A General Family of Two Step Collocation Methods for Ordinary Differential Equations

R. D'Ambrosio*, M. Ferro† and B. Paternoster**

*Dipartimento di Matematica e Informatica, Universitá di Salerno, Italy, rdambrosio@unisa.it
†Dipartimento di Matematica e Applicazioni, Universitá di Napoli, Italy, maria.ferro@dma.unina.it
**Dipartimento di Matematica e Informatica, Universitá di Salerno, Italy, beapat@unisa.it

Abstract. We consider a general family of two step collocation methods for the numerical integration of Ordinary Differential Equations, which depends on the stage values at two consecutive step points. We describe two constructive techniques, discuss the order of the resulting methods, compute the nodes to obtain superconvergence and analyze their linear stability properties.

Keywords: Numerical methods for Ordinary Differential Equations, Collocation methods, Two step Runge–Kutta methods
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INTRODUCTION

We are concerned with an extension of multistep collocation methods for the numerical integration of an Initial Value Problem

$$\begin{cases} y'(x) = f(x,y(x)) \\ y(x_0) = y_0. \end{cases}$$

We assume that $x \in I := [x_0, X]$, $f : I \times \Omega \subset \mathbf{R}^d \to \mathbf{R}^d$. Let $I_h = \{x_h : x_0 < x_1 < ... < x_N = X\}$ be an uniform grid on I, where $h = \frac{X-x_0}{N}$.

Our aim is to derive a general class of multistep methods, which depends on the stage values at two consecutive step points, with the aim of higthen the order of the methods without increasing the computational cost.

The idea of collocation is old and well known in Numerical Analysis [1, 2, 3]. In order to advance from x_n to x_{n+1}, a collocation function is computed, usually an algebraic polynomial $P(x)$, which interpolates the numerical solution in the step point x_n, and satisfies the ODEs in the points $x_n + c_i h$, where $\{c_1, c_2, ..., c_m\}$ are m real numbers (typically between 0 and 1), called *collocation abscissas*, that is

$$P(x_n) = y_n, \tag{1}$$
$$P'(x_n + c_i h) = f(x_n + c_i h, P(x_n + c_i h)), \qquad i = 1, 2, ..., m. \tag{2}$$

A numerical method arises, which takes the solution in x_{n+1} by

$$y_{n+1} = P(x_{n+1}). \tag{3}$$

It is known that one step collocation methods are a subset of implicit Runge-Kutta methods, which can be represented by the Butcher array

$$\begin{array}{c|cccc} c_1 & a_{11} & a_{12} & & a_{1m} \\ c_2 & a_{21} & a_{22} & & a_{2m} \\ \dots & \dots & \dots & & \dots \\ c_m & a_{m1} & a_{m2} & & a_{mm} \\ \hline & b_1 & b_2 & & b_m \end{array}$$

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where

$$a_{ij} = \int_0^{c_i} L_j(t)dt, \quad i = 1,2,...,m \quad b_j = \int_0^1 L_j(t)dt, \quad j = 1,2,...,m$$

and $L_j(t)$, $j = 1,...,m$, are fundamental Lagrange polynomials. Moreover the maximum attainable order is at most $2m$, and it's obtained by using Gaussian collocation points [1, 2].

Guillou & Soulé introduced multistep collocation methods [4], by adding interpolation conditions in the previous k step points, so that the collocation polynomial is defined by

$$P(x_{n+i}) = y_{n+i} \quad i = 0,1,...,k-1; \tag{4}$$

$$P'(x_{n+k-1} + c_j h) = f(x_{n+k-1} + c_j h, P(x_{n+k-1} + c_j h)) \quad j = 1,...,m. \tag{5}$$

The numerical solution is then

$$y_{n+k} = P(x_{n+k}) \tag{6}$$

In [3] it is proved that the method is equivalent to a multistep Runge–Kutta method, and the points which guarantee the superconvergence are called Radau points. Lie and Norsett analyzed the order of the resulting methods [5].

In this paper we extend the idea of multistep collocations methods, by considering the case of $k = 2$, and by adding some extra collocation conditions, so that the resulting methods depend on stage values at two consecutive step points. More in details, the collocation polynomial is then defined by the following conditions:

$$P(x_n) = y_n, \qquad P(x_{n+1}) = y_{n+1} \tag{7}$$

$$P'(x_{n-1} + c_i h) = f(x_{n-1} + c_i h, P(x_{n-1} + c_i h)), \qquad i = 1,2,...,m \tag{8}$$

$$P'(x_n + c_i h) = f(x_n + c_i h, P(x_n + c_i h)), \qquad i = 1,2,...,m. \tag{9}$$

The previous problem constitutes a Hermite interpolation problem with incomplete data. Then the collocation polynomials takes the form

$$P(x_n + sh) = \phi_0(s) y_{n-1} + \phi_1(s) y_n + h \sum_{j=1}^{m} \Big[\chi_j(s) f(P'(x_{n-1} + c_j h)) + \psi_j(s) f(P'(x_n + c_j h))\Big], \tag{10}$$

with $s \in [0,1]$, $\quad n = 1,2,...,N-1$.

CONSTRUCTION OF THE METHOD

Introducing the dimensionless coordinate $t = \frac{x - x_{n+1}}{h}$, the expression of the basis functions $\phi_i(t)$ $(i = 0,1)$, $\psi_j(t)$ and $\chi_j(t)$ in (10) $(j = 1,2,...m)$ can be obtained by applying the conditions (7), (8), (9) of interpolation and collocation, i.e.

$$\begin{array}{rcl} \phi_i(t_j) & = & \delta_{ij} \qquad i,j = 0,1 \\ \phi_i'(c_j - 1) & = & 0, \qquad \phi_i'(c_j) = 0 \qquad i = 0,1, \qquad j = 1,...,m \\ \chi_i(t_j) & = & 0 \qquad i = 1,...,m, \qquad j = 0,1 \\ \chi_i'(c_j - 1) & = & \delta_{ij}, \qquad \chi_i'(c_j) = 0 \qquad i,j = 1,...,m \\ \psi_i(t_j) & = & 0 \qquad i = 1,...,m, \qquad j = 0,1 \\ \psi_i'(c_j - 1) & = & 0, \qquad \psi_i'(c_j) = \delta_{ij} \qquad i,j = 1,...,m \end{array} \tag{11}$$

where $t_0 = -1$, $t_1 = 0$.

From the previous conditions some linear systems arise, whose unknowns are the coefficients of the polynomials of the generalized Lagrange basis. These linear systems can be solved (apart from some exceptional values for collocation abscissa), giving the expressions of the collocation poylnomial $P(x)$.

It is possible to prove that $P(x)$ provides a uniform approximation to the solution of order $2m+1$ for any choice of the collocation abscissas $\{c_1, c_2, ..., c_m\}$.

Theorem 1.
The method above defined is equivalent to a General Two–Step Runge–Kutta (GTSRK) method [6, 7]

$$
\begin{aligned}
Y^j_{n+k} &= u_{j,1}y_{n+k-2} + u_{j,2}y_{n+k-1} + h\sum_{s=1}^{m}[a_{js}f(x_{n+k-2}+c_sh, Y^s_{n+k-1}) + b_{js}f(x_{n+k-1}+c_sh, Y^s_{n+k}),\\
y_{n+k} &= \theta_1 y_{n+k-2} + \theta_2 y_{n+k-1} + h\sum_{j=1}^{m}[v_j f(x_{n+k-2}+c_jh, Y^j_{n+k-2}) + w_j f(x_{n+k-1}+c_jh, Y^j_{n+k-1})
\end{aligned}
\tag{12}
$$

where

$$
\begin{aligned}
\theta_i &= \phi_i(1), \qquad i=1,2, \qquad j=1,...,m;\\
u_{j,i} &= \phi_i(c_j), \qquad j,s=1,...,m;\\
a_{js} &= \chi_j(c_s), \qquad j,s=1,...,m;\\
b_{js} &= \psi_j(c_s), \qquad j,s=1,...,m;\\
v_j &= \chi_j(1), \qquad j=1,...,m;\\
w_j &= \psi_j(1), \qquad j=1,...,m.
\end{aligned}
$$

◇

Butcher's array of the new method is

$$
\begin{array}{c|c|c}
u & A & B\\
\hline
\theta & v^T & w^T
\end{array}
=
\begin{array}{c|cccc|cccc}
u_1 & a_{11} & a_{12} & & a_{1m} & b_{11} & b_{12} & & b_{1m}\\
u_2 & a_{21} & a_{22} & & a_{2m} & b_{21} & b_{22} & & b_{2m}\\
. & . & . & & . & . & . & & .\\
. & . & . & & . & . & . & & .\\
. & . & . & & . & . & . & & .\\
u_m & a_{m1} & a_{m2} & & a_{mm} & b_{m1} & b_{m2} & & b_{mm}\\
\hline
\theta & v_1 & v_2 & & v_m & w_1 & w_2 & & w_m
\end{array}
$$

where $c = (A+B)e - u$, and $e = [1, ..., 1]^T \in R^m$.

The GTSRK method can also be regarded as a General Linear Method [8, 9].

The reason of interest in methods GTSRK lies in the fact that, advancing from x_i to x_{i+1} we only have to compute Y^j_{n+k}, because Y^j_{n+k-1} have alreay been evaluated in the previous step. Therefore the computational cost of the method depends on the matrix $B = (b_{ij})$, while the matrix $A = (a_{ij})$ adds extra degrees of freedom, without need for extra function evaluation. TSRK methods based on algebraic and trigonometric polynomials have been considered in [10]; the extension of TSRK methods for $y'' = f(x,y)$ have been introduced in [11].

The method can be derived by extending the technique used by Hairer and Wanner [3], and by Lie and Norsett [5]. Moreover, following the technique described in [5], we can compute the so–called Radau nodes, to obtain superconvergence.

LINEAR STABILITY ANALYSIS

By applying the method to the test equation $y' = \lambda y, \qquad Re(\lambda) < 0$, the following recurrence arises

$$\begin{aligned} P(x_n + c_j h) &= Q[\phi_0(c_j) y_{n-1} + \phi_1(c_j) y_n + zAP(x_{n-1} + c_j h)] \\ y_{n+1} &= (\varphi_0(1) + zw^T Q\varphi_0(c)) y_{n-1} + (\varphi_1(1) + zw^T Q\varphi_0(c)) y_n + z(v^T + zw^T QA) P(t_{n-1} + ch) \end{aligned}$$

where $A = [\chi_j(c_i)]_{i,j=1}^m$, $B = [\psi_j(c_i)]_{i,j=1}^m$, $z = h\lambda$, $Q = (I - zB)^{-1}$.

In compact form,

$$\begin{pmatrix} y_{n+1} \\ y_n \\ P(x_n + c_j h) \end{pmatrix} = M(z) \begin{pmatrix} y_n \\ y_{n-1} \\ P(x_{n-1} + c_j h) \end{pmatrix}$$

where the expression of the stability matrix is given by

$$M(z) = \begin{pmatrix} \varphi_1(1) + zw^T Q\varphi_1(c) & \varphi_0(1) + zw^T Q\varphi_0(c) & z(v^T + zw^T QA) \\ 1 & 0 & 0 \\ Q\varphi_1(c) & Q\varphi_0(c) & zQA \end{pmatrix}$$

The method is A–stable if the eigenvalues of the stability matrix are less 1 for $Re(\lambda) < 0$.

We didn't find A–stable methods within this class, but wide stability regions exist, so that the methods are suitable for the numerical integration of stiff system.

The knowledge of the collocation polynomial, which provides a continuous approximation of the solution, allows a cheap variable stepsize implementation. In [12] we introduce a modification in the technique, by relaxing some of the collocation conditions, in order to have free parameters in the method, to be used to get A-stability.

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