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Continuous Two-Step Runge–Kutta Methods for Ordinary Differential Equations

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Abstract New classes of continuous two-step Runge-Kutta methods for the numerical solution of ordinary differential equations are derived. These methods are developed imposing some interpolation and collocation conditions, in order to obtain desirable stability properties such as A-stability and L-stability. Particular structures of the stability polynomial are also investigated.

Keywords Two-step collocation methods · A-stability · L-stability · quadratic stability functions · Runge-Kutta stability.

1 Introduction

It is the purpose of this paper to develop a special family of two-step continuous methods of the type

$$\begin{cases} P(t_n + sh) = \varphi_0(s)y_{n-1} + \varphi_1(s)y_n \\ \quad + h\displaystyle\sum_{j=1}^{m}\Big(\chi_j(s)f(P(t_{n-1}+c_jh)) + \psi_j(s)f(P(t_n+c_jh))\Big), \\ y_{n+1} = P(t_{n+1}), \end{cases} \tag{1.1}$$

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with $s \in (0,1]$, $n = 1,2,\dots,N-1$, where N is the number of grid points, for the numerical solution of initial value problems based on ordinary differential equations

$$\begin{cases} y'(t) = f(y(t)), \quad t \in [t_0, T], \\ y(t_0) = y_0 \in \mathbb{R}^d, \end{cases} \tag{1.2}$$

with $f : [t_0, T] \times \mathbb{R}^d \to \mathbb{R}^d$. It is assumed that the function f is sufficiently smooth, in such a way that the problem (1.2) is well-posed. Setting

$$Y_j^{[n-1]} = P(t_{n-1} + c_j h), \quad Y_j^{[n]} = P(t_n + c_j h), \quad j = 1,2,\dots,m,$$

the method (1.1) can be written as two-step Runge-Kutta (TSRK) method of the form

$$\begin{cases} y_{n+1} = \theta y_{n-1} + \widetilde{\theta} y_n + h \sum_{j=1}^{m} \Big(v_j f(Y_j^{[n-1]}) + w_j f(Y_j^{[n]}) \Big), \\ Y_i^{[n]} = u_i y_{n-1} + \widetilde{u}_i y_n + h \sum_{j=1}^{m} \Big(a_{ij} f(Y_j^{[n-1]}) + b_{ij} f(Y_j^{[n]}) \Big), \end{cases} \tag{1.3}$$

with $i = 1,2,\dots,m$, $n = 1,2,\dots,N-1$ and

$$\begin{aligned} &\theta = \varphi_0(1), \quad \widetilde{\theta} = \varphi_1(1), \quad v_j = \chi_j(1), \quad w_j = \psi_j(1), \\ &u_i = \varphi_0(c_i),\ \widetilde{u}_i = \varphi_1(c_i),\ a_{ij} = \chi_j(c_i),\ b_{ij} = \psi_j(c_i). \end{aligned}$$

TSRK methods were introduced by Jackiewicz and Tracogna [22] and further investigated in [1], [3], [9], [10], [11], [18], [21],[24], [30], and [31]. Continuous methods (1.1) provide an approximation to the solution $y(t)$ of (1.2) on the whole interval of integration, and not only in the gridpoints $\{t_n\}$ as in the case of discrete TSRK methods (1.3).

Different approaches to the construction of continuous TSRK methods are presented in [4], [6] and [23]. TSRK methods for delay differential equations are considered in [2], [5] and for Volterra integral equations in [12].

The continuous approximant $P(t_n + sh)$ in (1.1), expressed as linear combination of the basis functions

$$\{\varphi_0(s),\ \varphi_1(s),\ \chi_j(s),\ \psi_j(s),\ j = 1,2,\dots,m\},$$

is an algebraic polynomial satisfying some appropriate interpolation and collocation conditions. We relax some of these conditions, in order to derive continuous TSRK methods of order $p = m$ up to 4, having strong stability properties, such as A-stability and L-stability. We also discuss the conditions to impose on the parameters of the methods in order to obtain Runge–Kutta stability (i.e. the stability matrix has only one nonzero eigenvalue, see [8,21] and references therein) or quadratic stability polynomials [11].

The paper is organized as follows: in section 1 we discuss the general strategy we follow in order to derive continuous A-stable and L-stable methods with appropriate properties, e.g. θ=0, $\mathbf{u} = \mathbf{0}$ and FSAL (i.e. the first stage of the next step is the same as the last stage of the current step, see [8,25]). In sections 2, 3, 4 and 5 we derive methods of order $p = m = 1,2,3$ and 4, possessing the mentioned properties. In section 6 we discuss the conditions to obtain Runge–Kutta stability. Section 7 contains some examples of methods derived using the results obtained in the previous sections. In section 8 we make some concluding remarks.

2 Construction of methods

The methods we aim to derive can be recognized as special multistep collocation methods [13], [14], [15] [16], [17], [20], [26], in the sense that, in order to advance from the point t_n to the point t_{n+1}, the continuous approximant (1.1) is derived imposing some appropriate interpolation and collocation conditions, only with respect to the points t_n and $t_n + c_j h$, i.e.

$$P(t_n) = y_n, \tag{2.1}$$
$$P'(t_n + c_j h) = f(P(t_n + c_j h)), \quad j = 1, 2, \ldots, m. \tag{2.2}$$

These conditions imply that

$$\varphi_0(0) = 0, \quad \varphi_1(0) = 1, \quad \chi_j(0) = 0, \quad \psi_j(0) = 0, \tag{2.3}$$
$$\varphi\prime_0(c_i) = 0, \quad \varphi\prime_1(c_i) = 0, \quad \chi\prime_j(c_i) = 0, \quad \psi\prime_j(c_i) = \delta_{ij}, \tag{2.4}$$

for i, j= 1, 2, ..., m, where δ_{ij} is the usual Kronecker delta. We propose two different classes of continuous methods of this form. First, we impose the interpolation condition (2.1) only, obtaining a family of *interpolation based TSRK methods*. Then we follow the more general appoach and impose the whole set of conditions (2.1), (2.2), achieving a class of *interpolation-collocation based TSRK methods*. The strategy we follow in the construction of these methods can be summarized as follows.

First of all, we fix the polynomials $\varphi_0(s)$, $\chi_j(s)$, j=1, 2, ..., m. Interpolation based TSRK methods with $m = 1$ (which can be exploited in a complete systematic way) are derived imposing

$$\varphi_0(s) = p_0 s, \quad \chi(s) = q_0 s, \tag{2.5}$$

with $p_0,\ q_0 \in \mathbb{R}$, while for interpolation-collocation TSRK methods with $m = 1$, we infer from (2.4) that

$$\varphi_0'(s) = \chi_j'(s) = (s - c),$$

therefore, if $\pi_1(s)$ is the primitive function of $\varphi_0'(s)$ and $\chi'(s)$ such that $\varphi_0(0) = \chi(0) = 0$, we impose

$$\varphi_0(s) = \alpha_0 \pi_1(s), \qquad \chi(s) = \beta_0 \pi_1(s), \tag{2.6}$$

with $\alpha_0,\ \beta_0 \in \mathbb{R}$. In the case $m \geq 2$, in order to carry out a more general analysis, we also ask for methods such that $\theta = 0$ and $\mathbf{u} = \mathbf{0}$. This choice is desirable in order to simplify the systems of order conditions, without loss in terms of stability and order, as Jackiewicz and Tracogna themselves stated in [22], where they first introduced TSRK methods. As a consequence of these choices, new conditions on $\varphi_0(s)$ arise, i.e.

$$\varphi_0(1) = 0, \quad \varphi_0(c_i) = 0, \quad i = 1, 2, \ldots, m, \tag{2.7}$$

which can be fulfilled in the following ways:

- for interpolation based methods, we choose $c_1 = 0,\ c_m = 1$, in order to obtain a family of FSAL methods with

  $$\varphi_0(s) = s(s - c_2) \cdots (s - c_{m-1})(s - 1), \quad \chi_j(s) = q_j s, \quad j = 1, 2, \ldots, m, \tag{2.8}$$

  with $q_j \in \mathbb{R}$, $j = 1, 2, \ldots, m$. It is well known that FSAL methods are suitable for efficient implementation (see [8,25]).

- for interpolation-collocation based methods with $m \geq 2$, we impose

$$\varphi_0(s) = 0, \quad \chi_j(s) = \beta_j \pi_m(s), \quad j = 1, 2, \ldots, m, \tag{2.9}$$

with $\beta_j \in \mathbb{R}$, $j = 1, 2, \ldots, m$, where $\pi_m(s)$ is the primitive function of $\chi_j'(s)$ such that $\chi_j(0) = 0$.

As we aim for methods of order $p = m$, we impose the corresponding set of order conditions, stated in the following theorem (see [13], [14]).

Theorem 2.1 *Assume that the function $f(y)$ is sufficiently smooth. Then the method (1.1) has uniform order p if the following conditions are satisfied*

$$\begin{cases} \varphi_0(s) + \varphi_1(s) = 1, \\ \dfrac{(-1)^k}{k!}\varphi_0(s) + \displaystyle\sum_{j=1}^{m} \left(\chi_j(s) \frac{(c_j - 1)^{k-1}}{(k-1)!} + \psi_j(s) \frac{c_j^{k-1}}{(k-1)!} \right) = \frac{s^k}{k!}, \end{cases} \tag{2.10}$$

$s \in [0, 1]$, $k = 1, 2, \ldots, p$. $\square$

The unknown basis functions $\varphi_1(s)$, $\psi_j(s)$, j=1, 2, …, m, can then be computed as solutions of (2.10).

We next compute the stability polynomial of the obtained methods, i.e. the characteristic polynomial of the stability matrix related to the class of methods (1.1). The general expression of the stability matrix of (1.1) has already been derived in [13], [14] and takes the following form

$$M(z) = \begin{bmatrix} M_{11}(z) & M_{12}(z) & M_{13}(z) \\ 1 & 0 & 0 \\ Q\varphi_1(c) & Q\varphi_0(c) & zQA \end{bmatrix} \in \mathbb{C}^{(m+2)\times(m+2)}, \tag{2.11}$$

where

$$\begin{aligned} M_{11}(z) &= \varphi_1(1) + zw^T Q\varphi_1(c), \\ M_{12}(z) &= \varphi_0(1) + zw^T Q\varphi_0(c), \\ M_{13}(z) &= z(v^T + zw^T QA), \end{aligned}$$

and

$$Q = (I - zB)^{-1} \in \mathbb{C}^{m\times m},$$

in order to investigate the stability properties of the developed methods. It is possible to prove that, in correspondence of the above stated choices of the basis functions, the stability polynomial is of the type

$$p(\omega, z) = \omega^m (p_2(z)\omega^2 + p_1(z)\omega + p_0(z)), \tag{2.12}$$

and, therefore, the stability properties of the corresponding methods depend on the quadratic function (cfr. [11])

$$\widetilde{p}(\omega, z) = p_2(z)\omega^2 + p_1(z)\omega + p_0(z). \tag{2.13}$$

Such an expression of the stability polynomial is suitable and desirable for many reasons. In fact, it is well known that the construction of high order methods which are A-stable (i.e. the roots ω_1, ω_2 of the polynomial $\widetilde{p}(\omega, z)$ defined by (2.13) lie in the unit circle, for all $z \in \mathbb{C}$ such that $\mathrm{Re}(z) \leq 0$) and L-stable (i.e. the roots of the polynomial $\widetilde{p}(\omega, z)/p_2(z)$ tend to zero as $z \to -\infty$) is a strongly nontrivial task. A very helpful tool in this context is the Schur criterion [29] (see also [25]), which allows us to determine the parameters of A-stable methods. The Schur criterion for a general k^{th} polynomial can be formulated as follows. Consider the polynomial

$$\phi(w) = d_k w^k + d_{k-1} w^{k-1} + \cdots + d_1 w + d_0,$$

where d_i are complex coefficients, $d_k \neq 0$ and $d_0 \neq 0$. $\phi(w)$ is said to be a Schur polynomial if all its roots w_i, $i = 1, 2, \ldots, k$, are inside of the unit circle. Define

$$\widehat{\phi}(w) = \overline{d}_0 w^k + \overline{d}_1 w^{k-1} + \cdots + \overline{d}_{k-1} w + \overline{d}_k,$$

where $\overline{d}_i$ is the complex conjugate of d_i. Define also the polynomial

$$\phi_1(w) = \frac{1}{w}\Big(\widehat{\phi}(0)\phi(w) - \phi(0)\widehat{\phi}(w)\Big)$$

of degree at most $k-1$. We have the following theorem.

Theorem 2.2 *(Schur [29]). $\phi(w)$ is a Schur polynomial if and only if*

$$|\widehat{\phi}(0)| > |\phi(0)|$$

and $\phi_1(w)$ is a Schur polynomial.

Roughly speaking, the Schur criterion allows us to investigate the stability properties of a k^{th} degree polynomial, looking at the roots of a polynomial of lower degree (i.e. $k-1$). Iterating this process, the last step consists in the investigation of the root of a linear polynomial, plus some additional conditions. Of course, we can succeed in finding A-stable methods implementing the Schur criterion in a symbolic environment only if the stability polynomial has low degree. For TSRK methods, it is natural to investigate the conditions to impose in order to force the stability properties to depend on a polynomial of degree 2, as discussed in [11], or on a linear polynomial. For this reason, we also discuss the conditions to accomplish in order to achieve the so-called Runge–Kutta stability (i.e. the stability matrix has one nonzero eigenvalues and, therefore, the stability properties of the corresponding methods depend on a linear polynomial).

Once we have obtained A-stability, L-stable is obtained by requiring that the parameters satisfy the nonlinear system of equations

$$\lim_{z \to -\infty} \frac{p_0(z)}{p_2(z)} = 0, \quad \lim_{z \to -\infty} \frac{p_1(z)}{p_2(z)} = 0. \tag{2.14}$$

In the following sections, we discuss the details of the construction of highly stable m-stage methods, with $m = 1, 2, 3, 4$.

3 Analysis of methods with $m = 1$

In this section we focus our attention on one stage continuous methods (1.1). We first assume that the polynomial $P(t_n + sh)$ in (1.1) satisfies the interpolation condition (2.1) only. As a consequence, we need to impose conditions (2.3) on the basis functions, i.e.

$$\varphi_0(0) = 0,\ \varphi_1(0) = 1,\ \chi(0) = 0,\ \psi(0) = 0. \tag{3.1}$$

Correspondingly, according to the assumption (2.5), we fix

$$\varphi_0(s) = p_0 s, \quad \chi(s) = q_0 s, \tag{3.2}$$

and derive $\varphi_1(s)$ and $\psi(s)$ imposing the order conditions (2.10), obtaining

$$\varphi_1(s) = 1 - p_0 s, \quad \psi(s) = (1 + p_0 - q_0)s. \tag{3.3}$$

Therefore, the basis functions depend on p_0 and q_0, which must be determined in order to achieve high stability properties. We know from the general theory of TSRK (see [22]), that a TSRK method is zero–stable if and only if $-1 < \theta \leq 1$: in our case, as $\theta = \varphi_0(1) = p_0$, it must be

$$-1 < p_0 \leq 1. \tag{3.4}$$

We next compute the stability function (2.12) of the method: in this case we have

$$p(\omega, z) = p_0(z) + p_1(z)\omega + p_2(z)\omega^2 \tag{3.5}$$

with

$$p_0(z) = -p_0 - (1 - c)q_0 z, \tag{3.6}$$
$$p_1(z) = -1 + p_0 + (-1 + c - p_0 + cp_0 + q_0 - 2cq_0)z, \tag{3.7}$$
$$p_2(z) = 1 - c(1 + p_0 + q_0)z. \tag{3.8}$$

In order to determine the values of the parameters p_0, q_0 and c achieving A-stability, we apply the Schur criterion to the stability function (3.5), obtaining the following result.

Theorem 3.1 *One stage interpolation based continuous methods (1.1), with basis functions (3.2), (3.3) are A-stable if and only if the parameters p_0, q_0 and c satisfy the following system of inequalities*

$$\begin{cases} -1 < p_0 < 1, \\ p_0 + 2c(1 + p_0) > 1 + 2q_0, \\ (-1 + 2c)(1 + p_0 - 2q_0) > 0, \\ (c + cp_0 - q_0)\big(c(1 + p_0 - 2q_0) + q_0\big) > 0. \end{cases} \tag{3.9}$$

Proof. In order to achieve A-stability, the roots of the polynomial (3.5) must lie inside the unit circle for any $z \in \mathbb{C}$ such that $\mathrm{Re}(z) < 0$. By the maximum principle, this is the case if

1. the polynomial (3.5) has no poles in the negative half plane;
2. the roots of $p(\omega, iy)$ are inside the unit circle $\forall y \in \mathbb{R}$.

Condition 1 is trivially satisfied. We analyse condition 2 applying the Schur criterion described in theorem 2.2 to the polynomial $p(\omega, iy)$ that we will next denote as $p(\omega, y)$. In order to use the results in theorem 2.2, we first compute the polynomial

$$\widehat{p}(\omega, y) = \overline{p_2}(z)\omega^2 + \overline{p_1}(z)\omega + \overline{p_0}(z)$$

where $\overline{p_2}(z)$, $\overline{p_1}(z)$ and $\overline{p_0}(z)$ are the complex conjugate polynomials associated to $p_2(z)$, $p_1(z)$ and $p_0(z)$ respectively. We next compute the polynomial

$$\alpha(\omega, y) = \frac{1}{\omega}\left(\widehat{p}(0, y)\widetilde{p}(\omega, y) - \widetilde{p}(0, y)\widehat{p}(0, y)\right)$$

of degree 1. According to Schur criterion, $p(\omega, y)$ is a Schur polynomial if and only if

$$|\widehat{p}(0, y)| > |p(0, y)| \tag{3.10}$$

and $\alpha(\omega, y)$ is a Schur polynomial. Condition (3.10) is satisfied if and only if

$$1 - p_0^2 > 0, \qquad (c + cp_0 - q_0)\big(c(1 + p_0 - 2q_0) + q_0\big) > 0. \tag{3.11}$$

In order to investigate on the polynomial $\alpha(\omega, y)$, we apply the same procedure, i.e. we derive the corresponding polynomials $\widehat{\alpha}(\omega, y)$ and

$$\beta(\omega, y) = \frac{1}{\omega}\left(\widehat{\alpha}(0, y)\widetilde{\alpha}(\omega, y) - \widetilde{\alpha}(0, y)\widehat{\alpha}(0, y)\right),$$

and the conditions imposed by the Schur criterion are satisfied for

$$-1 + p_0 + 2c(1 + p_0) - 2q_0 > 0, \qquad (-1 + 2c)(1 + p_0 - 2q_0) > 0. \tag{3.12}$$

Conditions (3.11) and (3.12) together give the system of inequalities (3.9). □

Fig. 3.1 shows some regions of A-stability in the parameter space (p_0, q_0), in correspondence of some values of the collocation abscissa c.

$c = \frac{3}{4}$ $\quad$ $c = 1$

Fig. 3.1 Regions of A-stability in the (p_0, q_0)-plane for two-step methods (1.1) with $p = m = 1$, for somevalues of the abscissa c.

We next look for L-stable methods: in this case, conditions (2.14) take the form

$$q_0(1 - c) = 0, \qquad 1 - c + p_0(1 - c) - q_0(1 - 2c) = 0,$$

whose solution is $(p_0, q_0) = (-1, 1)$, which is not acceptable because it violates the zero-stability requirement (3.4). However, if we set $c = 1$ and $q_0 = 0$, the above system is automatically satisfied, for any $p_0 \in (-1, 1]$, i.e. the corresponding methods are L-stable. In other words, if $c = 1$ and the basis polynomial $\chi(s)$ is identically zero, the resulting methods are all L-stable.

We now assume that the polynomial $P(t_n + sh)$ in (1.1) satisfies the whole set of conditions (2.3), (2.4), i.e.

$$\varphi_0(0) = 0,\ \varphi_1(0) = 1,\ \chi(0) = 0,\ \psi(0) = 0,$$
$$\varphi_0'(c) = 0,\ \varphi_1'(c) = 0,\ \chi(c) = 0,\ \psi(c) = 1.$$

Correspondingly, we assume

$$\varphi_0(s) = \alpha\pi_1(s), \quad \chi(s) = \beta\pi_1(s), \tag{3.13}$$

and compute $\varphi_1(s)$ and $\psi(s)$ from the order conditions (2.10), obtaining

$$\varphi_1(s) = 1 + \alpha c s^2 - \frac{1}{2}\alpha s^3, \quad \psi(s) = s - (\alpha - \beta)cs^2 + \frac{1}{2}(\alpha - \beta)s^3. \tag{3.14}$$

As $\theta = \varphi_0(1) = \alpha(\frac{1}{2} - c)$, zero-stability is accomplished if and only if

$$-1 < \alpha(\frac{1}{2} - c) \leq 1. \tag{3.15}$$

We now investigate on the stability properties of the methods in analysis, considering the stability function (2.12). Following the lines drawn in Theorem 3.1, we obtain the following result.

Theorem 3.2 *One stage interpolation-collocation based continuous methods (1.1), with basis functions (3.13), (3.14) are A-stable if and only if the parameters α, β and c satisfy the following system of inequalities*

$$\begin{cases} \alpha^2(1-2c)^2 < 4 \\ (-\beta + 2c + \alpha c + 2\beta c - 2\alpha c^2)(\beta + 2c - \alpha c - 2\beta c + 2\alpha c^2 - 2\alpha c^3 + 2\beta c^3) > 0 \\ (-2 - \alpha + 2\alpha c)(2 + \alpha - 2\beta - 4c + 4\beta c - 4\alpha c^2 + 4\alpha c^3 - 4\beta c^3) > 0 \\ (-1 + 2c)(\alpha - 2(1+\beta) + 2\alpha c)(2 + \alpha(-1 + 2c)) < 0 \end{cases}$$

We have drawn some regions of A-stability in the parameter space (α, β), using the results stated in Theorem 3.2: fig. 3.2 shows the results we have obtained for particular values of c.

$c = \frac{3}{4}$ $\quad c = 1$

Fig. 3.2 Regions of A-stability in the (α, β)-plane for interpolation-collocation based methods (1.1) with $p = m = 1$, for different values of the abscissa c.

We finally compute the values of α and β achieving L-stability, solving the system (2.14): it can be easily proved that those values are

$$\alpha = 2\frac{-1 + c + c^2}{1 - 2c - c^2 + 2c^3}, \quad \beta = \frac{2c}{-1 + c + 2c^2}.$$

4 Analysis of methods with $m = 2$

We now consider two-step continuous methods (1.1) with $p = m = 2$ and general abscissa vector, satisfying the interpolation condition (2.1). We assume

$$\varphi_0(s) = 0, \quad \chi_1(s) = q_1 s, \quad \chi_2(s) = q_2 s, \tag{4.1}$$

solve the system of order conditions (2.10) with respect to $\varphi_1(s)$, $\psi_1(s)$, $\psi_2(s)$, and compute the corresponding stability function (2.13), where $p_0(z)$, $p_1(z)$, $p_2(z)$ are polynomials of degree 2 with respect to z. In this case, the system (2.14) can be solved setting $c_2 = 1$ and $q_2 = -q_1$. We finally apply the Schur criterion in order to localize the whole set of possible values of c_1 and q_1 such that the corresponding methods are L-stable. The results are shown in fig. 4.1.

Fig. 4.1 Region of L-stability in the (c_1, q_1)-plane for two-step methods (1.1) with $p = m = 2$ and $c_2 = 1$.

We next consider interpolation-collocation methods of type (1.1) with $p = m = 2$. According to assumption (2.9), we set

$$\varphi_0(s) = 0, \quad \chi_1(s) = \beta_1 \pi_2(s), \quad \chi_2(s) = \beta_2 \pi_2(s),$$

and derive $\varphi_1(s)$, $\psi_1(s)$ and $\psi_2(s)$ from the set of order conditions (2.10). We compute the stability function (2.13) where $p_0(z)$, $p_1(z)$ and $p_2(z)$ are polynomials of degree 2 with respect to z, and look for values of the parameters β_1 and β_2 achieving L-stability, solving the system (2.14), obtaining

$$\beta_1 = \frac{3}{\gamma}(-1 + c_1)(-1 + c_2)^2, \quad \beta_2 = \frac{3}{\gamma}(-1 + 2c_1 - c_1^2 + c_2 - 2c_1c_2 + c_1^2 c_2),$$

where

$$\gamma = 4 - 8c_1 + 5c_1^2 - c_1^3 - 8c_2 + 14c_1c_2 - 7c_1^2c_2 + 2c_1^3c_2 + 5c_2^2 - 7c_1c_2^2 - c_2^3 + 2c_1c_2^3.$$

Finally, we draw the L–stability region in the parameter space (c_1, c_2), performing a computer search based on the Schur criterion. Fig. 4.2 shows the result we obtained.

Fig. 4.2 Region of L-stability in the (c_1, c_2)-plane for two-step methods (1.1) with $p = m = 2$.

5 Construction of methods with $m = 3$

We now consider three stage continuous methods (1.1) with $p = m = 3$. Let us first derive FSAL interpolation based methods of order 3, corrisponding to the abscissa vector $(c_1,\ c_2,\ c_3) = (0,\ c_2,\ 1)$. Following the assumptions in (2.8), we set

$$\varphi_0(s) = s(s - c_2)(s - 1), \quad \chi_1(s) = q_1 s, \quad \chi_2(s) = q_2 s, \quad \chi_3(s) = q_3 s,$$

and derive $\varphi_1(s),\ \psi_1(s),\ \psi_2(s),\ \psi_3(s)$, imposing the set of conditions (2.10). We omit for brevity the expression of the resulting basis polynomials, which can be easily recognized. We then derive the stability function (2.13), where $p_0(z),\ p_1(z),\ p_2(z)$ are polynomials of degree 3 with respect to z, and next look for the values of the parameters achieving L–stability. In this case, the solutions of the system (2.14) are

$$q_1 = -\frac{1-c_2^2}{18c_2}, \quad q_2 = \frac{1-2c_2}{18c_2}.$$

By using the Schur criterion, it can be recognized that (2.13) is a Schur polynomial if and only if $c_2 > \frac{1}{2}$. We notice that q_3 has no influence in the construction, so we set it equal to 0.

We next develop interpolation-collocation based methods, assuming

$$\varphi_0(s) = 0, \quad \chi_1(s) = \beta_1 \pi_3(s), \quad \chi_2(s) = \beta_2 \pi_3(s), \quad \chi_3(s) = \beta_3 \pi_3(s).$$

We impose the set of order conditions (2.10), derive $\psi_1(s)$, $\psi_2(s)$, $\psi_3(s)$, and compute β_1 and β_2 as solutions of the system (2.14), in order to gain L-stability. Finally, using the Schur criterion, we draw some regions of L–stability in the parameters space (c_1, c_2), for different values of β_3 and $c_3 = 1$, as shown in fig. 5.1.

$\beta_3 = 0$ $\quad$ $\beta_3 = \frac{1}{2}$

$\beta_3 = 1$ $\quad$ $\beta_3 = 2$

Fig. 5.1 Regions of L-stability in the (c_1, c_2)-plane for two-step methods (1.1) with $p = m = 3$, $c_3 = 1$ and different values of the parameter β_3.

6 Construction of methods with $m = 4$

We now focus our attention on the development of two-step continuous methods (1.1) with $p = m = 4$, first considering interpolation based methods. According to assumptions (2.8), we fix

$$\begin{aligned} &\varphi_0(s) = s(s-c_2)(s-c_3)(s-1), \; \chi_1(s) = sq_1, \\ &\chi_2(s) = sq_2, \qquad \chi_3(s) = sq_3, \qquad \chi_4(s) = sq_4, \end{aligned}$$

and derive $\varphi_1(s)$, $\psi_1(s)$, $\psi_2(s)$, $\psi_3(s)$, $\psi_4(s)$, imposing the set of order conditions (2.10) for $p = 4$. We omit for brevity the expression of the resulting basis polynomials, which can be easily recognized also in this case. We next derive the stability polynomial (2.13), where $p_0(z)$, $p_1(z)$, $p_2(z)$ are polynomials of degree 4 with respect to z, and look for the values of the parameters achieving L–stability, solving the system (2.14), with respect to q_1 and q_2. Fig. 6.1 shows some regions of L-stability in the parameter space (c_3, q_3), for some values of c_2, drawn using the Schur criterion. We notice that q_4 does not play any rule in the derivation of the methods, so it can be put equal to zero.

$c_2 = \frac{1}{2}$ $c_2 = \frac{3}{5}$

$c_2 = \frac{13}{20}$ $c_2 = \frac{2}{3}$

Fig. 6.1 Regions of L-stability in the (c_3, q_3)-plane for two-step methods (1.1), for specific values of the abscissa c_2.

We now consider continuous four stage methods (1.1), obtained by imposing interpolation and collocation conditions (2.1), (2.2), and asking for $\theta = 0$, $\mathbf{u} = \mathbf{0}$. In line with assumptions (2.9), we impose

$$\begin{aligned} &\varphi_0(s) = 0, \quad \chi_1(s) = \beta_1 \pi_4(s), \quad \chi_2(s) = \beta_2 \pi_4(s), \\ &\chi_3(s) = \beta_3 \pi_4(s), \quad \chi_4(s) = \beta_3 \pi_4(s), \end{aligned}$$

and, from the set of order conditions (2.10), we derive $\psi_1(s)$, $\psi_2(s)$, $\psi_3(s)$ and $\psi_4(s)$. We compute the stability function (2.13) and determine β_2 and β_3 as solutions of (2.14), in order to gain L-stability. Finally, using the Schur criterion, we draw some regions of L-stability in the parameters space (c_2, c_3), for different values of β_1, as shown in fig. 6.2.

$\beta_1 = 0$ $\beta_1 = \frac{1}{2}$

$\beta_1 = 1$ $\beta_1 = 2$

Fig. 6.2 Regions of L-stability in the (c_2, c_3)-plane for two-step methods (1.1) with $p = m = 4$, for specific values of the parameter β_1.

7 Runge–Kutta stability

In this section we investigate the existence of continuous TSRK methods having the so-called Runge-Kutta stability, i.e. methods such that the stability matrix has one nonzero eigenvalue only, which is in general a very complicate requirement. We restrict our attention to one-stage continuous methods, whose general construction has been treated in section 3. We infer the following result.

Theorem 7.1 *For* $c \in [0, 1)$*, the only interpolation based TSRK methods with* $m = 1$ *having Runge–Kutta stability are Runge–Kutta methods themselves. In the special case* $c = 1$*,* $p_0 = 0$*, all the corresponding methods posses Runge–Kutta stability, for any* $q_0 \in \mathbb{R}$*.*

Proof. It is sufficient to annihilate $p_0(z)$ in (3.6), in order to obtain a stability polynomial of the form

$$p(w, z) = w^2 \Big(p_2(z) w + p_1(z) \Big), \tag{7.1}$$

having only one nonzero root. The conditions to impose in order to have $p_0(z)$ identically equal to 0 are

$$p_0 = 0, \quad (1-c)q_0 = 0.$$

The solution of this system is $p_0 = q_0 = 0$ and, correspondingly, the basis polynomials take the form

$$\varphi_0(s) = 0, \quad \varphi_1(s) = 1, \quad \chi(s) = 0, \quad \psi(s) = s,$$

and the continuous approximant (1.1) is

$$P(t_n + sh) = y_n + hsf\Big(P(t_n + ch)\Big). \tag{7.2}$$

The last part of the thesis is achieved simply setting $c = 1$ and $p_0 = 0$ in (3.6). □

With similar considerations, we can state an analogous result for interpolation-collocation TSRK methods with $m = 1$.

Theorem 7.2 *For $c \in [0, 1]$ and $c \neq \frac{1}{2}$, the only interpolation-collocation methods (1.1) having Runge-Kutta stability are Runge-Kutta methods themselves. For $c = \frac{1}{2}$ and $\alpha = \beta$, the corrisponding TSRK methods have Runge–Kutta stability.*

8 Examples of methods

In this section we derive examples of A-stable and L-stable continuous TSRK methods (2.13) for $m = 1$, 2, 3, and 4. It is always assumed that $\theta = 0$ and $\mathbf{u} = \mathbf{0}$.

8.1 Examples of interpolation based methods

Example 1. Assuming $p = m = 1$, we set $c = \frac{3}{4}$ and obtain an A-stable ethod of type (1.1), with

$$\varphi_0(s) = 0, \quad \varphi_1(s) = 1, \quad \chi(s) = -s, \quad \psi(s) = 2s.$$

We notice that the L-stable method with $c = 1$, $q_0 = 0$ and $p_0 = 0$ is equivalent to the backward Euler method.

Example 2. Referring to the results derived in section 4, the basis functions of the L-stable method of order $p = m = 2$ corresponding to the abscissa vector $c = [\frac{3}{4}, 1]^T$ are

$$\begin{aligned}
&\varphi_0(s) = 0, \quad \varphi_1(s) = 1, \quad \chi_1(s) = s, \quad \chi_2(s) = -s, \\
&\psi_1(s) = s(9 - 2s), \quad \psi_2(s) = 2s(s-1).
\end{aligned}$$

We also show an example of A-stable FSAL method, setting $q_0 = 1$, $q_1 = 1$ and $q_2 = 0$, i.e.

$$\begin{aligned}
&\varphi_0(s) = s(s-1), \quad \varphi_1(s) = 1 + s - s^2, \quad \chi_1(s) = s, \quad \chi_2(s) = 0, \\
&\psi_1(s) = \tfrac{1}{2}s(-5 + 2s), \quad \psi_2(s) = \tfrac{9}{2}s.
\end{aligned}$$

Example 3. Following the results contained in section 5, we show the basis functions of a FSAL L-stable method with $p = q = m = 3$, corresponding to the abscissa vector $c = [0, \frac{3}{4}, 1]^T$:

$$\varphi_0(s)=\tfrac{s}{4}(s-1+s)(4s-3),\quad \varphi_1(s)=1-\tfrac{3}{4}s+\tfrac{7}{4}s^2-s^3$$
$$\chi_1(s)=-\tfrac{7s}{216},\quad \chi_2(s)=-\tfrac{s}{27},\quad \chi_3(s)=0$$
$$\psi_1(s)=s\Big(\tfrac{685}{216}-\tfrac{413}{72}s+\tfrac{55}{18}s^2\Big),\quad \psi_2(s)=-s\Big(\tfrac{101}{27}-\tfrac{94}{9}s+\tfrac{56}{9}s^2\Big)$$
$$\psi_3(s)=s\Big(\tfrac{43}{18}-\tfrac{155}{24}s+\tfrac{25}{6}s^2\Big).$$

Example 4. We next derive a FSAL L-stable method (1.1) with $p=m=4$, corresponding to abscissa the vector $c=[0,\frac{1}{2},\frac{3}{4},1]^T$. The basis functions of this method are

$$\varphi_0(s)=\tfrac{s}{8}(s-1)(2s-1)(4s-3),\quad \varphi_1(s)=1+\tfrac{3}{8}s-\tfrac{13}{8}s^2+\tfrac{9}{4}s^3-s^4,$$
$$\chi_1(s)=-\tfrac{149}{264}s,\quad \chi_2(s)=\tfrac{233}{132}s,\quad \chi_3(s)=s,\quad \chi_4(s)=0,$$
$$\psi_1(s)=-s\Big(\tfrac{2435}{528}-\tfrac{117}{16}s+\tfrac{89}{8}s^2-\tfrac{31}{6}s^3\Big)$$
$$\psi_2(s)=s\Big(\tfrac{235}{33}-\tfrac{305}{12}s+\tfrac{205}{6}s^2-\tfrac{46}{3}s^3\Big)$$
$$\psi_3(s)=-s\Big(\tfrac{169}{33}-\tfrac{88}{3}s+\tfrac{112}{3}s^2-16s^3\Big)$$
$$\psi_4(s)=s\Big(\tfrac{547}{528}-\tfrac{461}{48}s+\tfrac{289}{24}s^2-\tfrac{29}{6}s^3\Big).$$

8.2 Examples of interpolation-collocation based methods.

Example 5. Referring to the results derived in section 3, we set $c=\frac{3}{4}$, obtaining a L-stable interpolation-collocation method with $p=m=1$, whose basis functions are

$$\varphi_0(s)=s^2\Big(\tfrac{15}{7}-\tfrac{10}{7}s\Big),\quad \varphi_1(s)=1-\tfrac{15}{7}s^2+\tfrac{10}{7}s^3,$$
$$\chi(s)=-s^2\Big(\tfrac{9}{7}-\tfrac{6}{7}s\Big),\quad \psi(s)=1+\tfrac{24}{7}s^2-\tfrac{16}{7}s^3.$$

Example 6. Referring to the results derived in section 4, we show the basis functions of the L-stable interpolation-collocation method (1.1) with $p=m=2$, corresponding to the abscissa vector $c=[\frac{1}{2},\frac{9}{10}]^T$, i.e.

$$\varphi_0(s)=0,\quad \varphi_1(s)=1,$$
$$\chi_1(s)=-s\Big(\tfrac{3}{4}-\tfrac{7}{60}s+\tfrac{1}{18}s^2\Big),\quad \chi_2(s)=-s\Big(\tfrac{3}{8}-\tfrac{7}{12}s+\tfrac{5}{18}s^2\Big),$$
$$\psi_1(s)=s\Big(\tfrac{69}{20}-\tfrac{187}{60}s+\tfrac{8}{9}s^2\Big),\quad \psi_2(s)=-s\Big(2-\tfrac{29}{12}s+\tfrac{5}{9}s^2\Big).$$

Example 7. Following the results contained in section 5 we show the basis functions of a FSAL L-stable interpolation-collocation method (1.1) with $p=m=3$, corresponding to the abscissa vector $c=[0,\frac{3}{5},1]^T$:

$$\varphi_0(s)=0,\quad \varphi_1(s)=1,\quad \chi_1(s)=-s^3\Big(\tfrac{240}{337}-\tfrac{1280}{1011}s+\tfrac{200}{337}s^2\Big),$$
$$\chi_2(s)=s^3\Big(\tfrac{1125}{337}-\tfrac{2000}{337}s+\tfrac{1857}{674}s^2\Big),\quad \chi_3(s)=s^3\Big(\tfrac{3}{10}-\tfrac{8}{15}s+\tfrac{1}{4}s^2\Big),$$
$$\psi_1(s)=s\Big(1-\tfrac{4}{3}s-\tfrac{113299}{30330}s^2+\tfrac{115688}{15165}s^3-\tfrac{14461}{4044}s^4\Big),$$
$$\psi_2(s)=25s^2\Big(\tfrac{1}{12}+\tfrac{113}{6066}s-\tfrac{400}{3033}s^2+\tfrac{125}{2022}s^3\Big)$$
$$\psi_3(s)=-s^2\Big(\tfrac{3}{4}-\tfrac{695}{2022}s-\tfrac{880}{1011}s^2+\tfrac{275}{674}s^3\Big).$$

Example 8. We now consider the FSAL L-stable method (1.1) with $p = m = 4$, $\theta = 0$, $\mathbf{u} = \mathbf{0}$, corresponding to the abscissa vector $c = [0, \frac{7}{10}, \frac{9}{10}, 1]^T$, whose basis functions are

$$
\begin{aligned}
&\varphi_0(s) = 0, \quad \varphi_1(s) = 1, \quad \chi_1(s) = -s^3\Big(\tfrac{63}{100} - \tfrac{223}{150}s + \tfrac{13}{10}s^2 - \tfrac{2}{5}s^3\Big),\\
&\chi_2(s) = \tfrac{125840873}{10156165010}s^3\Big(189 - 446s + 390s^2 - 120s^3\Big),\\
&\chi_3(s) = \tfrac{313000831}{6093699006}s^3(189 - 446s + 390s^2 - 120s^3), \quad \chi_4(s) = 0,\\
&\psi_1(s) = s\Big(1 - \tfrac{223}{126}s - \tfrac{110596774973233}{9597575934450}s^2 + \tfrac{48055456715852}{1599595989075}s^3\\
&\qquad\qquad\qquad - \tfrac{2838443145187}{106639732605}s^4 + \tfrac{873367121596}{106639732605}s^5\Big),\\
&\psi_2(s) = s^2\Big(\tfrac{75}{7} - \tfrac{131546117771291}{6398383956300}s + \tfrac{671254535668}{35546577535}s^2 - \tfrac{80390326549}{7109315507}s^3 + \tfrac{24735485092}{7109315507}s^4\Big),\\
&\psi_3(s) = -s^2\Big(\tfrac{175}{9} - \tfrac{2867265551881}{54843291054}s + \tfrac{575594042414}{9140548509}s^2 - \tfrac{130770083795}{3046849503}s^3 + \tfrac{40236948860}{3046849503}s^4\Big),\\
&\psi_4(s) = s^2\Big(\tfrac{21}{2} - \tfrac{28900702732187}{914054850900}s + \tfrac{2081690316751}{50780825050}s^2 - \tfrac{290054503193}{10156165010}s^3 + \tfrac{44623769722}{5078082505}s^4\Big).
\end{aligned}
$$

9 Numerical experiments

In this section we will demonstrate that continuous TSRK methods of order p and stage order do not suffer from order reduction in the integration of stiff differential systems, which is the case for classical Runge-Kutta formulae. This phenomenon, in fact, does not occur for continuous TSRK methods because they possess high stage order equal to their uniform order of convergence over the entire integration interval. On the other hand, Runge-Kutta methods do not possess the same feature, because their stage order is only equal to m, where m is the number of stages. To illustrate this we have applied the two-stage Runge-Kutta-Gauss method of order $p = 4$ and stage order $q = 2$ and the continuous TSRK method of uniform order $p = 4$ given in the Example 8 of Section 8 to the following problems:

1. the Prothero-Robinson problem [28]

$$
\begin{cases} y'(t) = \lambda\big(y(t) - F(t)\big) + F'(t), \; t \in [t_0, T],\\ y(t_0) = y_0, \end{cases}
$$

   where $\mathrm{Re}(\lambda) < 0$ and $F(t)$ is a slowly varying function on the interval $[t_0, T]$. In our experiments, we have considered $F(t) = \sin(t)$. As observed by Hairer and Wanner [20] in the context of Runge-Kutta methods this equation provides much new insight into the behavoiur of numerical methods for stiff problems. This equation with $t_0 = 0$, $F(t) = \exp(\mu t)$, and $y_0 = 1$, was also used by Butcher [7] to investigate order reduction for Runge-Kutta-Gauss methods of order $p = 2s$;
2. the van der Pol oscillator (see VDPOL problem in [20])

$$
\begin{cases} y_1' = y_2, & y_1(0) = 2,\\ y_2' = \big((1 - y_1^2)y_2 - y_1\big)/\epsilon, & y_2(0) = -2/3, \end{cases} \tag{9.1}
$$

   $t \in [0, T]$, with stiffness parameter ϵ;

	$\lambda = -10^3$			$\lambda = -10^5$	
k	$e_h^{RKG}(T)$	p	k	$e_h^{RKG}(T)$	p
10	$1.55 \cdot 10^{-5}$				
11	$7.80 \cdot 10^{-7}$	3.89	7	$1.11 \cdot 10^{-3}$	
12	$4.94 \cdot 10^{-8}$	3.98	8	$2.78 \cdot 10^{-4}$	2.00
13	$3.09 \cdot 10^{-9}$	3.99	9	$6.80 \cdot 10^{-5}$	2.02
14	$1.93 \cdot 10^{-10}$	4.00	10	$1.68 \cdot 10^{-5}$	2.01

Table 9.1 Numerical results for Runge-Kutta-Gauss method of order $p = 4$ and stage order $q = 2$ for the Prothero-Robinson problem

3. the Hires problem in [20]

$$\begin{cases} y_1' = -1.71y_1 + 0.43y_2 + 8.32y_3 + 0.0007 \\ y_2' = 1.71y_1 - 8.75y_2 \\ y_3' = -10.03y_3 + 0.43y_4 + 0.035y_5 \\ y_4' = 8.32y_2 + 1.71y_3 - 1.12y_4 \\ y_5' = -1.745y_5 + 0.43y_6 + 0.43y_7 \\ y_6' = -280y_6y_8 + 0.69y_4 + 1.71y_5 - 0.43y_6 + 0.69y_7 \\ y_7' = 280y_6y_8 - 1.81y_7 \\ y_8' = -280y_6y_8 + 1.81y_7 \\ y_1(0) = 1, \quad y_2(0) = \ldots = y_7(0) = 1 \quad y_8(0) = 0.0057, \end{cases}$$

with $t \in [0, 321.8122]$.

For each problem, we have implemented both methods with a fixed stepsize

$$h = (T - t_0)/2^k,$$

with several integer values of k, and listed norms of errors $\|e_h^{RKG}(T)\|$ and $\|e_h^{TSRK}(T)\|$ at the endpoint of integration T and the observed order of convergence p computed from the formula

$$p = \frac{\log\left(\|e_h(T)\|/\|e_{h/2}(T)\|\right)}{\log(2)},$$

where $e_h(T)$ and $e_{h/2}(T)$ are errors corresponding to stepsizes h and $h/2$ for Runge-Kutta-Gauss and TSRK methods.

Let us first consider the results obtained for the Prothero-Robinson problem in the interval $[0, 50]$, which are presented in Table 9.1 and Table 9.2, for the Runge-Kutta-Gauss method and the continuous TSRK one respectively, in correspondence of several values for the stiffness parameter λ.

We can observe that in the case $\lambda = -10^3$, for which the Prothero-Robinson problem is mildly stiff, both methods are convergent with expected order $p = 4$. However,

	$\lambda=-10^3$			$\lambda=-10^5$	
k	$e_h^{TSRK}(T)$	p	k	$e_h^{TSRK}(T)$	p
10	$3.29 \cdot 10^{-11}$				
11	$2.11 \cdot 10^{-12}$	3.97	7	$1.12 \cdot 10^{-9}$	
12	$1.34 \cdot 10^{-13}$	3.98	8	$7.75 \cdot 10^{-11}$	3.86
13	$8.43 \cdot 10^{-15}$	3.98	9	$4.97 \cdot 10^{-12}$	3.96
14	$5.55 \cdot 10^{-16}$	3.92	10	$3.03 \cdot 10^{-13}$	4.03

Table 9.2 Numerical results for continuous TSRK method of uniform order $p = 4$ for the Prothero-Robinson problem

	$\epsilon = 10^{-1}$		$\epsilon = 10^{-3}$		$\epsilon = 10^{-6}$	
k	$\|e_h^{RKG}(T)\|$	p	$\|e_h^{RKG}(T)\|$	p	$\|e_h^{RKG}(T)\|$	p
6	$1.88 \cdot 10^{-8}$		$2.25 \cdot 10^{-4}$		$1.49 \cdot 10^{-3}$	
7	$1.18 \cdot 10^{-9}$	4.00	$1.68 \cdot 10^{-5}$	3.74	$3.71 \cdot 10^{-4}$	2.01
8	$8.21 \cdot 10^{-11}$	3.84	$1.11 \cdot 10^{-6}$	3.93	$8.84 \cdot 10^{-5}$	2.07
9	$1.43 \cdot 10^{-11}$	2.52	$7.02 \cdot 10^{-8}$	3.98	$1.87 \cdot 10^{-5}$	2.24

Table 9.3 Numerical results for Runge-Kutta-Gauss method of order $p = 4$ and stage order $q = 2$ for the Van der Pol problem

for $\lambda = -10^5$, the problem in stiff and the Runge-Kutta-Gauss method exhibits the order reduction phenomenon and its order of convergence drops to about $p = 2$ which corresponds to the stage order $q = 2$. This is not the case for TSRK method which preserves order of convergence $p = q = 4$, which leads to higher accuracy.

We next consider the Van der Pol oscillator, which is observed in the interval $[0, 3/4]$, i.e. for the slowly varying parts of the solution, where the problem is stiff for small values of the parameter ϵ (the problem is not stiff on the interval where the solution is changing rapidly). The results are presented in Table 9.3 and Table 9.4, for several values of the parameter ϵ.

Also in this case we can observe that for the values of $\epsilon = 10^{-1}$ and $\epsilon = 10^{-3}$ for which the problem (9.1) is not stiff and mildly stiff both methods are convergent with expected order $p = 4$. However, for small values of ϵ ($\epsilon = 10^{-6}$) for which the van der Pol oscillator is stiff the Runge-Kutta-Gauss method exhibits order reduction phenomenon and its order of convergence drops to about $p = 2$ which corresponds to the stage order $q = 2$. This is not the case for TSRK method which preserves order of convergence $p = q = 4$, which leads to higher accuracy.

We conclude our analysis presenting the results obtained for the Hires problem, included in Table 9.5 and Table 9.6. Also in this case, the order reduction phenomenon is evident for the Runge-Kutta-Gauss method, while it is not present on the continuous TSRK method considered.

k	$\epsilon = 10^{-1}$		$\epsilon = 10^{-3}$		$\epsilon = 10^{-6}$	
	$\|e_h^{TSRK}(T)\|$	p	$\|e_h^{TSRK}(T)\|$	p	$\|e_h^{TSRK}(T)\|$	p
6	$5.82 \cdot 10^{-8}$		$1.58 \cdot 10^{-5}$		$1.54 \cdot 10^{-5}$	
7	$3.66 \cdot 10^{-9}$	3.99	$1.17 \cdot 10^{-6}$	3.75	$1.09 \cdot 10^{-6}$	3.81
8	$2.32 \cdot 10^{-10}$	3.98	$7.85 \cdot 10^{-8}$	3.90	$7.34 \cdot 10^{-8}$	3.90
9	$1.46 \cdot 10^{-11}$	3.99	$4.80 \cdot 10^{-9}$	4.03	$4.75 \cdot 10^{-9}$	3.94

Table 9.4 Numerical results for TSRK method of order $p = 4$ and stage order $q = 4$ for the Van der Pol problem

k	$\|e_h^{RKG}(T)\|$	p
8	$3.05 \cdot 10^{-6}$	
9	$6.42 \cdot 10^{-7}$	2.25
10	$1.47 \cdot 10^{-7}$	2.12
11	$3.52 \cdot 10^{-8}$	2.06
12	$8.62 \cdot 10^{-9}$	2.03

Table 9.5 Numerical results for Runge-Kutta-Gauss method of order $p = 4$ and stage order $q = 2$ for the Hires problem

k	$\|e_h^{TSRK}(T)\|$	p
6	$4.85 \cdot 10^{-5}$	
7	$3.31 \cdot 10^{-6}$	3.87
8	$2.16 \cdot 10^{-7}$	3.93

Table 9.6 Numerical results for TSRK method of order $p = 4$ and stage order $q = 4$ for the Hires problem

Additional results which confirm that continuous TSRK methods constructed in this paper preserve the order of convergence for stiff problems are given in [13].

It would be also interesting to compare these methods in variable stepsize implementations. However before this can be done, the investigation of many implementation issues is required, such as a choice of appropriate starting procedures, estimation of local discretization errors for small and large stepsizes, filtering error estimates for stiff problems, the design of stepsize and order changing strategies, and the design of strategies for efficient solution of nonlinear systems of equations at each integration step. All these implementation issues require different techniques than the ones employed in this paper, which is only devoted to the construction of highly stable continuous methods, and their investigation and comparison of methods in variable stepsize environments is subject of [16].

10 Conclusions and future work

We proposed new families of highly–stable continuous two-step m-stage methods for the numerical solution of ordinary differential equations, which is, in general, a nontrivial task. These methods are of uniform order p equal to the stage order and as a result they do not suffer from order reduction phenomenon (see [7], [8], [20]) persistent with methods of low stage order. They are constructed using the some interpolation and collocation conditions, deriving the parameters such that the resulting methods are A-stable and L-stable. Runge–Kutta stability is also discussed and example of methods with $m = 1,\ 2,\ 3,\ 4$ are also given. Future work will address various implementation issues such as the choice of appropriate starting procedures, stepsize and order changing strategy, solving nonlinear systems of equations by modified Newton methods and local error estimation for large stepsizes, in order to efficiently implement stiff differential systems [16].

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