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Two-step almost collocation methods for ordinary differential equations

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Abstract A new class of two-step Runge-Kutta methods for the numerical solution of ordinary differential equations is proposed. These methods are obtained using the collocation approach by relaxing some of the collocation conditions to obtain methods with desirable stability properties. Local error estimation for these methods is also discussed.

Keywords Two-step collocation methods · order conditions · absolute stability · A-stability · local error estimation.

1 Introduction

It is the purpose of this paper to discuss the construction of highly stable two-step collocation methods for the numerical solution of initial value problem for the system of ordinary differential equations (ODEs)

$$\begin{cases} y'(t) = f(y(t)),\ t \in [t_0, T], \\ y(t_0) = y_0. \end{cases} \tag{1.1}$$

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Here, $f : R^d \to R^d$ is assumed to be sufficiently smooth, and $y_0 \in R^d$ is a given initial value. Let $h > 0$ be a constant stepsize and define the grid

$$t_n = t_0 + nh, \quad n = 0, 1, \ldots, N,$$

where $Nh = T - t_0$. Assume that the continuous approximation of sufficiently high order $P(t_0 + sh)$ to the solution $y(t_0 + sh)$ of (1.1) is already computed for $s \in [0, 1]$ which corresponds to the initial interval $[t_0, t_1]$. Then the two-step continuous approximation to the solution of (1.1) is defined by

$$\begin{cases} P(t_n + sh) = \varphi_0(s) y_{n-1} + \varphi_1(s) y_n \\ \quad + h \sum_{j=1}^{m} \Big(\chi_j(s) f(P(t_{n-1} + c_j h)) + \psi_j(s) f(P(t_n + c_j h)) \Big), \\ y_{n+1} = P(t_{n+1}), \end{cases} \tag{1.2}$$

$s \in (0, 1]$, $n = 1, 2, \ldots, N-1$. Here, $c = [c_1, \ldots, c_m]^T$ is the abscissa vector and $\varphi_0(s)$, $\varphi_1(s)$, $\chi_j(s)$, and $\psi_j(s)$, $j = 1, 2, \ldots, m$, are polynomials which define the method. This formula defines the polynomial $P(t)$ on the current step from t_n to t_{n+1}, while $P(t_{n-1} + c_j h)$ corresponds to the polynomial values computed in the previous step from t_{n-1} to t_n. This method requires the starting procedure to compute the approximate solution on the initial interval $[t_0, t_1]$. For this purpose we can use, for example, the continuous Runge-Kutta methods constructed by Owren and Zennaro [21], [22], [23].

Putting

$$Y_j^{[n-1]} = P(t_{n-1} + c_j h), \quad Y_j^{[n]} = P(t_n + c_j h), \quad j = 1, 2, \ldots, m,$$

the method (1.2) corresponding to $s = c_i$, $i = 1, 2, \ldots, m$, can be written as two-step Runge-Kutta (TSRK) method of the form

$$\begin{cases} y_{n+1} = \theta y_{n-1} + \tilde{\theta} y_n + h \sum_{j=1}^{m} \Big(v_j f(Y_j^{[n-1]}) + w_j f(Y_j^{[n]}) \Big), \\ Y_i^{[n]} = u_i y_{n-1} + \tilde{u}_i y_n + h \sum_{j=1}^{m} \Big(a_{ij} f(Y_j^{[n-1]}) + b_{ij} f(Y_j^{[n]}) \Big), \end{cases} \tag{1.3}$$

$i = 1, 2, \ldots, m$, $n = 1, 2, \ldots, N-1$, with

$$\begin{array}{l} \theta = \varphi_0(1), \quad \tilde{\theta} = \varphi_1(1), \quad v_j = \chi_j(1), \quad w_j = \psi_j(1), \\ u_i = \varphi_0(c_i), \, \tilde{u}_i = \varphi_1(c_i), \, a_{ij} = \chi_j(c_i), \, b_{ij} = \psi_j(c_i). \end{array}$$

This general class of TSRK methods was introduced by Jackiewicz and Tracogna [16] and further investigated in [1], [3], [9], [10], [11], [14], [18], [28], and [29]. The special case of collocation methods (1.2) provide continuous approximation to the solution $y(t)$ of (1.1) on the whole interval of integration, and not only at the gridpoints $\{t_n\}$ as is the case for the methods defined by (1.3).

Different approach to the construction of continuous two-step Runge-Kutta methods is presented in [17], [4] and [6]. Continuous two-step Runge-Kutta methods for delay differential equations are considered in [2], [5] and for Volterra integral equations in [12].

The organization of this paper is as follows. In Section 2 we derive the order conditions so that the method (1.2) has uniform order p and stage order $q = p$. In Section 3 we derive the recurrence relation which are needed to analyze linear stability properties of these methods. In Section 4 the estimation of the principal part of the local error is discussed. The analysis of methods with $m = 1$ and $m = 2$ is given in Sections 5 and 6, where the examples of A-stable and L-stable methods are also listed. Finally, in Section 7 some concluding remarks are given and plans for future research are briefly outlined.

2 Order conditions

In this section we derive continuous order conditions for (1.2) assuming that $P(t_n + sh)$ is a uniform approximation to $y(t_n + sh)$, $s \in (0, 1]$, of order p. As the result the stage values $P(t_n + c_j h)$ have (stage) order $q = p$. To this end we investigate the local discretization error $\xi(t_n + sh)$ of (1.2) which is defined as the residuum obtained by replacing $P(t_n + sh)$ by $y(t_n + sh)$, $P(t_n + c_j h)$ by $y(t_n + c_j h)$, $j = 1, 2, \ldots, m$, y_{n-1} by $y(t_{n-1})$ and y_n by $y(t_n)$, where $y(t)$ is the solution to (1.1). This leads to

$$\begin{aligned} \xi(t_n + sh) &= y(t_n + sh) - \varphi_0(s) y(t_n - h) - \varphi_1(s) y(t_n) \\ &\quad - h \sum_{j=1}^{m} \Big(\chi_j(s) y'(t_n + (c_j - 1)h) + \psi_j(s) y'(t_n + c_j h) \Big), \end{aligned} \tag{2.1}$$

$s \in (0, 1]$, $n = 1, 2, \ldots, N - 1$. We have the following theorem.

Theorem 1 *Assume that the function $f(y)$ is sufficiently smooth. Then the method (1.2) has uniform order p if the following conditions are satisfied*

$$\begin{cases} \varphi_0(s) + \varphi_1(s) = 1, \\ \dfrac{(-1)^k}{k!} \varphi_0(s) + \displaystyle\sum_{j=1}^{m} \left(\chi_j(s) \frac{(c_j - 1)^{k-1}}{(k-1)!} + \psi_j(s) \frac{c_j^{k-1}}{(k-1)!} \right) = \dfrac{s^k}{k!}, \end{cases} \tag{2.2}$$

$s \in [0, 1]$, $k = 1, 2, \ldots, p$. Moreover, the local discretization error (2.1) takes the form

$$\xi(t_n + sh) = h^{p+1} C_p(s) y^{(p+1)}(t_n) + O(h^{p+2}), \tag{2.3}$$

as $h \to 0$, where the error function $C_p(s)$ is defined by

$$C_p(s) = \frac{s^{p+1}}{(p+1)!} - \frac{(-1)^{p+1}}{(p+1)!} \varphi_0(s) - \sum_{j=1}^{m} \left(\chi_j(s) \frac{(c_j - 1)^p}{p!} + \psi_j(s) \frac{c_j^p}{p!} \right). \tag{2.4}$$

Proof . Expanding $y(t_n+sh)$, $y(t_n-h)$, $y'(t+(c_j-1)h)$ and $y(t_n+c_jh)$ into Taylor series around the point t_n and collecting terms with the same powers of h we obtain

$$\begin{aligned}\xi(t_n+sh) = {} & \big(1-\varphi_0(s)-\varphi_1(s)\big)y(t_n) \\ & + \sum_{k=1}^{p+1}\left(\frac{s^k}{k!}-\frac{(-1)^k}{k!}\varphi_0(s)\right)h^k y^{(k)}(t_n) \\ & - \sum_{k=1}^{p+1}\sum_{j=1}^{m}\left(\chi_j(s)\frac{(c_j-1)^{k-1}}{(k-1)!}+\psi_j(s)\frac{c_j^{k-1}}{(k-1)!}\right)h^k y^{(k)}(t_n) \\ & + O(h^{p+2}).\end{aligned}$$

Equating to zero the terms of order k, $k=0,1,\ldots,p$, we obtain order conditions (2.2). Comparing the terms of order $p+1$ we obtain (2.3) with error function $C_p(s)$ defined by (2.4). □

The condition

$$\varphi_0(s)+\varphi_1(s)=1, \quad s\in[0,1],$$

is the generalization of preconsistency conditions for TSRK methods (1.3), compare [15]. This condition implies that θ, $\tilde{\theta}$, u_j and $\tilde{u}_j$ appearing in (1.3) satisfy the conditions

$$\theta+\tilde{\theta}=1, \quad u_j+\tilde{u}_j=1, \quad j=1,2,\ldots,m.$$

We are mainly interested in methods corresponding to $p=m+r$, where $r=1,2,\ldots,m+1$, and the next theorem examines the solvability of the linear systems of equations (2.2) corresponding to these orders.

Theorem 2 *Assume that $c_i\neq c_j$, and $c_i\neq c_j-1$ for $i\neq j$. Then the system of continuous order conditions (2.2) corresponding to $p=m+r$, where $r=1,2,\ldots,m$, has a unique solution $\varphi_1(s)$, $\chi_j(s)$, $j=m-r+1,m-r+2,\ldots,m$, and $\psi_j(s)$, $j=1,2,\ldots,m$, for any given polynomials $\varphi_0(s)$ and $\chi_j(s)$, $j=1,2,\ldots,m-r$. The system (2.2) corresponding to $p=2m+1$ has a unique solution $\varphi_0(s)$, $\varphi_1(s)$, $\chi_j(s)$, and $\psi_j(s)$, $j=1,2,\ldots,m$, which are polynomials of degree $\leq 2m+1$.*

Proof . Observe that the polynomial $\varphi_1(s)$ is uniquely determined from the first equation of (2.2). The proof of the first part of the theorem for $p=m+r$, $r=1,2,\ldots,m$, follows from the fact that the matrices of these systems (2.2) corresponding to $\chi_j(s)$, $j=m-r+1,m-r+2,\ldots,m$, are Vandermonde matrices. The second part of the theorem corresponding to $p=2m+1$ is technically more complicated and the details are given in [13]. □

The next result shows that the polynomials $\varphi_0(s)$, $\varphi_1(s)$, $\chi_j(s)$, and $\psi_j(s)$, $j=1,2,\ldots,m$, corresponding to the methods of order $p=2m+1$ satisfy some interpolation and collocation conditions.

Theorem 3 *Assume that* $\varphi_0(s)$, $\varphi_1(s)$, $\chi_j(s)$, *and* $\psi_j(s)$, $j = 1, 2, \ldots, m$, *satisfy (2.2) for* $p = 2m + 1$. *Then these polynomials satisfy the interpolation conditions*

$$\begin{aligned} &\varphi_0(0) = 0, \quad \varphi_1(0) = 1, \quad \chi_j(0) = 0, \quad \psi_j(0) = 0, \\ &\varphi_0(-1) = 1, \varphi_1(-1) = 0, \chi_j(-1) = 0, \psi_j(-1) = 0, \end{aligned} \tag{2.5}$$

and the collocation conditions

$$\begin{aligned} &\varphi_0'(c_i) = 0, \qquad \varphi_1'(c_i) = 0, \qquad \chi_j'(c_i) = 0, \qquad \psi_j'(c_i) = \delta_{ij}, \\ &\varphi_0'(c_i - 1) = 0, \varphi_1'(c_i - 1) = 0, \chi_j'(c_i - 1) = \delta_{ij}, \psi_j'(c_i - 1) = 0, \end{aligned} \tag{2.6}$$

$i, j = 1, 2, \ldots, m$. *Here,* δ_{ij} *is the Kronecker delta, i.e.,* $\delta_{ii} = 1$ *and* $\delta_{ij} = 0$ *for* $i \neq j$.

Proof . The conditions (2.5) follow immediately by substituting $s = 0$ and $s = -1$ into (2.2) corresponding to $p = 2m + 1$. To show (2.6) we differentiate (2.2) to get

$$\begin{cases} \varphi_0'(s) + \varphi_1'(s) = 0, \\ \dfrac{(-1)^k}{k!}\varphi_0'(s) + \displaystyle\sum_{j=1}^{m}\left(\chi_j'(s)\frac{(c_j-1)^{k-1}}{(k-1)!} + \psi_j'(s)\frac{c_j^{k-1}}{(k-1)!}\right) = \frac{s^{k-1}}{(k-1)!}, \end{cases} \tag{2.7}$$

$k = 1, 2, \ldots, 2m + 1$. Substituting $s = c_i$ and $s = c_i - 1$, $i = 1, 2, \ldots, m$, into (2.7) we obtain (2.6). □

It follows from (2.5) and (2.6) that the polynomial $P(t)$ defined by (1.2) satisfies the interpolation conditions

$$P(t_n) = y_n, \quad P(t_{n-1}) = y_{n-1},$$

and the collocation conditions

$$P'(t_n + c_i h) = f(P(t_n + c_i h)), \quad P'(t_{n-1} + c_i h) = f(P(t_{n-1} + c_i h)),$$

$i = 1, 2, \ldots, m$. It also follows from (2.5) that the methods described in Theorem 3 satisfy the conditions $C_p(-1) = 0$ and $C_p(0) = 0$.

For the methods of order $p = m + r$, $r = 1, 2, \ldots, m$, we will choose $\varphi_0(s)$ and $\chi_j(s)$, $j = 1, 2, \ldots, m - r$, as polynomials of degree $\leq m + r$ which satisfy the interpolation conditions

$$\varphi_0(0) = 0, \quad \chi_j(0) = 0, \quad j = 1, 2, \ldots, m - r, \tag{2.8}$$

and the collocation conditions

$$\varphi_0'(c_i) = 0, \quad \chi_j'(c_i) = 0, \quad j = 1, 2, \ldots, m - r. \tag{2.9}$$

This leads to the polynomials $\varphi_0(s)$ and $\chi_j(s)$, $j = 1, 2, \ldots, m - r$, of the form

$$\varphi_0(s) = s\big(q_0 + q_1 s + \cdots + q_{m+r-1}s^{m+r-1}\big),$$

$$\chi_j(s) = s\big(r_{j,0} + r_{j,1}s + \cdots + r_{j,m+r-1}s^{m+r-1}\big),$$

$j = 1, 2, \ldots, m - r$, where

$$q_0 + 2q_1 c_i + \cdots + (m + r) q_{m+r-1} c_i^{m+r-1} = 0,$$

$$r_{j,0} + 2r_{j,1} c_i + \cdots + (m + r) r_{j,m+r-1} c_i^{m+r-1} = 0,$$

$j = 1, 2, \ldots, m - r$, $i = 1, 2, \ldots, m$. The methods obtained in this way satisfy some of the interpolation and collocation conditions (2.5) and (2.6). We have the following theorem.

Theorem 4 *Assume that $\varphi_0(s)$ and $\chi_j(s)$, $j = 1, 2, \ldots, m - r$, satisfy (2.8) and (2.9). Then the solution $\varphi_1(s)$, $\chi_j(s)$, $j = m - r + 1, m - r + 2, \ldots, m$, and $\psi_j(s)$, $j = 1, 2, \ldots, m$ of (2.2) satisfy the interpolation conditions*

$$\begin{array}{l} \varphi_1(0) = 1, \quad \chi_j(0) = 0, \quad j = m - r + 1, m - r + 2, \ldots, m, \\ \psi_j(0) = 0, \quad j = 1, 2, \ldots, m, \end{array} \tag{2.10}$$

and the collocation conditions

$$\begin{array}{l} \varphi_1'(c_i) = 0, \quad \chi_j'(c_i) = 0, \quad j = m - r + 1, m - r + 2, \ldots, m, \\ \psi_j'(c_i) = \delta_{ij}, \quad j = 1, 2, \ldots, m, \end{array} \tag{2.11}$$

$i = 1, 2, \ldots, m$.

Proof . Substituting $s = 0$ into (2.2) corresponding to $p = m + r$, $r = 1, 2, \ldots, m$, and taking into account that the solution to (2.2) is unique the condition (2.10) follows. Differentiating (2.2) with respect to s and substituting $s = c_i$, $i = 1, 2, \ldots, m$, into the resulting relations for $k = 1, 2, \ldots, m + r$, we obtain (2.11). This completes the proof. □

The formulas obtained by imposing the conditions (2.8) and (2.9) will be then called two-step almost collocation methods. It follows from Theorem 4 that the polynomial $P(t)$ defined by the method (1.2) of order $p = m + r$, $r = 1, 2, \ldots, m$, satisfies the interpolation condition

$$P(t_n) = y_n$$

and the collocation conditions at the points c_i, i.e.,

$$P'(t_n + c_i h) = f(P(t_n + c_i h)), \quad i = 1, 2, \ldots, m.$$

However, in general, these methods do not satisfy the interpolation condition

$$P(t_{n-1}) = y_{n-1}$$

and the collocation conditions

$$P'(t_{n-1} + c_i h) = f(P(t_{n-1} + c_i h)), \quad i = 1, 2, \ldots, m.$$

In our search for highly stable methods (A-stability, L-stability) we will be mainly concerned with methods of order $p = 2m$ and $p = 2m - 1$. The advantage of these methods as compared, for example, with methods of low stage order, consists of the fact that they provide a uniform approximation $P(t)$ of order $p = 2m$ to the solution $y(t)$ of (1.1) over the entire interval of integration $[t_0, T]$. As a result these methods do not suffer from the order reduction phenomenon [7]. This is in contrast to implicit Runge-Kutta methods with m stages of order $p = 2m$, $p = 2m-1$, or $p = 2m-2$ for which the continuous approximation to $y(t)$ is only of (stage) order m. This leads to the reduction of order for stiff systems of ODEs for which the effective order is equal only to the stage order m.

3 Linear stability analysis

To analyze the stability properties of the methods (1.2) we will use the standard test equation

$$y' = \lambda y, \quad t \geq 0, \tag{3.1}$$

where λ is a complex parameter. Applying (1.2) to (3.1) and computing the resulting expression at the points $s = c_i$, $i = 1, 2, \ldots, m$, and $s = 1$ we obtain

$$\begin{cases} P(t_n + c_i h) = \varphi_0(c_i) y_{n-1} + \varphi_1(c_i) y_n \\ \quad + h\lambda \displaystyle\sum_{j=1}^{m} \Big(\chi_j(c_i) P(t_{n-1} + c_j h) + \psi_j(c_i) P(t_n + c_j h) \Big), \\ y_{n+1} = \varphi_0(1) y_{n-1} + \varphi_1(1) y_n \\ \quad + h\lambda \displaystyle\sum_{j=1}^{m} \Big(\chi_j(1) P(t_{n-1} + c_j h) + \psi_j(1) P(t_n + c_j h) \Big), \end{cases} \tag{3.2}$$

$i = 1, 2, \ldots, m$, $n = 1, 2, \ldots, N-1$. Introducing the notation $z = h\lambda$,

$$P(t_n + ch) = \begin{bmatrix} P(t_n + c_1 h) \\ \vdots \\ P(t_n + c_m h) \end{bmatrix}, \quad \varphi_0(c) = \begin{bmatrix} \varphi_0(c_1) \\ \vdots \\ \varphi_0(c_m) \end{bmatrix}, \quad \varphi_1(c) = \begin{bmatrix} \varphi_1(c_1) \\ \vdots \\ \varphi_1(c_m) \end{bmatrix},$$

$$v^T = \begin{bmatrix} \chi_1(1) \cdots \chi_m(1) \end{bmatrix}^T, \quad w^T = \begin{bmatrix} \psi_1(1) \cdots \psi_m(1) \end{bmatrix}^T,$$

and

$$A = [\chi_j(c_i)]_{i,j=1}^{m}, \quad B = [\psi_j(c_i)]_{i,j=1}^{m},$$

(compare also Section 1 for the definition of v, w, A, and B) the relation (3.2) can be written in a vector form

$$\begin{cases} P(t_n + ch) = \varphi_0(c) y_{n-1} + \varphi_1(c) y_n + z\Big(AP(t_{n-1} + ch) + BP(t_n + ch) \Big), \\ y_{n+1} = \varphi_0(1) + \varphi_1(1) y_n + z\Big(v^T P(t_{n-1} + ch) + w^T P(t_n + ch) \Big), \end{cases} \tag{3.3}$$

$n = 1, 2, \ldots, N-1$. Hence,

$$P(t_n + ch) = \big(I - zB\big)^{-1} \Big(\varphi_0(c) y_{n-1} + \varphi_1(c) y_n + zAP(t_{n-1} + ch) \Big) \tag{3.4}$$

and substituting this relation into the equation for y_{n+1} leads to

$$\begin{aligned} y_{n+1} = &\Big(\varphi_0(1) + zw^T(I-zB)^{-1}\varphi_0(c)\Big)y_{n-1} \\ &+ \Big(\varphi_1(1) + zw^T(I-zB)^{-1}\varphi_1(c)\Big)y_n \\ &+ z\Big(v^T + zw^T(I-zB)^{-1}A\Big)P(t_{n-1}+ch). \end{aligned} \tag{3.5}$$

The relations (3.4) and (3.5) are equivalent to

$$\begin{bmatrix} y_{n+1} \\ y_n \\ P(t_n+ch) \end{bmatrix} = \begin{bmatrix} M_{11}(z) & M_{12}(z) & M_{13}(z) \\ 1 & 0 & 0 \\ Q\varphi_1(c) & Q\varphi_0(c) & zQA \end{bmatrix} \begin{bmatrix} y_n \\ y_{n-1} \\ P(t_{n-1}+ch) \end{bmatrix}, \tag{3.6}$$

where

$$\begin{aligned} M_{11}(z) &= \varphi_1(1) + zw^TQ\varphi_1(c), \\ M_{12}(z) &= \varphi_0(1) + zw^TQ\varphi_0(c), \\ M_{13}(z) &= z(v^T + zw^TQA), \end{aligned}$$

and

$$Q = (I-zB)^{-1} \in C^{m\times m}.$$

The matrix appearing in (3.6) is called stability matrix of the method (1.2), and will be denoted by $M(z)$. We have $M(z) \in C^{(m+2)\times(m+2)}$. We also define the stability function of the method (1.2) as

$$p(w,z) = \det\Big(wI - M(z)\Big). \tag{3.7}$$

We will be mainly interested in methods which are A-stable. This means that all the roots $w_1, w_2, \dots, w_{m+2}$ of the polynomial $p(w,z)$ defined by (3.7) are in the unit circle for all $z \in C$ such that $\mathrm{Re}(z) \le 0$. By the maximum principle this will be the case if the denominator of $p(w,z)$ does not have poles in the negative half plane C_- and if the roots of $P(w,iy)$ are in the unit circle for all $y \in R$. This last condition will be investigated using the Schur theorem [25] (see also [19]). This criterion for a polynomial of any degree k can be formulated as follows. Consider the polynomial

$$\phi(w) = c_kw^k + c_{k-1}w^{k-1} + \cdots + c_1w + c_0,$$

where c_i are complex coefficients, $c_k \neq 0$ and $c_0 \neq 0$. $\phi(w)$ is said to be a Schur polynomial if all its roots w_i, $i = 1, 2, \dots, k$, are inside of the unit circle. Define

$$\hat{\phi}(w) = \bar{c}_0w^k + \bar{c}_1w^{k-1} + \cdots + \bar{c}_{k-1}w + \bar{c}_k,$$

where $\bar{c}_i$ is the complex conjugate of c_i. Define also the polynomial

$$\phi_1(w) = \frac{1}{w}\Big(\hat{\phi}(0)\phi(w) - \phi(0)\hat{\phi}(w)\Big)$$

of degree at most $k-1$. We have the following theorem.

Theorem 5 *(Schur [25]). $\phi(w)$ is a Schur polynomial if and only if*

$$|\hat{\phi}(0)| > |\phi(0)|$$

and $\phi_1(w)$ is a Schur polynomial.

We will be also interested in methods which are L-stable, i.e., methods which are A-stable and all the roots of the stability function $p(w, z)$ given by (3.7) are equal to zero as $z \to -\infty$. Examples of such methods will be given in Section 5 and in Section 6.

4 Local error estimation

It was demonstrated in Section 2 that the local discretization error at the point t_{n+1} of the m-stage method (1.2) of order p or is given by

$$\xi(t_{n+1}) = C_p(1)h^{p+1}y^{(p+1)}(t_n) + O(h^{p+2}), \tag{4.1}$$

where the error constant $C_p(1)$ is defined by (2.4) for $s = 1$. We will also consider local error $\text{le}(t_{n+1})$ defined by

$$\text{le}(t_{n+1}) = C_p(1)h^{p+1}\tilde{y}^{(p+1)}(t_n) + O(h^{p+2}), \tag{4.2}$$

where $\tilde{y}(t)$ is the so-called local solution, i.e., the solution to the initial-value problem

$$\begin{cases} \tilde{y}'(t) = f(\tilde{y}(t)), \ t \in [t_n, t_{n+1}], \\ \tilde{y}(t_n) = y_n. \end{cases} \tag{4.3}$$

Assuming that the function $f(y)$ appearing in (1.1) and (4.1) satisfies the Lipschitz condition of the form

$$\|f(y) - f(z)\| \le L\|y - z\|,$$

with a constant $L \ge 0$, subtracting the integral forms of (1.1) and (4.1) we obtain

$$\|y(t) - \tilde{y}(t)\| \le \|y(t_n) - y_n\| + L\int_{t_n}^{t} \|y(s) - \tilde{y}(s)\| ds,$$

$t \in [t_n, t_{n+1}]$. Using Gronwall's lemma (compare for example [27]) yields

$$\|y(t) - \tilde{y}(t)\| \le \|y(t_n) - y_n\| e^{L(t-t_n)}.$$

Hence,

$$\|y(t) - \tilde{y}(t)\| = O(h^p), \quad t \in [t_n, t_{n+1}].$$

Assuming that the function $f(y)$ is sufficiently smooth we have similar conclusion for the derivatives of $y(t)$ and $\tilde{y}(t)$

$$\|y^{(i)}(t) - \tilde{y}^{(i)}(t)\| = O(h^p), \quad t \in [t_n, t_{n+1}], \quad i = 1, 2, \ldots,$$

compare [20], [26]. Therefore, we can conclude that the principal parts, i.e., terms of order $p+1$, of the local discretization error (4.1) and the local error (4.2) are the same.

In the remainder of this section we will look for estimates of $h^{p+1}\tilde{y}^{(p+1)}(t_n)$ of the form

$$\begin{aligned} h^{p+1}\tilde{y}^{(p+1)}(t_n) &= \alpha_0 y_{n-1} + \alpha_1 y_n + \\ &+ h\sum_{j=1}^{m}\Big[\beta_j f(P(t_{n-1}+c_jh)) + \gamma_j f(P(t_n+c_jh))\Big]. \end{aligned} \tag{4.4}$$

We have the following theorem.

Theorem 6 *Assume that the solution $\tilde{y}(t)$ to (4.3) is sufficiently smooth. Then the constants α_0, α_1, β_j, and γ_j, $j=1,2,\ldots,m$ appearing in (4.4) satisfy the system of equations*

$$\begin{cases} \alpha_0 + \alpha_1 = 0, \\ \dfrac{(-1)^k}{k!}\alpha_0 + \displaystyle\sum_{j=1}^{m}\left(\beta_j \frac{(c_j-1)^{k-1}}{(k-1)!} + \gamma_j \frac{c_j^{k-1}}{(k-1)!}\right) = 0, \\ \qquad k = 1,2,\ldots,p, \\ \left(\dfrac{(-1)^{p+1}}{(p+1)!} - C_p(-1)\right)\alpha_0 + \displaystyle\sum_{j=1}^{m}\left(\beta_j \frac{(c_j-1)^p}{p!} + \gamma_j \frac{c_j^p}{p!}\right) = 1. \end{cases} \tag{4.5}$$

Proof . Since $\tilde{y}(t) = y_n$ (compare (4.3)), and the method (1.2) is of order p it is locally of order $p+1$ and we have

$$y_{n-1} = \tilde{y}(t_{n-1}) - C_p(-1)h^{p+1}\tilde{y}^{(p+1)}(t_n) + O(h^{p+2}).$$

We have also

$$P(t_n+sh) = \tilde{y}(t_n+sh) + O(h^{p+1}), \quad s\in[-1,1].$$

Substituting these relations and $y_n = \tilde{y}(t_n)$ into (4.4) we obtain

$$\begin{aligned} h^{p+1}\tilde{y}(t_n) &= \alpha_0\Big(\tilde{y}(t_n-h) - C_p(-1)h^{p+1}\tilde{y}^{(p+1)}(t_n)\Big) + \alpha_1\tilde{y}(t_n) \\ &+ h\sum_{j=1}^{m}\Big(\beta_j\tilde{y}'(t_n+(c_j-1)h) + \gamma_j\tilde{y}'(t_n+c_jh)\Big). \end{aligned}$$

Expanding $\tilde{y}(t_{n-1})$, $\tilde{y}(t_{n+1})$, $\tilde{y}'(t_n+(c_j-1)h)$, and $\tilde{y}'(t_n+c_jh)$ into Taylor series around the point t_n and comparing the terms of order $O(h^k)$ for $k=0,1,\ldots,p+1$ leads to the system (4.5). □

Observe that (4.5) constitutes a system of $p+2$ equations with respect to $2m+2$ unknown coefficients α_0, α_1, β_j, and γ_j, $j=1,2,\ldots,m$. We have the following theorem.

Theorem 7 *Assume that* $c_i \neq c_j$ *and* $c_i \neq c_j - 1$ *for* $i \neq j$*. Then the system (4.5) corresponding to* $p = m + r$*, where* $r = 1, 2, \ldots, m$*, has a family of solutions depending on* $m - r$ *free parameters which may be chosen as, for example,* $\beta_{r+1}, \beta_{r+2}, \ldots, \beta_m$ *or* $\gamma_{r+1}, \gamma_{r+2}, \ldots, \gamma_m$*. In particular, if* $r = m$ *then the solution to the system (4.5) is unique. This system does not have solutions if* $r = m + 1$*.*

Proof . The proof is similar to that of Theorem 2 and is therefore omitted. The interested reader can find complete details in [13]. □

Other choices of free parameters than those indicated in Theorem 7 are also possible. For example, if $r = m - 1 \geq 1$ there is one free parameter which may be chosen as α_1, if $r = m - 2 \geq 1$ there are two free parameters which may be chosen as α_1 and α_0, if $r = m - 3 \geq 1$ there are three free parameters which may be chosen as α_1, α_0, and β_1 or γ_1, and if $r = m - k \geq 1$, $k > 3$, there are k free parameters which may be chosen as α_1, α_0, and β_j or γ_j, $j = 1, 2, \ldots, k - 2$.

5 Analysis of methods with $m = 1$

Consider first the methods (1.2) of order $p = 2m + 1 = 3$. Solving the order conditions (2.2) corrsponding to $m = 1$ and $p = 3$ we obtain a one parameter family of two-step methods depending on the abscissa c. The coefficients of these methods are

$$\varphi_0(s) = \frac{s(6c(c-1) + 3(1-2c)s + 2s^2)}{1 - 6c^2},$$

$$\varphi_1(s) = -\frac{(1+s)(6c^2 - 1 + (1-6c)s + 2s^2)}{1 - 6c^2},$$

$$\chi(s) = -\frac{s(1+s)(2c + 3c^2 - (1+2c)s)}{1 - 6c^2},$$

$$\psi(s) = \frac{s(1+s)(1 - 4c + 3c^2 + (1-2c)s)}{1 - 6c^2},$$

and the error constant $C_3(1)$ is given by

$$C_3(1) = \frac{1 - 3c - 3c^2 + 12c^3 - 6c^4}{6(1 - 6c^2)},$$

with $c \neq \pm\sqrt{6}/6$. To investigate stability properties of (1.2) it is more convenient to work with the polynomial obtained by multipying the stability function (3.7) by its denominator. The resulting polynomial, which will be denoted by the same symbol $p(w, z)$, for this family of methods takes the form

$$p(w, z) = p_3(z)w^3 + p_2(z)w^2 + p_1(z)w + p_0(z), \tag{5.1}$$

where the polynomials $p_i(z),\ i=0,1,2,3$, assume the form

$$p_0(z) = -(c-1)^2c^2z,$$
$$p_1(z) = 5 - 12c + 6c^2 + (2 - 5c + 6c^2 - 6c^3 + 3c^4)z,$$
$$p_2(z) = -4 + 12c - 12c^2 + (4 - 8c - 3c^2 + 6c^3 - 3c^4)z,$$

and

$$p_3(z) = -1 + 6c^2 + (1 - 2c - 2c^2 + c^3)cz.$$

We will investigate next if there exist A-stable methods in this class of two-step formulas of order $p=3$. Let

$$\tilde{p}(w,y) := p(w,iy),$$

where $p(w,z)$ is the stability polynonial (5.1). We compute next the constant polynomial with respect to w, which will be denoted by $\tilde{p}_0(y)$, using the recursive procedure described at the end of Section 3. This polynomial takes the form

$$\tilde{p}_0(y) = \alpha(c)y^4 + \beta(c)y^6 + \gamma(c)y^8,$$

where $\alpha(c)$, $\beta(c)$ and $\gamma(c)$ are polynomials with respect to the abscissa c. It follows from the Schur criterion in Theorem 5 that the condition

$$\tilde{p}_0(y) \geq 0, \quad \text{for all} \quad y \geq 0,$$

is the necessary condition for A-stability. However, it can be verified that the polynomials $\alpha(c)$, $\beta(c)$ and $\gamma(c)$ are not simultaneously greater or equal to zero for any c. This proves that A-stable methods do not exist in this class of methods of order $p=3$. In fact the region of stability of such methods is bounded. This is illustrated in Fig. 1 for $m=1$ and $p=3$, where we have plotted, in the (c,z)-plane, the stability interval of the methods corresponding to each value of c, considering $c \geq \frac{1}{2}$ in order to be $-1 \leq \theta < 1$ for zero-stability.

Consider next the methods (1.2) of order $p = 2m = 2$. We choose the polynomial $\varphi_0(s)$ of degree less than or equal to two which satisfies the interpolation condition (2.8) and collocation condition (2.9), i.e., the conditions

$$\varphi_0(0) = 0 \quad \text{and} \quad \varphi_0'(c) = 0.$$

This leads to the polynomial $\varphi_0(s)$ of the form

$$\varphi_0(s) = q_0 s\left(1 - \frac{1}{2c}s\right), \tag{5.2}$$

where q_0 is a real parameter. Solving the order conditions (2.2) corresponding to $m=1$ and $p=2$, where $\varphi_0(s)$ is given by (5.2), we obtain a two-parameter family of two-step methods depending on the parameter q_0 and the abscissa c. The coefficients of these formulas are given by

$$\varphi_1(s) = 1 - q_0 s + \frac{q_0}{2c}s^2,$$

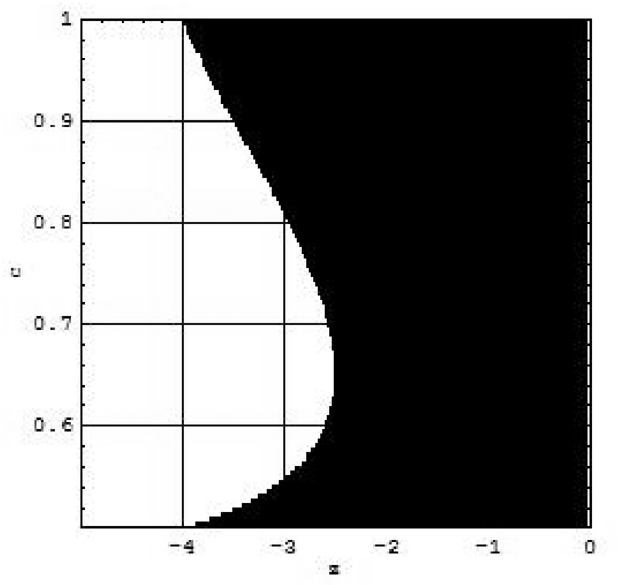

Fig. 1 Region of stability in the (c, z)-plane for the two-step methods (1.2) with $m = 1$ and $p = 3$.

$$\chi(s) = \left(c + \frac{q_0}{2} + cq_0\right)s - \left(\frac{1}{2} + \frac{q_0}{2} + \frac{q_0}{4c}\right)s^2,$$

$$\psi(s) = \left(1 - c + \frac{q_0}{2} - q_0 c\right)s + \left(\frac{1}{2} + \frac{q_0}{2} - \frac{q_0}{4c}\right)s^2,$$

and the error constant $C_2(1)$ takes the form

$$C_2(1) = \frac{10c - 24c^2 + 12c^3 + q_0 - 2q_0 c - 6q_0 c^2 + 12q_0 c^3}{24c}.$$

The stability polynomial of this family of methods is

$$p(w, z) = w\big(p_2(z)w^2 + p_1(z)w + p_0(z)\big), \tag{5.3}$$

where the polynomials $p_0(z)$, $p_1(z)$ and $p_2(z)$ are now given by

$$p_0(z) = 2q_0 - 4q_0 c + (2c - 4c^2 + 2c^3 + q_0 - 2q_0 c - q_0 c^2 + 2q_0 c^3)z,$$

$$p_1(z) = -4c - 2q_0 + 4q_0 c - (6c - 8c^2 + 4c^3 - q_0 + 2q_0 c - 2q_0 c^2 + 4q_0 c^3)z,$$

and

$$p_2(z) = 4c - c^2(4 - 2c + q_0 - 2q_0 c)z.$$

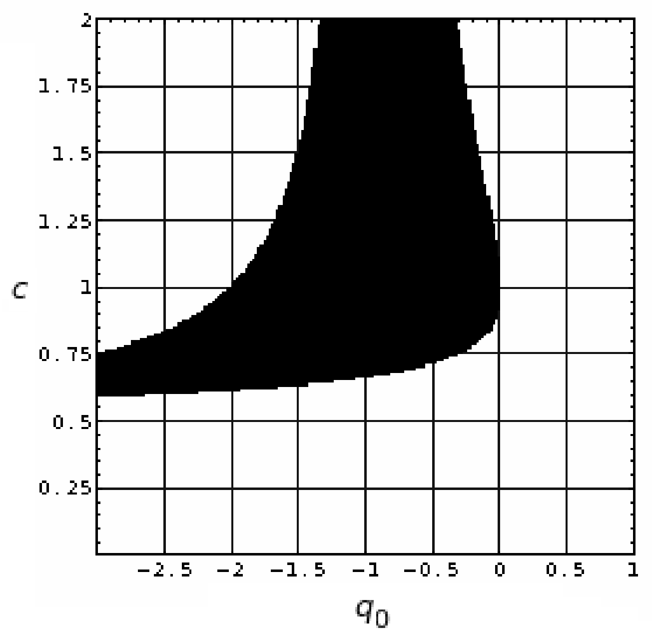

Fig. 2 Region of A-stability in the (q_0, c)-plane for the two-step methods (1.2) with $m = 1$ and $p = 2$.

We have performed a computer search based on the Schur criterion using the polynomial $p(w, z)$ given by (5.3) with $p_0(z)$, $p_1(z)$ and $p_2(z)$ defined above. This search was performed in the parameter space (q_0, c) and the results are presented in Fig. 2 for $-3 \leq q_0 \leq 1$ and $0 \leq c \leq 2$, where the shaded region corresponds to the A-stable formulas. Choosing, for example, $q_0 = -1$ and $c = \frac{3}{4}$ we obtain the A-stable two-step method with coefficients given by

$$\varphi_0(s) = \frac{(2s-3)s}{3}, \quad \varphi_1(s) = \frac{3+3s-2s^2}{3},$$
$$\chi(s) = \frac{(2s-3)s}{6}, \quad \psi(s) = \frac{(2s+3)s}{6}.$$

For this method the stability polynomial $p(w, z)$ is given by

$$p(w, z) = w\bigg(\Big(3 - \frac{27}{16}z\Big)w^2 - \Big(4 + \frac{5}{8}z\Big)w + \Big(1 + \frac{5}{16}z\Big)\bigg),$$

the error constant $C_2(1) = -\frac{17}{144}$ and the constants α_0, α_1, $\beta = \beta_1$ and $\gamma = \gamma_1$ appearing in the estimator of $h^3\tilde{y}^{(3)}(t_n)$ are

$$\alpha_0 = -\frac{288}{95}, \quad \alpha_1 = \frac{288}{95}, \quad \beta = -\frac{72}{19}, \quad \gamma = \frac{72}{95}.$$

We will look next for L-stable methods, i.e., methods for which all roots of the polynomial $p(w,z)/p_2(z))$, where $p(w,z)$ is given by (5.3), are equal to zero as $z \to -\infty$. Such methods correspond to the solutions of the nonlinear system of equations

$$\lim_{z->-\infty} \frac{p_0(z)}{p_2(z)} = 0, \qquad \lim_{z->-\infty} \frac{p_1(z)}{p_2(z)} = 0.$$

It can be verified that this system takes the form

$$\begin{cases} (c-1)(2c-2c^2+q_0-q_0c-2q_0c^2)=0, \\ 6c-8c^2+4c^3-q_0+2q_0c-2q_0c^2+4q_0c^3=0, \end{cases}$$

and has solutions

$$q_0 = -\frac{2}{3}, \quad c=1 \qquad \text{and} \qquad q_0 = -\frac{4}{9}, \quad c=2.$$

The coefficients of the method corresponding to the first set of the above parameters are

$$\varphi_0(s) = \frac{(s-2)s}{3}, \quad \varphi_1(s) = \frac{3+2s-s^2}{3}, \quad \chi(s)=0, \quad \psi(s) = \frac{(s+1)s}{3},$$

and the constants α_0, α_1, β and γ assume the values

$$\alpha_0 = -\frac{12}{5}, \quad \alpha_1 = \frac{12}{5}, \quad \beta = -\frac{18}{5}, \quad \gamma = \frac{6}{5}.$$

The coefficients of the method corresponding to the second set of the parameters q_0 and c are

$$\varphi_0(s) = \frac{s(s-4)}{9},\ \varphi_1(s) = \frac{9+4s-s^2}{9},\ \chi(s) = \frac{2(s-4)s}{9},\ \psi(s) = \frac{(s-1)s}{9},$$

and the constants α_0, α_1, β and γ have the values

$$\alpha_0 = -\frac{108}{115}, \quad \alpha_1 = \frac{108}{115}, \quad \beta = -\frac{54}{23}, \quad \gamma = \frac{162}{115}.$$

It can be verified that for $s=1$ both of the above methods reduce to backward differentiation method of order $p=2$, compare [19], [8].

6 Analysis of methods with $m = 2$

We consider first the methods (1.2) of order $p = 2m+1 = 5$. Solving the order conditions (2.2) corresponding to $m=2$ and $p=5$ we obtain a family of methods depending on the components of the abscissa vector c_1 and c_2. We have plotted in Fig. 1 the contour plots of error constant $C_5(1)$ of these formulas for $0 \le c_1 \le 1$ and $0 \le c_2 \le 1$. Choosing, for example, $c_1 = \frac{1}{2}$ and $c_2 = 1$ we obtain two-step formula of uniform order $p=5$ with coefficients given by

$$\varphi_0(s) = -\frac{(15-10s-30s^2+24s^3)s^2}{29},$$

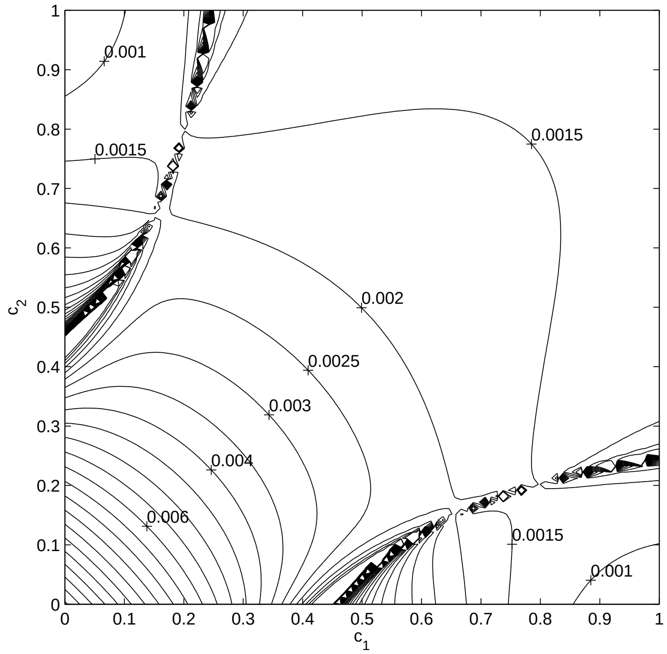

Fig. 1 Contour plots of error constant $C_5(1)$ for $0 \le c_1 \le 1$ and $0 \le c_2 \le 1$.

$$\varphi_1(s) = \frac{(1+s)(29-29s+44s^2-54s^3+24s^4)}{29},$$

$$\chi_1(s) = -\frac{s^2(1+s)(89-187s+96s^2)}{87},$$

$$\chi_2(s) = \frac{s(1+s)(29-31s-16s^2+20s^3)}{29},$$

$$\psi_1(s) = \frac{s^2(1+s)(19+7s-16s^2)}{29},$$

$$\psi_2(s) = -\frac{s^2(1+s)(7-2s-12s^2)}{87}.$$

The error constant of this method is $C_5(1) = \frac{113}{83520}$.

The stability polynomial of two parameter family of methods takes the form

$$p(w,z) = p_4(z)w^4 + p_3(z)w^3 + p_2(z)w^2 + p_1(z)w + p_0(z),$$

where $p_i(z)$, $i = 0,1,2,3,4$ are quadratic polynomials with respect to z. These polynomials depend also on c_1 and c_2. We have performed an extensive computer search based on Schur criterion in the two dimensional space (c_1, c_2) looking for methods with good stability properties but so far we were not able to find methods which are A-stable, because the region of stability of such methods is bounded, as it is illustrated in Fig. 2 for $m = 2$ and $p = 5$, where we have plotted, in the (c_1, z)-plane, the stability interval of the methods corresponding to each value of c_1, considering $c_2 = 1$ and taking $c_1 > \frac{-5+\sqrt{65}}{10}$, in order to satisfy the zero-stability requirement. We suspect that A-stable methods do not exist in the class of formulas with $m = 2$ and $p = 5$, also with respect to other values of c_2.

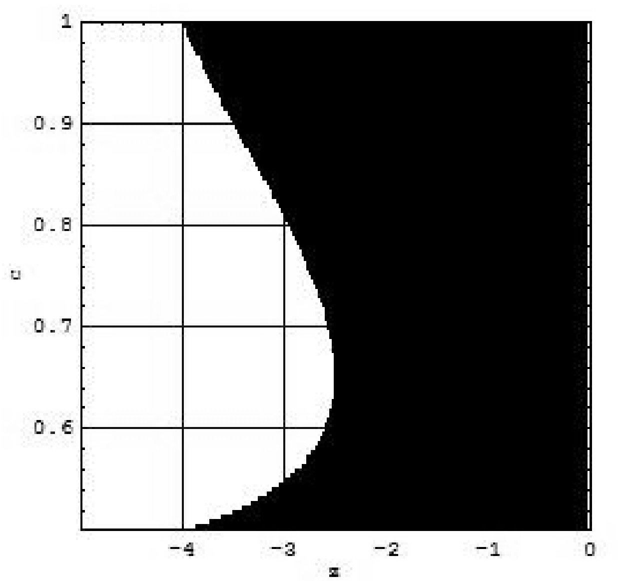

Fig. 2 Region of stability in the (c_1, z)-plane for the two-step methods (1.2) with $m = 2$, $p = 5$ and $c_2 = 1$.

We consider next the methods of order $p = 2m = 4$. We choose the polynomial $\varphi_0(s)$ which satisfies the interpolation condition (2.8) and collocation conditions (2.9), i.e., conditions of the form

$$\varphi_0(0) = 0 \quad \text{and} \quad \varphi_0'(c_i) = 0, \quad i = 1, 2.$$

This leads to the polynomial of the form

$$\varphi_0(s) = s(q_0 + q_1 s + q_2 s^2 + q_3 s^3),$$

where q_2 and q_3 are given by

$$q_2 = -\frac{7q_0 + 6q_1}{3}, \quad q_3 = \frac{3q_0 + 2q_1}{2}.$$

Choosing, for example, $c_1 = \frac{3}{4}$, $c_2 = 1$, $q_0 = q_1 = -1$, we obtain the method with coefficients given by

$$\varphi_0(s) = -\frac{s(27 + 27s - 79s^2 + 39s^3)}{27},$$

$$\varphi_1(s) = \frac{27 + 27s + 27s^2 - 79s^3 + 39s^4}{27},$$

$$\chi_1(s) = -\frac{2s(783 + 1026s - 2669s^2 + 1293s^3)}{405},$$

$$\chi_2(s) = \frac{s(783 + 756s - 2249s^2 + 1113s^3)}{162},$$

$$\psi_1(s) = -\frac{2s(27 + 18s - 97s^2 + 57s^3)}{27},$$

$$\psi_2(s) = \frac{s(837 + 594s - 2881s^2 + 1857s^3)}{810}.$$

The error constant of this method is $C_4(1) = \frac{1085}{248832}$ and the constants α_0, α_1, β_i, and γ_i, $i = 1, 2$, appearing in the estimator of $h^5\tilde{y}^{(5)}(t_n)$ and obtained by the solution of the system (4.5) corresponding to $m = 2$ and $p = 4$ are

$$\alpha_0 = \frac{1244160}{21127}, \quad \alpha_1 = -\frac{1244160}{21127}, \quad \beta_1 = \frac{4810752}{21127},$$

$$\beta_2 = -\frac{4769280}{21127}, \quad \gamma_1 = \frac{2488320}{21127}, \quad \gamma_2 = -\frac{1285632}{21127}.$$

The stability polynomial of the four parameter family of methods of order $p = 4$ takes the form

$$p(w, z) = w(p_3(z)w^3 + p_2(z)w^2 + p_1(z)w + p_0(z)),$$

where $p_i(z)$, $i = 0, 1, 2, 3$ are quadratic polynomials with respect to z. These polynomials depend also on the parameters q_0, q_1, c_1, and c_2. We have performed an extensive computer search based on the Schur criterion in the four dimensional space (q_0, q_1, c_1, c_2) but so far we were not able to find methods which are A-stable. We suspect again that such methods do not exist in this class of formulas with $m = 2$ and $p = 4$.

Finally, consider the methods of order $p = m + 1 = 3$. We choose the polynomials $\varphi_0(s)$ and $\chi_1(s)$ of degree less than or equal to three which satisfy conditions (2.8) and (2.9), i.e.,

$$\varphi_0(0) = 0, \quad \chi_1(0) = 0, \quad \varphi_0'(c_i) = 0, \quad \chi_1'(c_i) = 0, \quad i = 1, 2.$$

These polynomials take the form

$$\varphi_0(s) = s(q_0 + q_1 s + q_2 s^2), \quad \chi_1(s) = s(r_0 + r_1 s + r_2 s^2),$$

where

$$q_1 = r_1 = -\frac{(c_1 + c_2)q_0}{2c_1 c_2}, \quad q_2 = r_2 = \frac{q_0}{3c_1 c_2}.$$

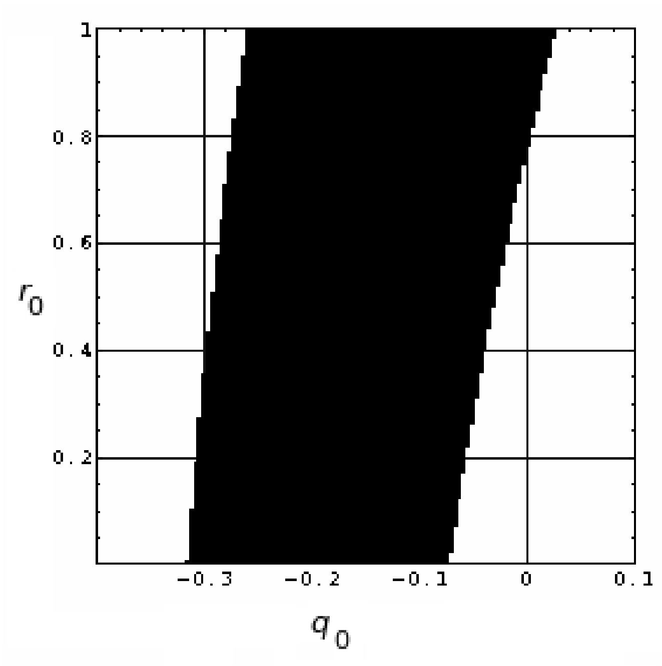

Fig. 3 Region of A-stability in the (q_0, r_0)-plane for the two-step methods (1.2) with $m = 2$ and $p = 3$.

Solving the order conditions (2.2) corresponding to $m = 2$ and $p = 3$ we obtain a four parameter family of methods (1.2) depending on q_0, r_0, c_1 and c_2. The stability polynomial of this family of methods is given by

$$p(w, z) = w^2(p_2(z)w^2 + p_1(z)w + p_0(z)),$$

where $p_i(z)$, $i = 0, 1, 2$, are polynomials of degree less than or equal to two with respect to z. These polynomials depend also on q_0, r_0, c_1 and c_2. We have performed again an extensive computer search looking for methods which are A-stable. We have found such methods only if both components of the abscissa vector are ouside of the interval $[0, 1]$. The results of this search for $c_1 = \frac{5}{2}$ and $c_2 = \frac{9}{2}$ are presented in Fig. 3 for $-0.4 \leq q_0 \leq 0.1$ and $0 \leq r_0 \leq 1$, where the shaded region corresponds to A-stable methods. The coefficients of the resulting methods with $m = 2$ and $p = 3$ are given by $c = [\frac{5}{2}, \frac{9}{2}]$,

$$\varphi_0(s) = \frac{q_0 s(135 - 42s + 4s^2)}{135}, \quad \varphi_1(s) = \frac{135 - 135q_0 s + 42q_0 s^2 - 4q_0 s^3}{135},$$

$$\chi_1(s) = \frac{r_0 s(135 - 42s + 4s^2)}{135},$$

$$\chi_2(s) = -\frac{(135 + 181q_0 - 36r_0)(135 - 42s + 4s^2)s}{1620},$$

$$\psi_1(s) = \left(\frac{63}{8}+\frac{241}{24}q_0-3r_0\right)s-\left(2+\frac{1687}{540}q_0-\frac{14}{15}r_0\right)s^2+\left(\frac{1}{6}+\frac{241}{810}q_0-\frac{4}{45}r_0\right)s^3.$$

$$\psi_2(s) = \left(\frac{35}{8}+\frac{145}{24}q_0-r_0\right)s-\left(\frac{3}{2}+\frac{203}{108}q_0-\frac{14}{45}r_0\right)s^2+\left(\frac{1}{6}+\frac{29}{162}q_0-\frac{4}{135}r_0\right)s^3.$$

The error constant $C_3(1)$ is

$$C_3(1) = \frac{4494825+6019723q_0-1229184r_0}{77760}.$$

For these methods there is a one parameter family of solutions to the system (4.5) which define coefficients α_0, α_1, β_i and γ_i, $i=1,2$. The solution to this system such that $\gamma_1=\gamma_2$ takes the form

$$\alpha_0 = -\alpha_1 = \frac{77760}{9019485+11232679q_0-2293632r_0},$$

$$\beta_1 = \frac{155520}{9019485+11232679q_0-2293632r_0},$$

$$\beta_2 = -\frac{706320}{9019485+11232679q_0-2293632r_0},$$

$$\gamma_1 = \gamma_2 = \frac{314280}{9019485+11232679q_0-2293632r_0}.$$

We have also found methods in this class which are L-stable. Such methods correspond to solutions of the nonlinear system

$$\lim_{z\to-\infty}\frac{p_0(z)}{p_2(z)} = 0, \quad \lim_{z\to-\infty}\frac{p_1(z)}{p_2(z)} = 0.$$

One such solution is

$$q_0 \approx -\frac{21225899}{77647080} \approx -0.273364, \quad r_0 \approx \frac{113887980}{163068619} \approx 0.698405,$$

and the resulting method is A-stable and L-stable. A faster damping of errors can be possibly achieved by the stronger property of stiff accuracy, considered in H. Podhaisky, B.A. Schmitt and R. Weiner [24]. However, this issue is not investigated in this paper.

7 Concluding remarks

We proposed a new class of continuous two-step m-stage methods for the numerical solution of ordinary differential equations. These methods are of uniform order p and stage order $q = p$ and as a result they do not suffer from order reduction phenomenon persistent with methods of low stage order. They are constructed using the collocation approach but by relaxing some of the collocation conditions to obtain methods with desirable stability properties. Local error estimation for these methods is also discussed. Examples of A-stable and L-stable methods are given with $m = 1$ and $p = 2$ and $m = 2$ and $p = 3$.

The construction of high order methods which are A-stable and L-stable is a highly nontrivial task. Future work will address the construction of such methods with $p = m$. The future work will also address various implementation issues such as the choice of appropriate starting procedures, stepsize and order changing strategy, solving nonlinear systems of equations by modified Newton methods and local error estimation for large stepsizes. We hope these methods will constitute building blocks of modern software for stiff differential systems.

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