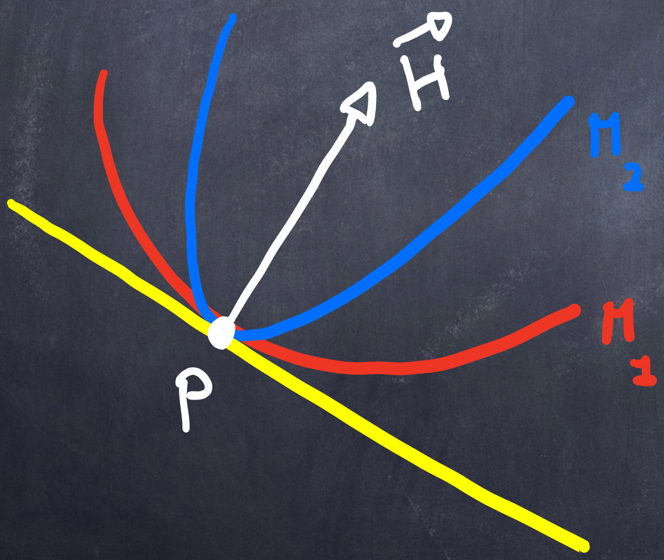
<u>Tools</u>.

- maximum principle for the elliptic PDE of constant mean curvature surfaces.
- Alexandrov reflection method (moving plane).

The geometric maximum principle

Let M_1, M_2 be two surfaces with the same mean curvature vector $\vec{H}$, tangent at a point p, with M_1 on one side of M_2 around p.



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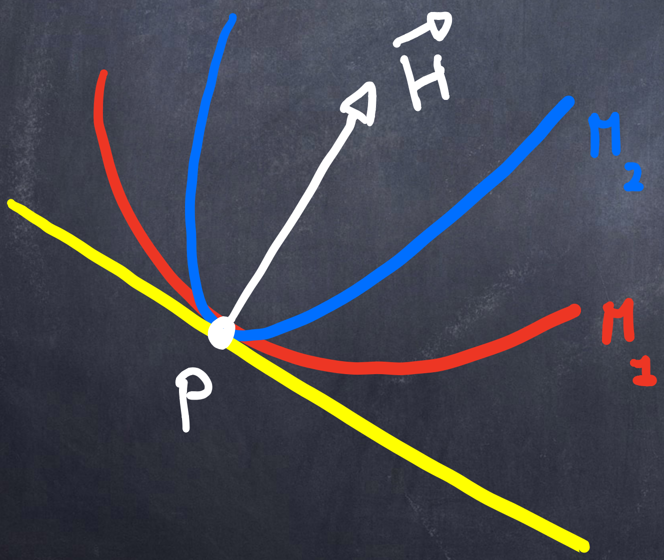
Let M_1, M_2 be two surfaces with the same mean curvature vector $\vec{H}$, tangent at a point p, with M_1 on one side of M_2 around p.

Then M_1 and M_2 coincide around p.



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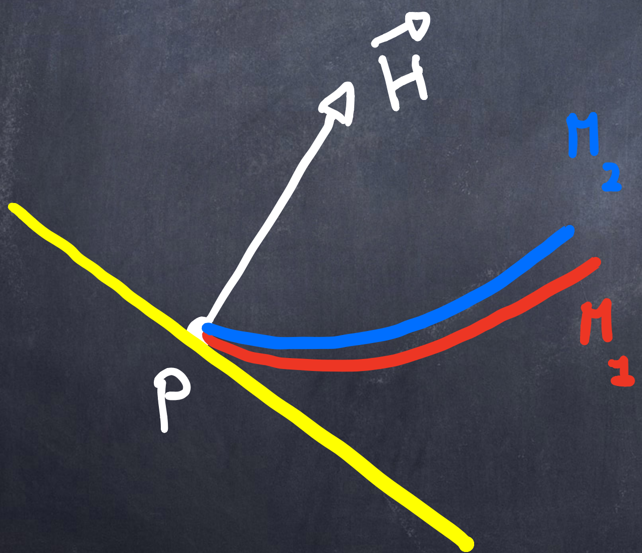
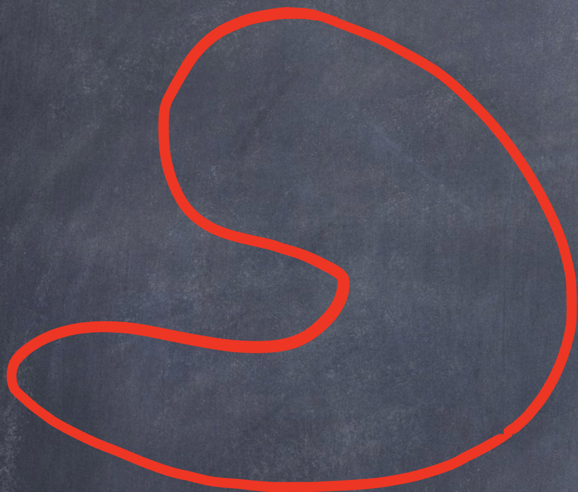


Then M_1 and M_2 coincide around p.

The same holds if p is a **boundary** point

Alexandrov Reflection method

$H = c$



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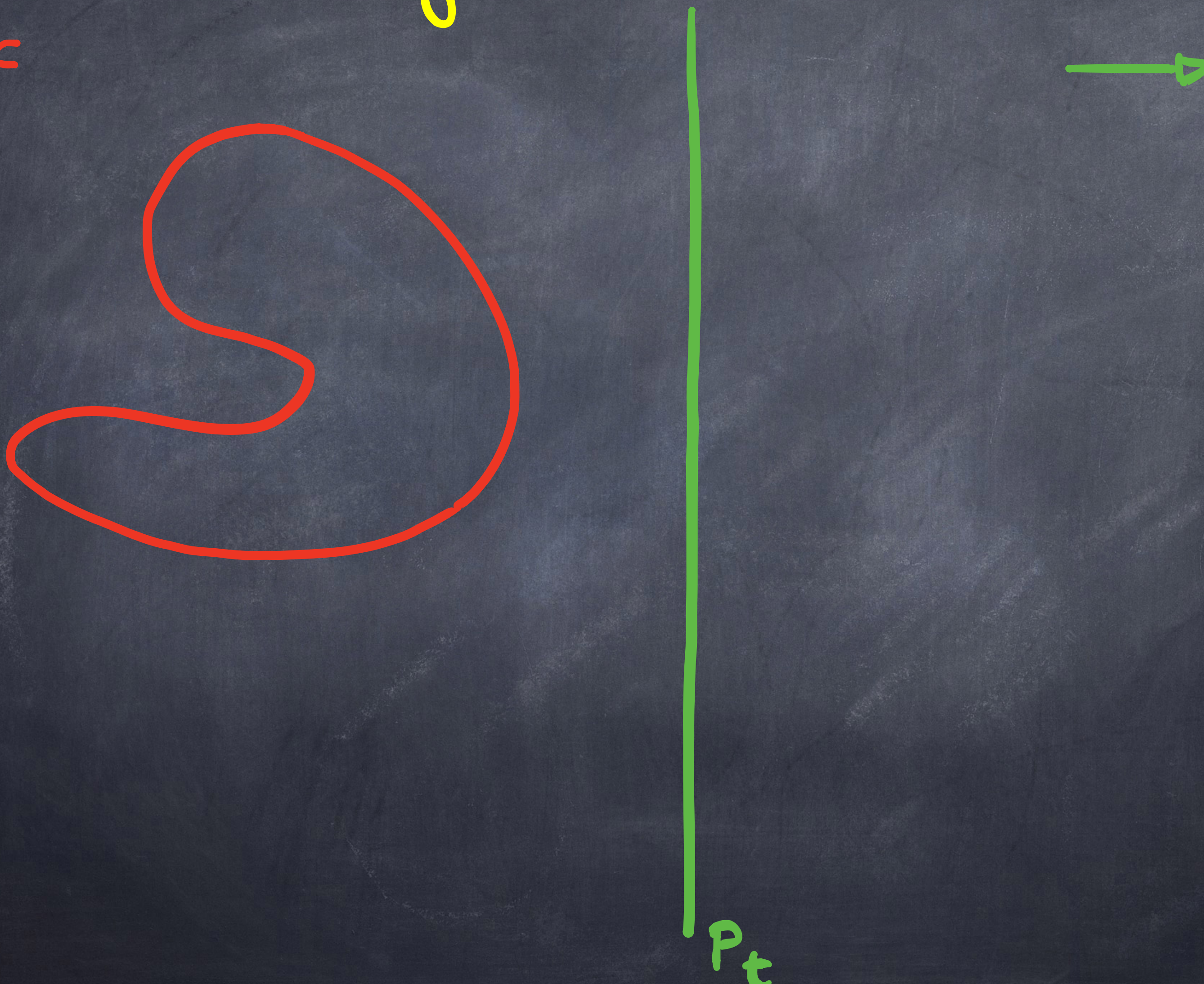

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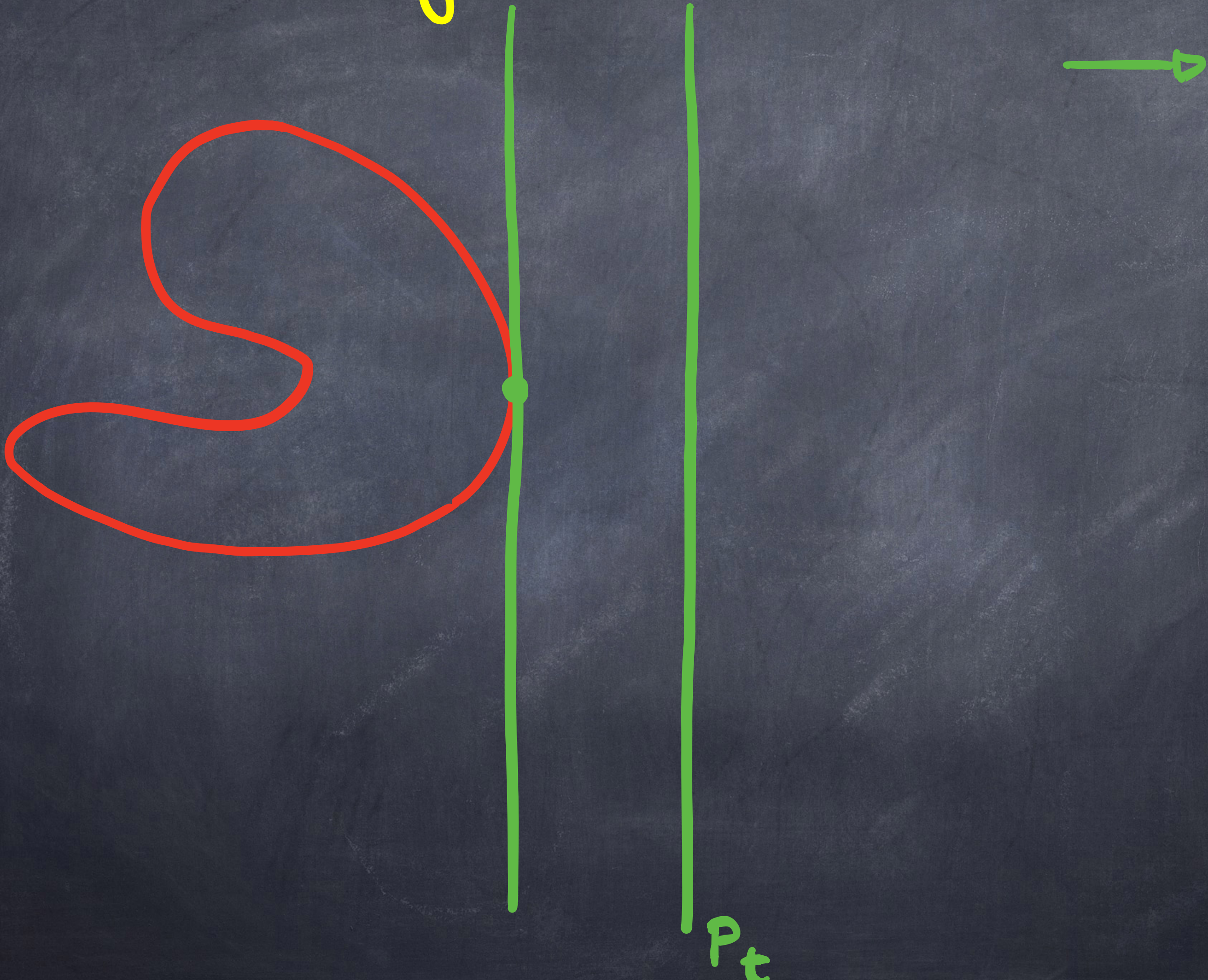

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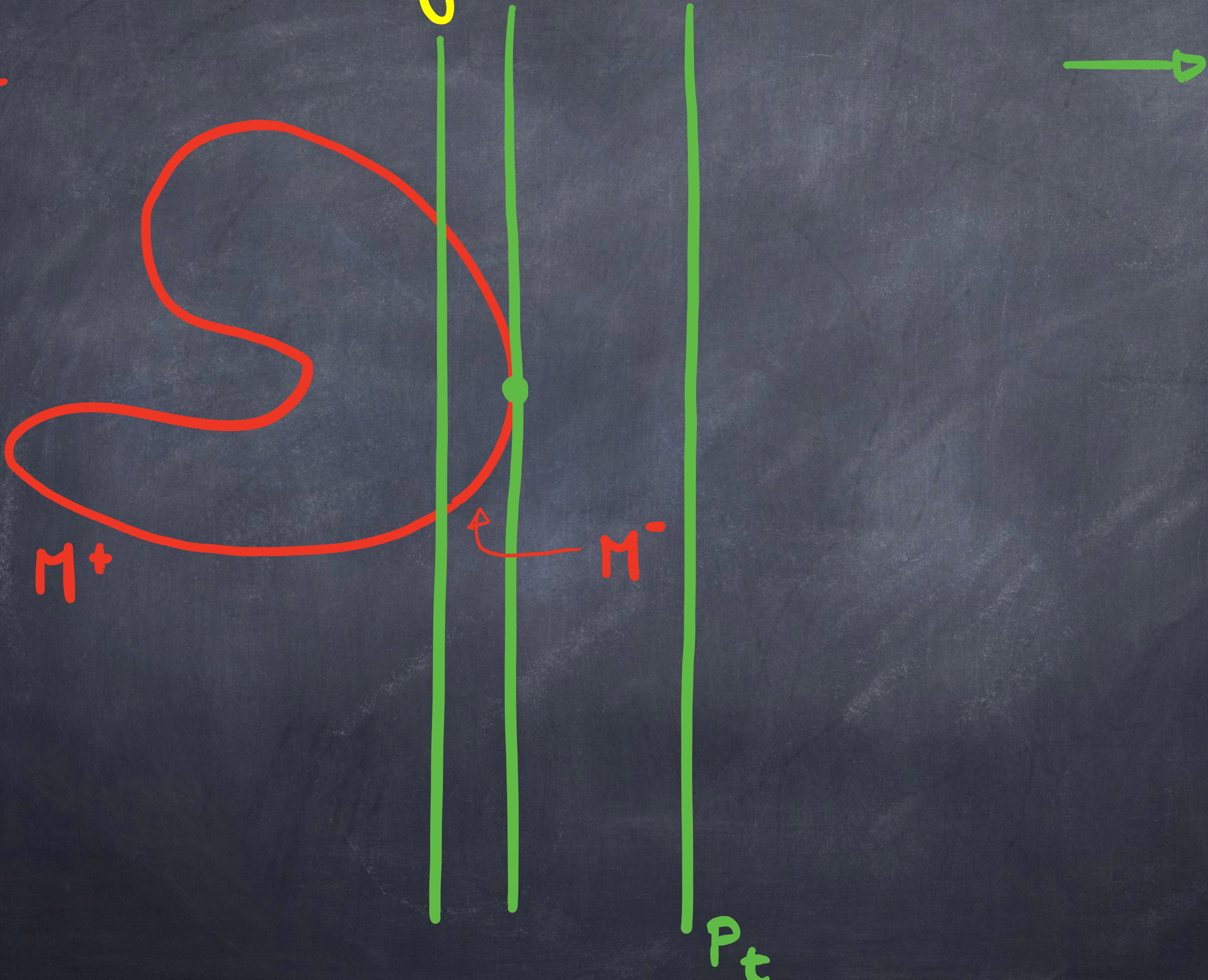

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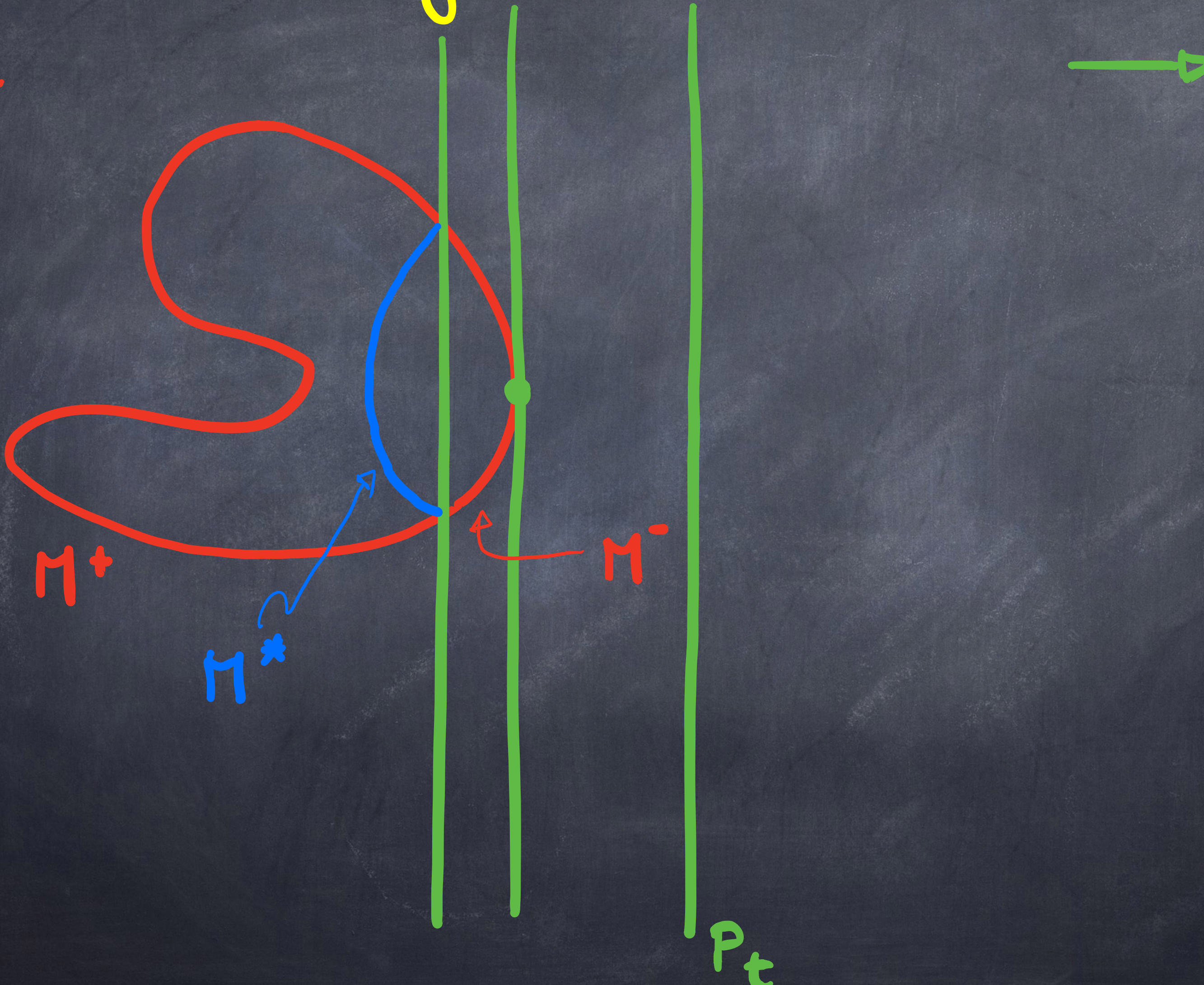

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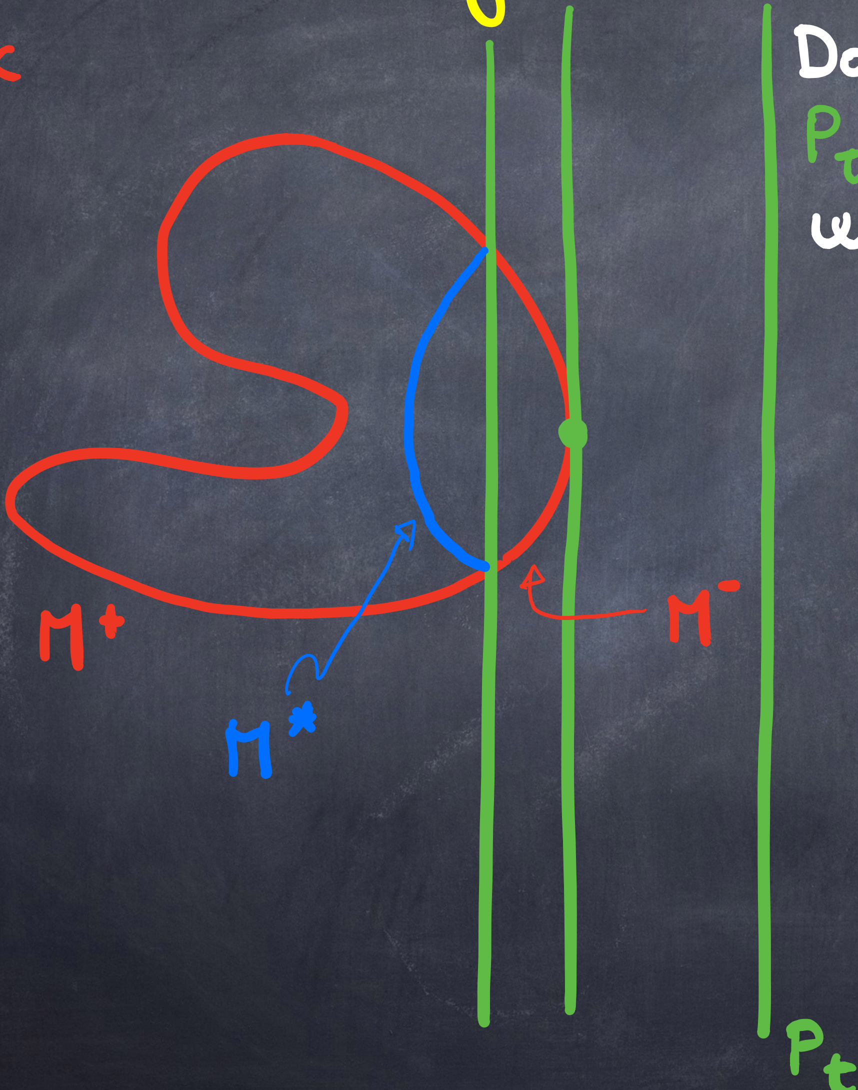
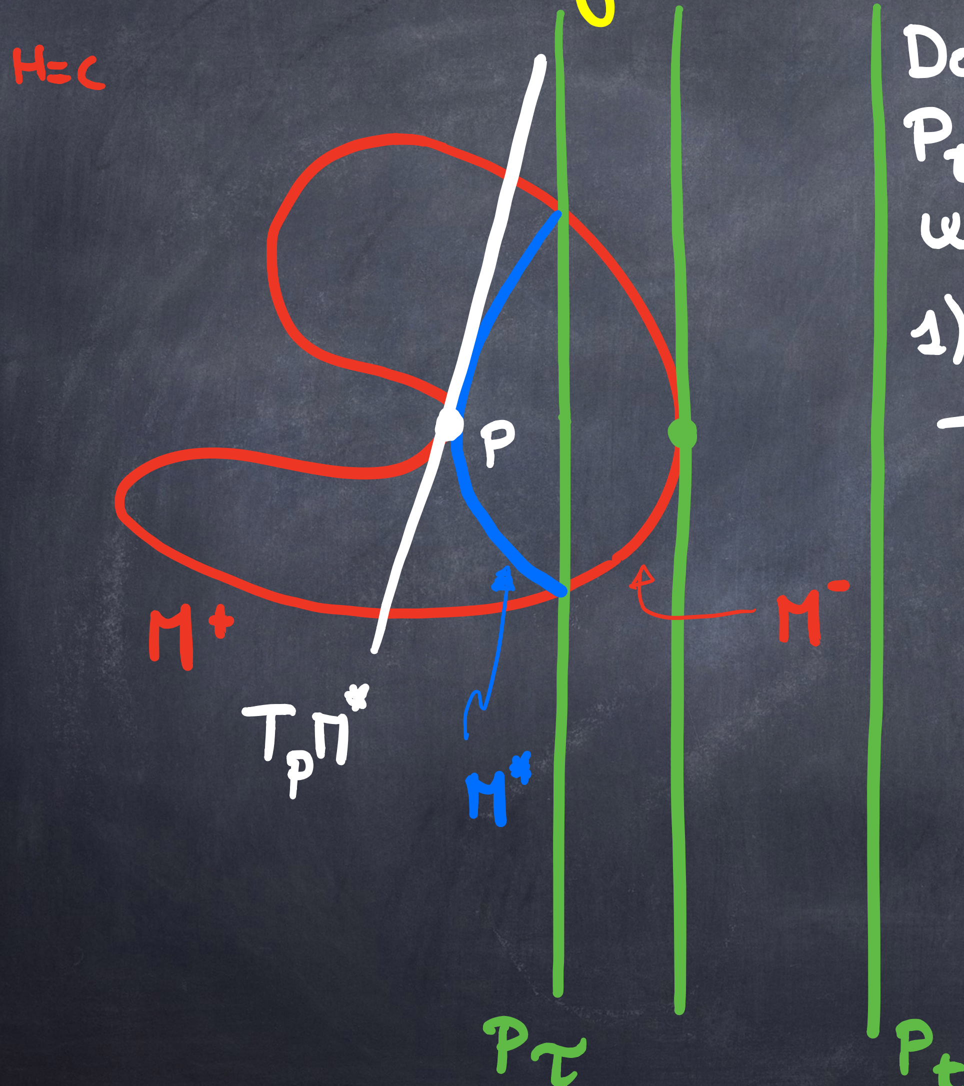


Do the reflection across P_τ till the first τ where

Alexandrov Reflection method



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$$T_pM^* = T_pM^+$$

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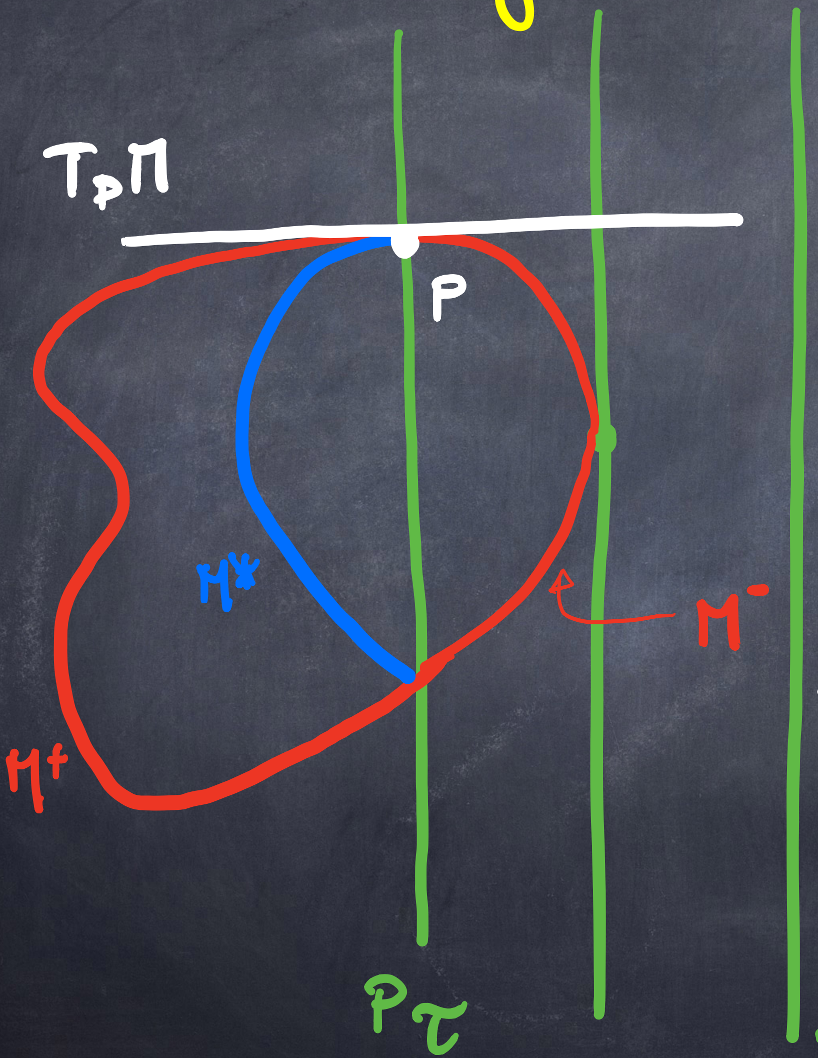

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$$T_p M^* = T_p M^+$$

or

2) $\exists \; p \in \partial M^- \cap P_\tau$ such that

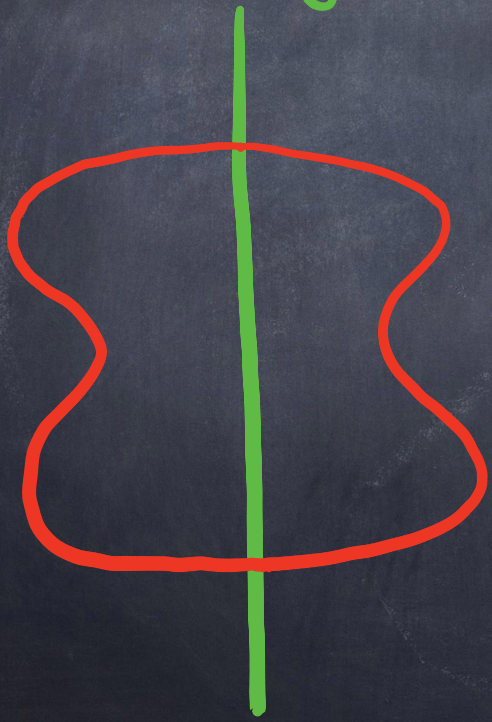
$$T_p M^* \equiv T_p M^+$$

(M^* stops to be a graph with bounded slope)

Alexandrov Reflection method

In both cases, one applies the maximum principle and obtain that $M^* \equiv M^+$

$\Rightarrow$ M is symmetric with respect to P_τ



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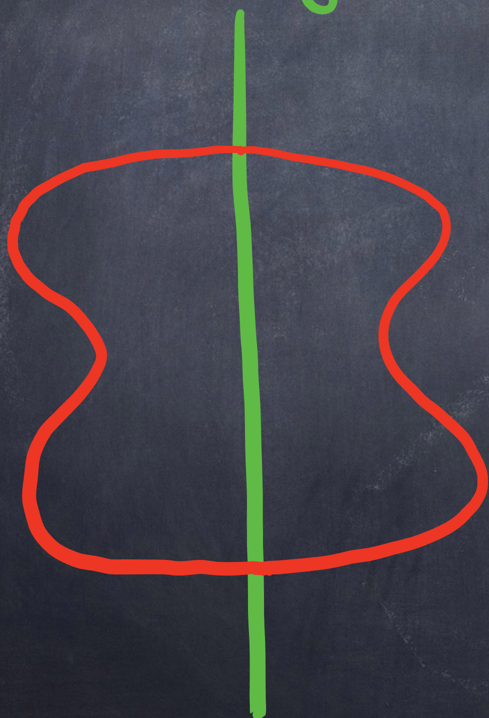

This can be performed in any direction.

Alexandrov Reflection method

In both cases, one applies the maximum principle and obtain that $M^* \equiv M^+$
$\Rightarrow$ M is symmetric with respect to P_τ

This can be performed in any direction.
Hence M is a round sphere.

Generalizations.

Alexandrov theorem holds in

- $\mathbb{R}^n$, $\mathbb{H}^n$ (hyperbolic space), $\mathbb{S}^n_+$ (half-sphere), for any n.

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Generalizations.

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- Some homogeneous simply connected Riemannian 3-mfolds $\mathbb{H}^2 \times \mathbb{R}$, $\mathbb{S}^2_+ \times \mathbb{R}$
- UNKNOWN in Nil_3, $\widetilde{PSL_2(\mathbb{R})}$, Berger spheres.

- Homogeneous simply connected 3-manifolds are: $\mathbb{R}^3$, $\mathbb{S}^3$, $\mathbb{H}^3$, the other 5 Thurston geometries ($\mathbb{H}^2\times\mathbb{R}$, $\mathbb{S}^2\times\mathbb{R}$, Nil_3, Sol_3, $\widetilde{PSL_2(\mathbb{R})}$), Berger spheres and some other Lie groups.

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- In the last 15 years, the study of constant mean curvature surfaces in homogeneous simply connected 3-manifolds has largely developed, producing a very rich theory in which geometric arguments interact with elliptic PDE, harmonic maps...

In manifolds different from $\mathbb{R}^3$, one looses the evident phisical interpretation of constant mean curvature surfaces in terms of soap bubbles.

BUT new interesting methods arise.

A side-effect is that the new method can be applied to problems in $\mathbb{R}^3$.

Another question one is able to answer

Bernstein theorem. A minimal entire graph in $\mathbb{R}^3$ is an affine plane.

[Bernstein 1915]

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[Almgreen '66] but there are counter-examples

for $n \geq 8$ [Bombieri-De Giorgi-Giusti '66]

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What about the shape of <u>stable</u> complete minimal surfaces?

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What about the shape of stable complete minimal surfaces?

N.B. A graph IS stable.

Theorem. A complete stable minimal surface in $\mathbb{R}^3$ is a plane.

[Do Carmo-Peng '79, Fischer Colbrie-Schoen '80, Pogorelov '81]

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Higher dimension?

UNKNOWN in $\mathbb{R}^n$, $3 < n \leq 7$.

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COUNTER-EXAMPLES for $n \geq 8$

[Bombieri-De Giorgi-Giusti '66]

$H \neq 0$

There is NO entire graph in $\mathbb{R}^n$ with constant mean curvature $H \neq 0$.

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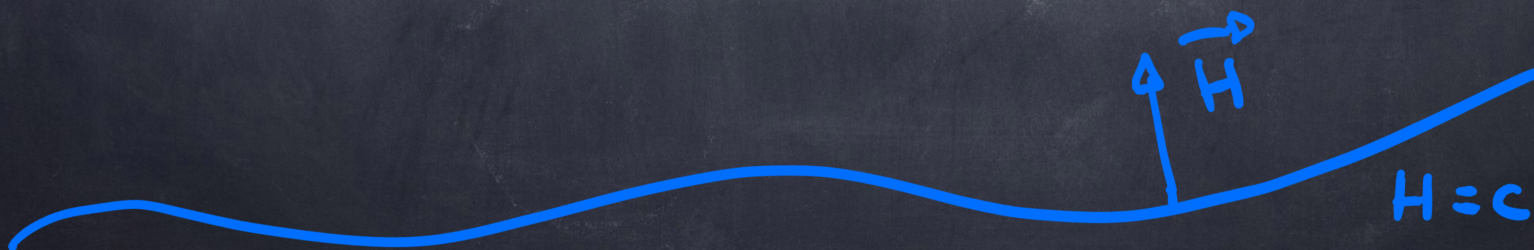

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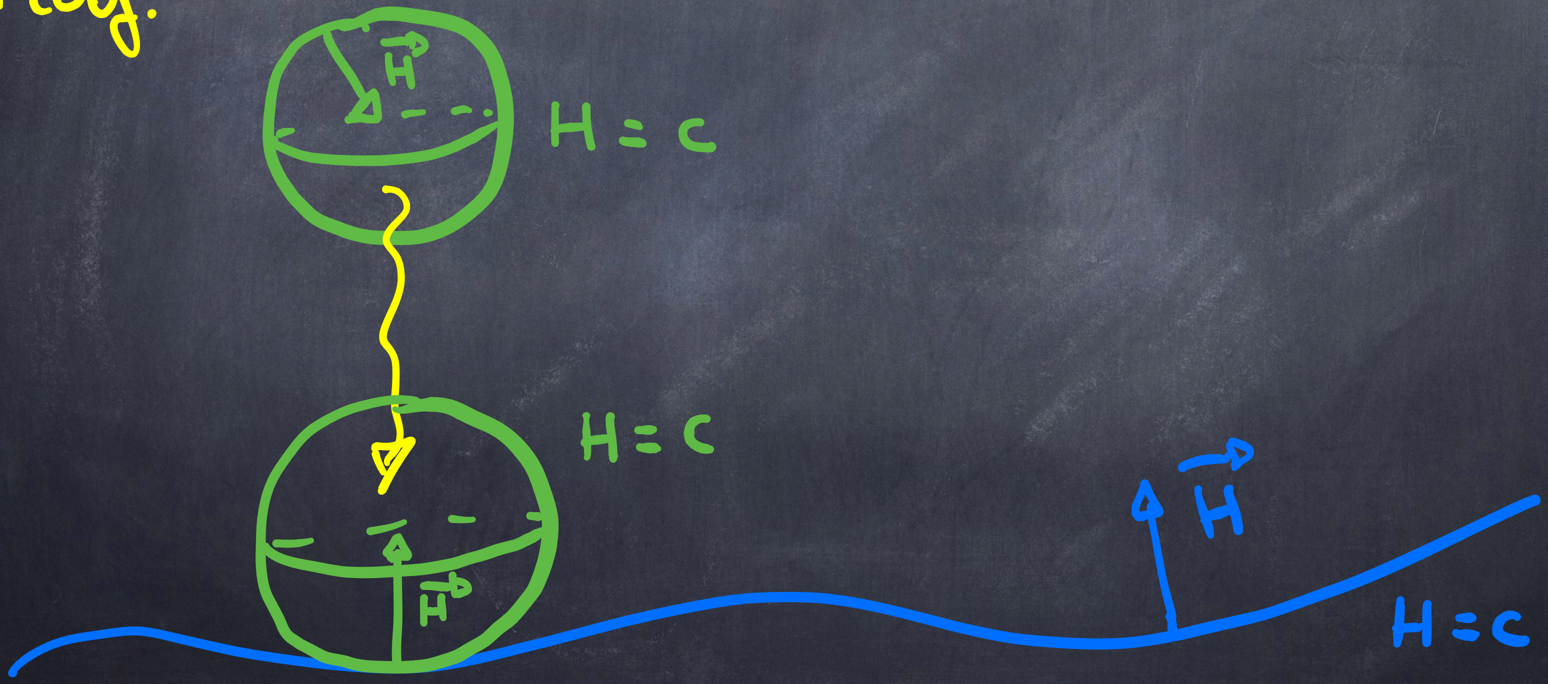


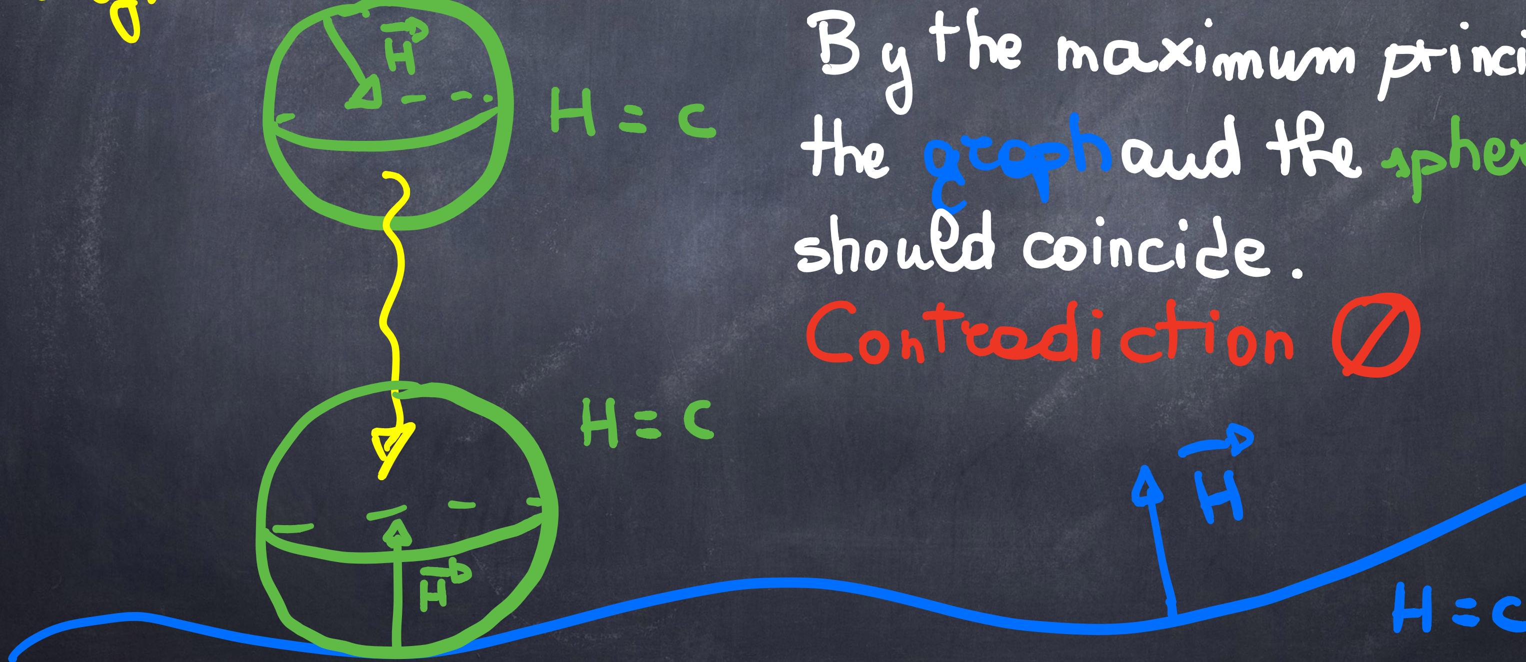
$H \neq 0$

There is **NO** entire graph in $\mathbb{R}^n$ with constant mean curvature $H \neq 0$.

Proof. Assume there is one.

By the maximum principle the graph and the sphere should coincide.

Contradiction $\emptyset$



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[n=3 Ros-Rosenberg 2001]

[n=4,5 Elbert-_-Rosenberg 2007]

PROCEEDINGS OF THE
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STABLE CONSTANT MEAN CURVATURE HYPERSURFACES

MARIA FERNANDA ELBERT, BARBARA NELLI, AND HAROLD ROSENBERG

(Communicated by Richard A. Wentworth)

Abstract. Let $\mathcal{N}^{n+1}$ be a Riemannian manifold with sectional curvatures uniformly bounded from below. When $n = 3, 4$, we prove that there are no complete (strongly) stable H-hypersurfaces, without boundary, provided $|H|$ is large enough. In particular, we prove that there are no complete strongly stable H-hypersurfaces in $\mathbb{R}^{n+1}$ without boundary, $H \neq 0$.

1. Introduction

Consider a Riemannian manifold $\mathcal{N}$ of dimension $n+1$ with sectional curvatures uniformly bounded from below; denote by $\sec(\mathcal{N})$ the infimum of the sectional curvatures of $\mathcal{N}$. Let M be an immersed submanifold of codimension one and let H be the mean curvature of M in the metric induced by the immersion. If H is constant, we call M an H-hypersurface. We prove the following diameter estimate.

Theorem 1. *Let $M^n \subset \mathcal{N}^{n+1}$ be a stable complete H-submanifold, $n = 3, 4$. There exists a constant $c = c(n, H, \sec(\mathcal{N}))$ such that for any $p \in M$ one has: $\mathrm{dist}_M(p, \partial M) \leq c$ whenever $|H| > 2\sqrt{|\min\{0, \sec(\mathcal{N})\}|}$.*

For the definition of stability, see Section 2. Particular cases of the previous Theorem in $\mathbb{R}^3$, $\mathbb{H}^3$, $\mathbb{H}^2 \times \mathbb{R}$ and any homogeneously regular three-manifold are proved in [9], [5], [7], [8], respectively.

We wonder if Theorem 1 holds in all dimensions.

Corollary 1. *Let M^n be a complete stable H-hypersurface of $\mathcal{N}^{n+1}$. If $n = 3, 4$ and $|H| > 2\sqrt{|\min\{0, \sec(\mathcal{N})\}|}$, then $\partial M \neq \emptyset$.*

In [12] it is proved that an H-hypersurface in $\mathbb{R}^{n+1}$, with finite total curvature, is minimal, so, if it is stable, it is a hyperplane (cf. [4]). For $n = 3, 4$, we are able to generalize this result in the following sense. We do not need the finite total curvature hypothesis on M, and the ambient space can be any manifold with uniformly bounded sectional curvature, provided the mean curvature $|H|$ is large enough (see Corollary 1).

As a consequence of the diameter estimate in Theorem 1, we have the Maximum Principle at Infinity.

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Theorem. There is NO complete stable hypersurface with constant mean curvature $H \neq 0$ in $\mathbb{R}^n$, $3 \leq n \leq 5$.

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[n=4,5 Elbert-_-Rosenberg 2007]

$n \geq 6$ UNKNOWN

[_ - Moraru, in progress]

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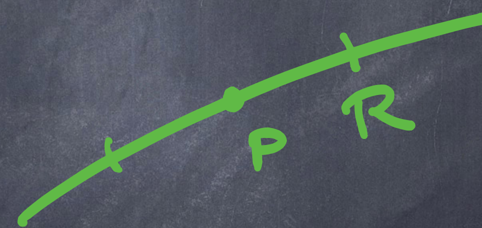
Idea of the proof. Assume such hypersurface M exists. Let $p \in M$ and $D_R(p)$ a disc contained in M.



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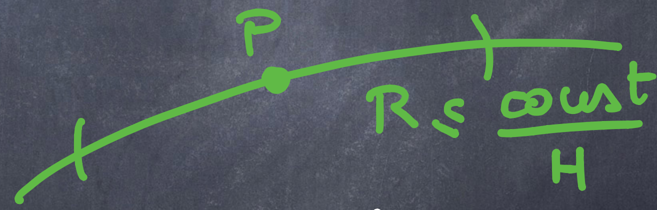

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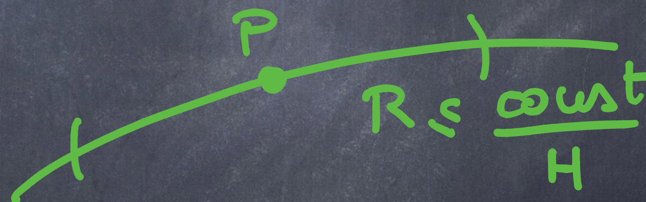

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The method looks impossible to extend in higher dimension.

Bernstein theorem and the Stability problem for Minimal Surfaces in the Heisenberg Space

Nil_3

Nil_3 is a 3-dimensional simply connected Lie group, endowed with a left invariant metric.

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- Nil_3 is a Riemannian submersion onto $\mathbb{R}^2$ (with constant bundle curvature).

A model for Nil_3 is $\mathbb{R}^3$ endowed with the Riemannian metric

$$ds^2 = dx_1^2 + dx_2^2 + \left(dx_3 + \tfrac{1}{2}(x_1 dx_2 - x_2 dx_1)\right)^2$$

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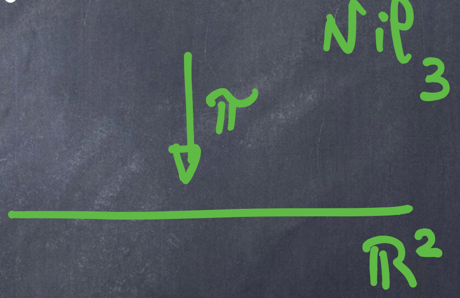
- The projection on the first two coordinates $\pi : Nil_3 \longrightarrow \mathbb{R}^2$ is a Riemannian submersion.



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$$ds^2 = dx_1^2 + dx_2^2 + \left(dx_3 + \tfrac{1}{2}(x_1 dx_2 - x_2 dx_1)\right)^2$$

- The projection on the first two coordinates $\pi : Nil_3 \longrightarrow \mathbb{R}^2$ is a Riemannian submersion.
- The fibers of π are geodesics and coincide with the integral curves of the Killing vector field ∂_3.
- $\dim(Iso(Nil_3)) = 4$

Minimal Graphs in Nil_3

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$$2H(u) := \operatorname{div}\left(\frac{Gu}{\sqrt{1+|Gu|^2}}\right) = 0$$

where div and $|\cdot|$ are in $\mathbb{R}^2$ and Gu is a vector field on Ω: $Gu = \nabla u + \frac{1}{2}(x_2 e_1 - x_1 e_2)$ and ∇u is in $\mathbb{R}^2$

Examples of minimal graphs

- The graph of $u(x_1, x_2) = ax_1 + bx_2 + c$, called umbrella.

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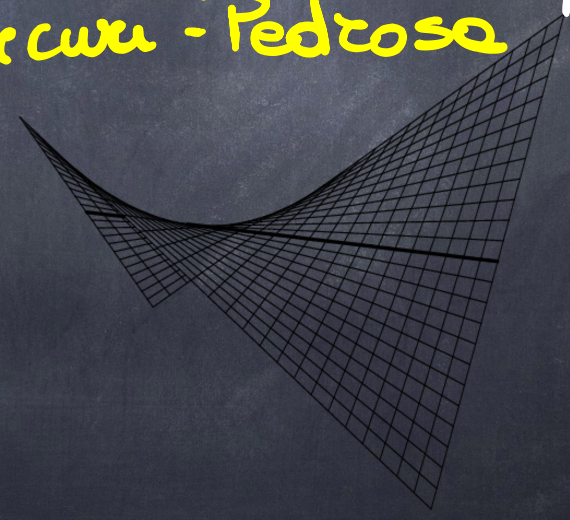
$$u(x_1,x_2)=\frac{1}{2}x_1x_2+\frac{\operatorname{sh}c}{2}\left[x_2\sqrt{1+x_2^2}+\operatorname{arcsh}x_2\right]$$

due to Figueroa-Mercuri-Pedrosa '99, Bekka-Sari '91.



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NO BERNSTEIN THEOREM

- Many entire minimal graphs with Abresch-Rosenberg differential on $\mathbb{C}$ or $\mathbb{D}$ (Fernandez-Mira 2011)

More examples.

- Vertical planes.
- Horizontal and vertical catenoids

[Daniel-Hauswirth 2009]

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- Horizontal and vertical catenoids [Daniel-Hauswirth 2009]
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Calc. Var. (2017) 56:27
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Calculus of Variations

CrossMark

Minimal graphs in Nil_3 : existence and non-existence results

B. Nelli[1] · R. Sa Earp[2] · E. Toubiana[3]

Received: 15 October 2016 / Accepted: 9 January 2017

Abstract We study the minimal surface equation in the Heisenberg space, Nil_3. A geometric proof of non existence of minimal graphs over non convex, bounded and unbounded domains is achieved for some prescribed boundary data (our proof holds in the Euclidean space as well). We solve the Dirichlet problem for the minimal surface equation over bounded and unbounded convex domains, taking bounded, piecewise continuous boundary value. We are able to construct a Scherk type minimal surface and we use it as a barrier to construct non trivial minimal graphs over a wedge of angle $\theta \in [\frac{\pi}{2}, \pi[$, taking non negative continuous boundary data, having at least quadratic growth. In the case of an half-plane, we are also able to give solutions (with either linear or quadratic growth), provided some geometric hypothesis on the boundary data are satisfied. Finally, some open problems arising from our work, are posed.

Mathematics Subject Classification 53A10 · 53C42 · 35J25

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The Stability Problem

Theorem. A complete, minimal, stable parabolic surface immersed in Nil_3 is either an entire graph or a vertical plane.

[Manzano-Perez-Rodriguez 2011]

- M is parabolic if every positive superharmonic function on M is constant.

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[Manzano-Perez-Rodriguez 2011]

- M is **parabolic** if every positive superharmonic function on M is constant.

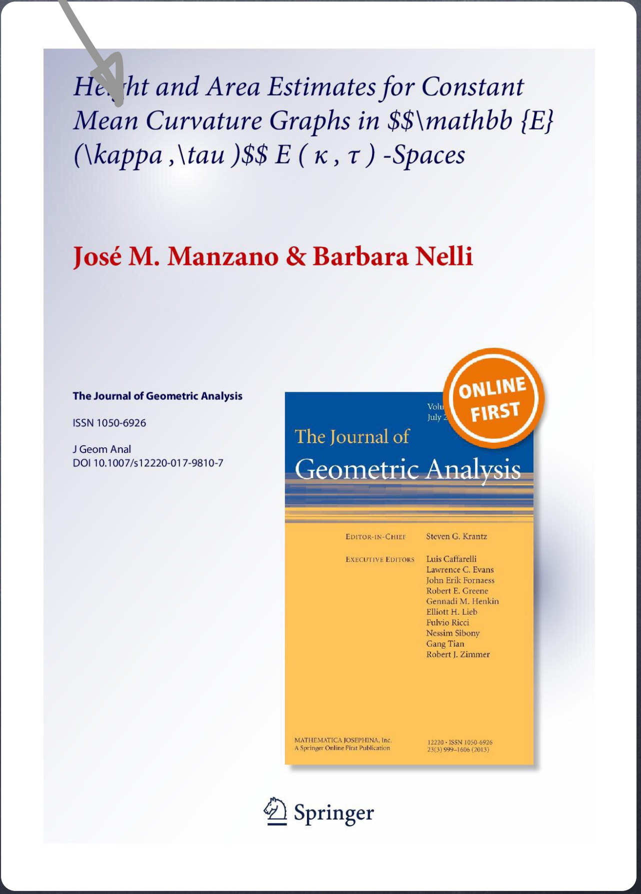
$\emptyset$ Every graph is stable BUT many graphs are <u>not</u> parabolic: umbrellas and many Fernandez-Mira graphs are hyperbolic.

Parabolicity ⟷ Area growth

- If a surface has quadratic area growth, then it is parabolic [Cheng-Yau 1976]

⟶ Study area growth of minimal graphs.

Theorem. An entire minimal graph M has extrinsic area growth between quadratic and cubic


Height and Area Estimates for Constant Mean Curvature Graphs in $$\mathbb {E}(\kappa ,\tau)$$ E (κ , τ) -Spaces

José M. Manzano & Barbara Nelli

The Journal of Geometric Analysis

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J Geom Anal
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Springer

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Ball of Nil_3


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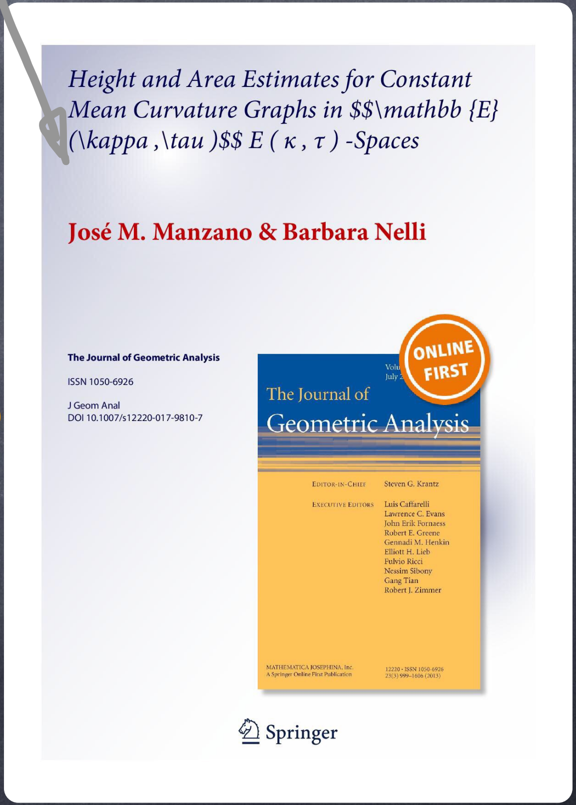
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CONJECTURE. The extrinsic growth of an entire minimal graph is **CUBIC**

[Manzano — 2017]

The proof of the area growth uses comparison arguments, integral estimates and gradient estimates, through a gradient estimates for entire space-like graphs with constant mean curvature in the Lorentz-Minkowsky space.

[Treibergs 1982] [Lee-Manzano 2017]

Using our estimate, we are able to understand the shape of some stable minimal surface in Nil_3.

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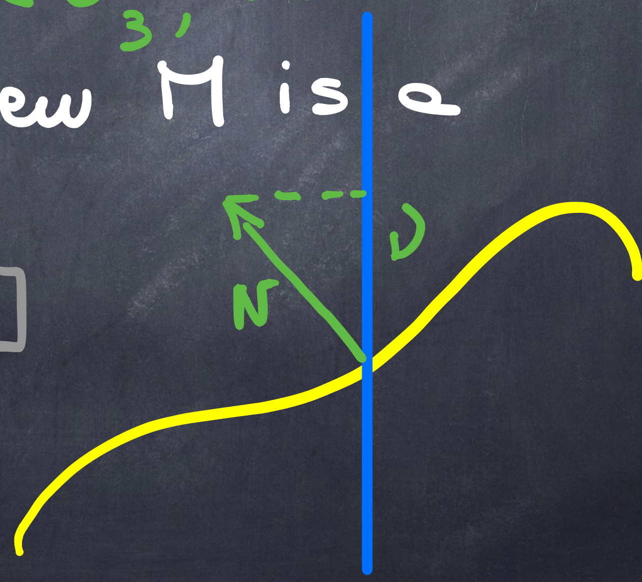
Theorem. Let M be a stable minimal surface immersed in Nil_3. If the angle function $\nu = \langle \partial_3, N \rangle$ is such that $\nu^2 \in L^1(M)$, then M is a vertical plane.

[Manzano - ___ 2017]



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If M is a graph:

$$\infty > \int_M \nu^2 \, dM = \int_{\mathbb{R}^2} \frac{d\sigma}{\sqrt{1+|Gu|^2}} \geq \int_0^\infty \frac{2\pi r \, dr}{\sqrt{1+B^2(1+r^2)^2}} = \infty$$

gradient estimates

Contradiction $\emptyset$

$\boxed{/}$

The classification of stable minimal surfaces in Nil_3 is related to the

Strong half-space conjecture

[Daniel-Meeks-Rosenberg 2011]

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Strong half-space CONJECTURE

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Strong half-space CONJECTURE

[Daniel-Meeks-Hauswirth 2011]

Two properly immersed minimal surfaces in Nil_3 that do not intersect are either two parallel vertical planes or an entire vertical graph and its image by a vertical translation.

The CONJECTURE is proved, provided:

- one of the surfaces is a vertical plane [Daniel-Hauswirth 2009]
- one of the surfaces is a graph [Daniel-Meeks-Rosenberg 2011]

Which is the relation between the classification of stable minimal surfaces in Nil_3 and the strong halfspace conjecture?

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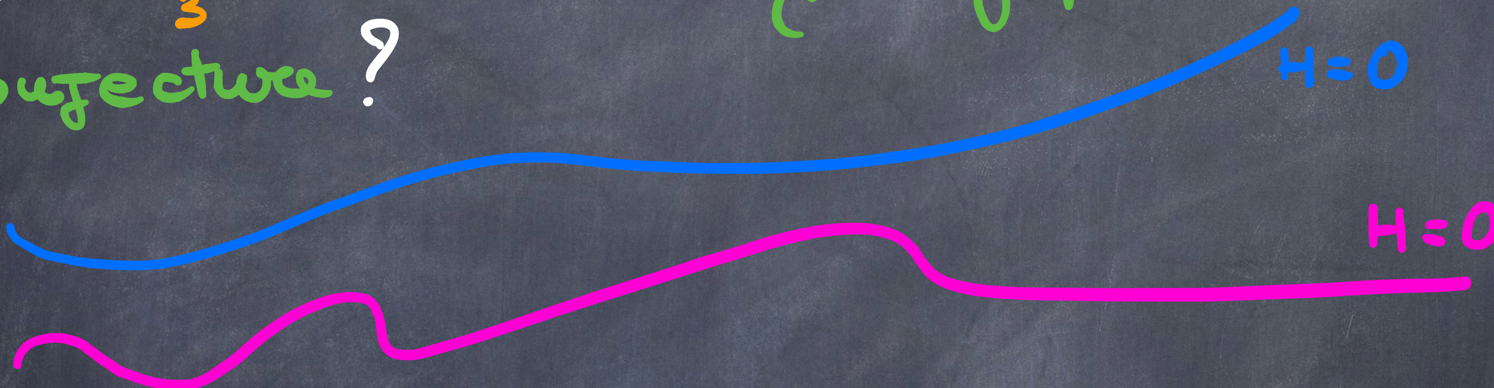

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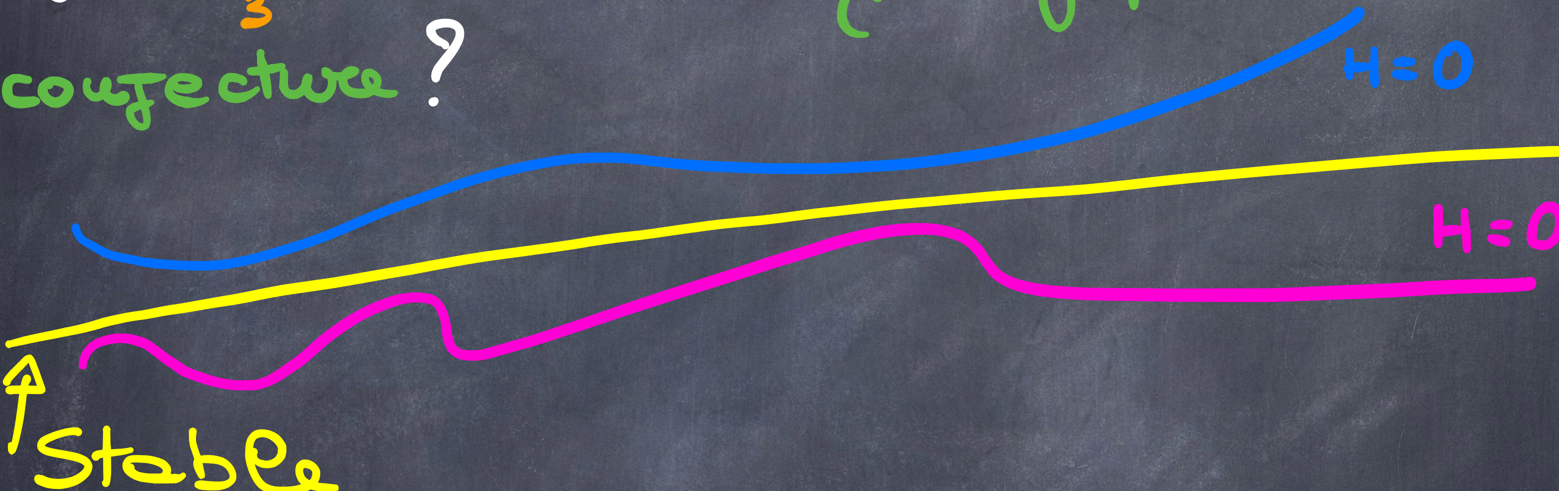

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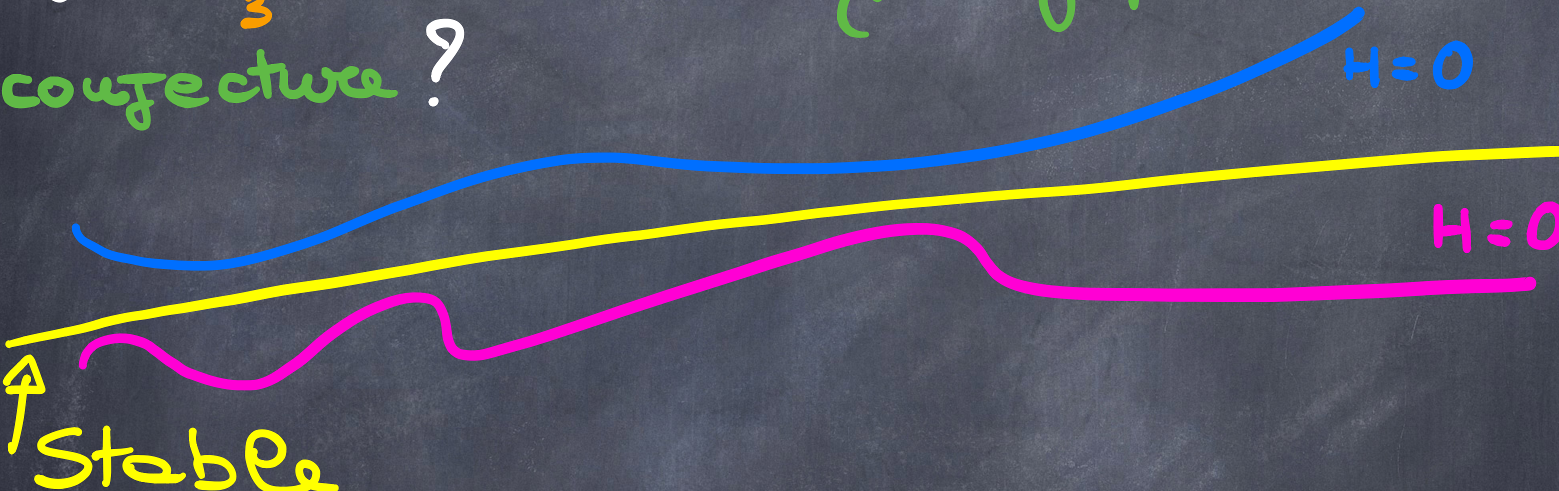

- If one proves that the stable surface is a graph of a vertical plane, then one gets the conjecture, using the partial results [DH] [DMR]

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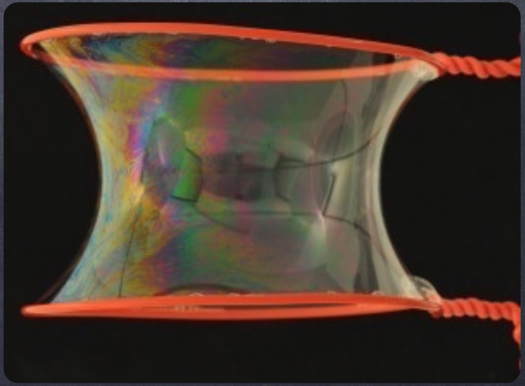
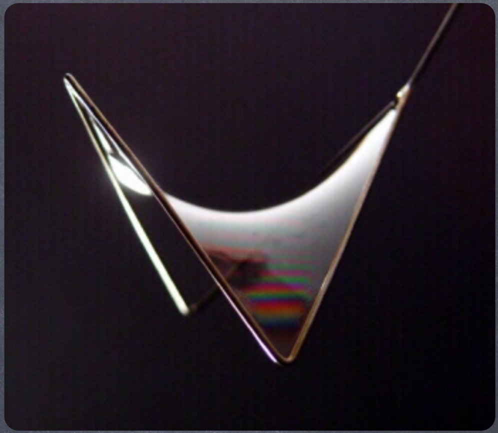
THANK YOU

Manchester



Catenoid

Nature

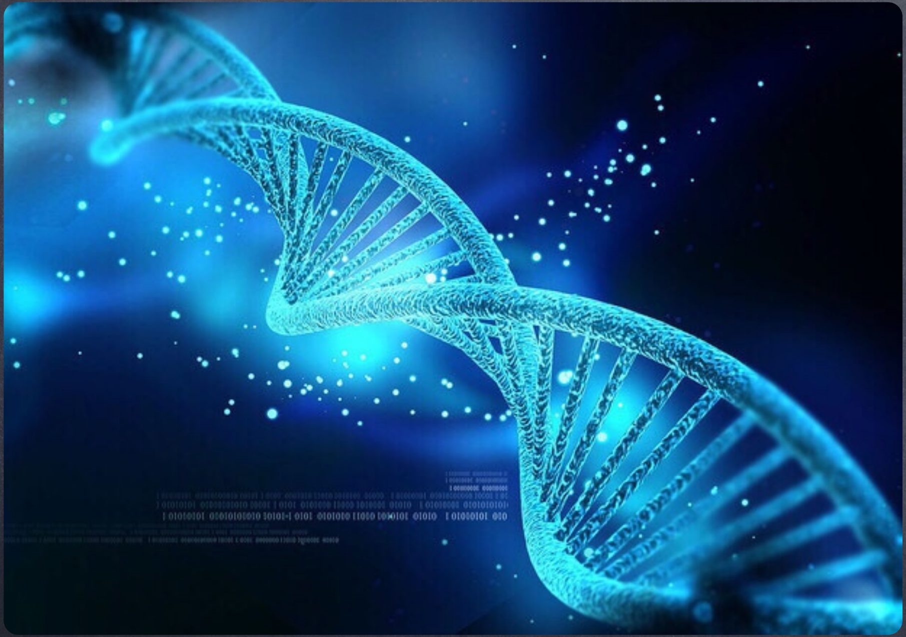


Saddle

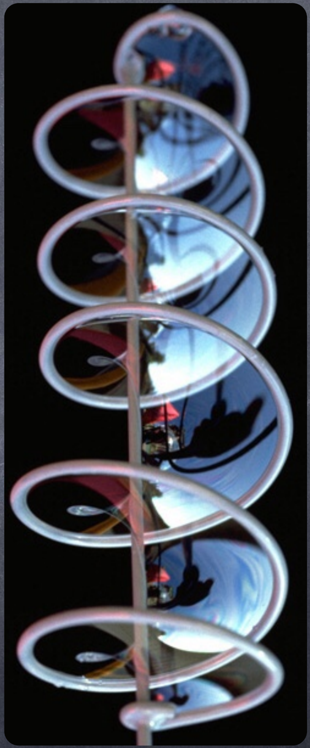
Tent

Thank you

Thank you



DNA



Helicoid