Modified Collocation Techniques for Volterra Integral Equations

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Abstract

The aim of this paper is the analysis of some new modified collocation based numerical methods for solving Volterra Integral Equations (VIEs), which turn out to be at the heart of many modern applications of Mathematics to natural phenomena and are used more and more for the description of complex systems, in particular evolutionary problems with memory. The developed methods have strong stability properties and higher order of convergence than the classical one-step collocation methods, without any increase of the computational cost, which is an important request in order to approach real problems.

Keywords: `Volterra Integral equations, two-step collocation methods, order conditions,` A`-stability.`

1. Introduction

In this paper we analyze the construction of high order, highly stable new two-step collocation methods for Volterra Integral Equations (VIEs) of the form

$$y(t) = g(t) + \int_0^t K(t, \eta, y(\eta))d\eta, \quad t \in I, \tag{1.1}$$

with $I \subseteq \mathbb{R}_+$, $K \in C(D \times \mathbb{R})$, $D = \{(t, \eta) : 0 \leq \eta \leq t \leq T\}$, K satisfying the uniform Lipschitz condition with respect to the third variable and $g \in C(I)$. VIEs are models of evolutionary problems with memory arising in many applications. In fact, the spread of diseases, the growth of biologic populations, the brain dynamics, elasticity and plasticity, wave problems, heat conduction, fluid dynamics, scattering theory, sismology, biomechanics, game theory, control, queuing theory, design of electronic filters and many other problems from physics, chemistry, pharmacology, medicine, economics can be modelled through systems of VIEs [1,15–18,20]. The following books and survey papers contain sections with various applications of VIEs in the physical and biological sciences and also include extensive lists of references: Brunner [4,5], Agarwal and O'Regan [2], Corduneanu and Sandberg [12], Zhao [25]. Due to the high variety of applications, it gets more and more important to develop efficient numerical methods in order to solve these problems and make some special requirements on these methods, such as high order and strong stability properties.

In the literature many authors (see [4,5] and references therein contained) have analyzed one-step collocation methods for VIEs. As it is well known, a collocation method is based on the idea of approximating the exact solution of a given integral equation with a suitable function belonging to a chosen finite dimensional space, usually a piecewise algebraic polynomial, which satisfies the integral equation exactly on a certain subset of the integration interval (called the set of collocation points). As done in [14] for Ordinary Differential Equations (ODEs), in the papers [9,10] we derived a general classe of m–stage r–step collocation methods for VIEs, with the aim of increasing the order of classical one-step collocation methods without any additional computational cost. The resulting high order methods had, however, bounded stability regions. For this reason, in [11], in analogy to the case of ODEs [13], we introduced a modification in the technique, leading to two-step almost-collocation methods. Such methods have been obtained by relaxing some of the collocation conditions and by introducing some previous stage values, in order to further increase the order and to get A-stability.

In this paper we analyze a modified class of high order two-step collocation methods, providing A-stable methods of uniform order $p = 2m$ on the whole integration interval, where m is the number of collocation points, without relaxing any interpolation or collocation condition.

The paper is organized as follows. In Section 2 we describe the new two-step collocation methods and analyze the order. In Section 3 we carry out the linear stability analysis. In Section 4 we provide examples of one-stage and two-stage A-stable methods. Finally in Section 5 some concluding remarks are given and plans for future research are briefly outlined.

2. Construction of the methods and order conditions

We divide the interval I in N subintervals of fixed length h, obtaining the set of *grid points* $I_h = \{t_n : 0 = t_0 < t_1 < ... < t_N = T\}$ and we define the set of *collocation points* $X_h = \{t_{n,j} := t_n + c_i h : 0 \leq c_1 < c_2 <$

$\cdots < c_m \leq 1$, $n = 0, 1, ..., N-1\}$. The equation (1.1) can be rewritten, by relating it to this mesh, as

$$y(t) = F_n(t) + \Phi_n(t), \quad t \in [t_n, t_{n+1}],$$

where $F_n(t) := g(t) + \int_0^{t_n} k(t, \tau, y(\tau)) d\tau$ and $\Phi_n(t) := \int_{t_n}^{t} k(t, \tau, y(\tau)) d\tau$ represent respectively the *lag term* and the *increment function*.

The collocation polynomial is considered of the form

$$P(t_n + sh) = \varphi(s) y_n + \sum_{j=1}^{m} \chi_j(s) Y_{n-1,j} + \sum_{j=1}^{m} \psi_j(s) Y_{n,j}, \tag{2.1}$$

with $s \in [0, 1]$, where

$$Y_{n-1,j} := P(t_{n-1,j}), \quad Y_{n,j} := P(t_{n,j}), \tag{2.2}$$

and the polynomials $\varphi(s)$, $\chi_j(s)$, $\psi_j(s)$ are determined by imposing the interpolation condition $P(t_n) = y_n$, and by satisfying (2.2). The collocation polynomial (2.1) differs from the polynomial introduced in [11], because we drop the previous time step y_{n-1}, mantaining only the previous stages $Y_{n-1,j}$, as it is usually done in two-step collocation and Runge-Kutta methods for ODEs, in order to get better stability properties and an efficient implementation, see [6–8,19,21,23,24].

By imposing the collocation conditions, i.e. that the collocation polynomial (2.1) exactly satisfies the VIE (1.1) at the collocation points $t_{n,i}$ and by computing $y_{n+1} = P(t_{n+1})$, the two-step collocation method takes the form

$$\begin{cases} Y_{n,i} = F_{n,i} + \Phi_{n,i} \\ y_{n+1} = \varphi(1) y_n + \sum_{j=1}^{m} \chi_j(1) Y_{n-1,j} + \sum_{j=1}^{m} \psi_j(1) Y_{n,j} \end{cases}, \tag{2.3}$$

The lag–term and increment–term approximations

$$F_{n,i} = g(t_{n,i}) + h \sum_{\nu=0}^{n-1} \sum_{l=1}^{\mu_1} b_l k(t_{n,i}, t_\nu + \xi_l h, P_\nu(t_\nu + \xi_l h)) \quad i = 1, ..., m \tag{2.4}$$

$$\Phi_{n,i} = h \sum_{l=1}^{\mu_0} w_{il} k(t_{n,i}, t_n + d_{il} h, P_n(t_n + d_{il} h)) \quad i = 1, ..., m \tag{2.5}$$

are obtained by using quadrature formulas of the form

$$(\xi_l, b_l)_{l=1}^{\mu_1}, \quad (d_{il}, w_{il})_{l=1}^{\mu_0}, \; i = 1, ..., m, \tag{2.6}$$

where the quadrature nodes ξ_l and d_{il} satisfy $0 \leq \xi_1 < ... < \xi_{\mu_1} \leq 1$ and $0 \leq d_{i1} < ... < d_{i\mu_0} \leq 1$, μ_0 and μ_1 are positive integers and w_{il}, b_l are suitable weights, as in [10].

The continuous order conditions and the convergence can be easily analyzed by looking at the method (2.3) as a subclass of the methods introduced in [11], with $\varphi_0(s) \equiv 0$, as shown in the following theorem.

Theorem 2.1. *Assume that the kernel $k(t, \eta, y)$ and the function $g(t)$ in (1.1) are sufficiently smooth. Then the method (2.3) has uniform order p for $s \in [0, 1]$, if the following conditions are satisfied*

$$\begin{cases} 1 - \varphi(s) - \displaystyle\sum_{j=1}^{m} \chi_j(s) - \sum_{j=1}^{m} \psi_j(s) = 0, \\ s^k - \displaystyle\sum_{j=1}^{m} (c_j - 1)^k \chi_j(s) - \sum_{j=1}^{m} c_j^k \psi_j(s) = 0, \end{cases} \tag{2.7}$$

$s \in [0, 1]$, $k = 1, 2, ..., p$. Assume moreover that $c_i \neq c_j$, $c_i \neq c_j - 1$, $c_i \neq 0, 1$. Then the system of continuous order conditions (2.7) is satisfied with $p = 2m$ if and only if the polynomials $\varphi(s)$, $\chi_j(s)$ and $\psi_j(s)$, $j = 1, 2, ..., m$ satisfy the interpolation conditions

$$\varphi(0) = 1, \quad \chi_j(0) = 0, \quad \psi_j(0) = 0 \tag{2.8}$$

and the collocation conditions

$$\varphi(c_i) = 0, \quad \chi_j(c_i) = 0, \quad \psi_j(c_i) = \delta_{ij} \tag{2.9}$$

$$\varphi(c_i - 1) = 0, \quad \chi_j(c_i - 1) = \delta_{ij}, \quad \psi_j(c_i - 1) = 0, \tag{2.10}$$

i=1,2,...,m.

Proof. The order conditions (2.7) can be derived from Theorem 2.1 in [11]. An argument similar to the proof of Theorem 2.2 in [11] leads to the a characterization (2.8)–(2.10) for the coefficients of the methods having order $p = 2m$. □

It can also be proved that the order of convergence is $2m$ if the conditions (2.8)–(2.10) are satisfied and the quadrature formulas (2.6) are of order at least $2m$.

3. Linear stability analysis

In this section we carry out the stability analysis of method (2.3) with respect to the basic test equation

$$y(t) = 1 + \lambda \int_0^t y(\eta)d\eta, \quad t \geq 0, \quad Re(\lambda) \leq 0, \tag{3.1}$$

usually employed in the literature for the stability analysis of numerical methods for VIEs (see [3,5] and their references). Let us consider the following vectors and matrices: $\mathbf{Y}_n = [Y_{n,1}, ..., Y_{n,m}]^T$, $\psi^T(1) = [\psi_1(1), ..., \psi_m(1)]$, $\chi^T(1) = [\chi_1(1), ..., \chi_m(1)]$, $\varphi(\xi) = [\varphi(\xi_1), ..., \varphi(\xi_{\mu_1})]^T$, $\mathbf{b} = [b_1, ..., b_m]^T$, $\beta = [\beta_1, ..., \beta_m]^T$, $\gamma = [\gamma_1, ..., \gamma_m]^T$, $\mathbf{v} = [v_1, ..., v_m]^T$, $\mathbf{\Omega} = [\Omega_{i,j}]_{i,j=1}^m$, $\mathbf{\Lambda} = [\Lambda_{i,j}]_{i,j=1}^m$, where

$$\beta_j = \sum_{l=1}^{\mu_1} b_l \chi_j(\xi_l), \quad \gamma_j = \sum_{l=1}^{\mu_1} b_l \psi_j(\xi_l),$$
$$v_i = \sum_{l=1}^{\mu_0} w_{il}\varphi(d_{il}), \quad \Omega_{i,j} = \sum_{l=1}^{\mu_0} w_{il}\chi_j(d_{il}) \quad \Lambda_{i,j} = \sum_{l=1}^{\mu_0} w_{il}\psi_j(d_{il})$$

and put $\mathbf{e} = [1, 1, .., 1]^T$. The following theorem provides the expression for the stability matrix of the method (2.3) with respect to the test equation (3.1).

Theorem 3.1. *The two-step collocation method (2.3), applied to the test equation (3.1), leads to the following matrix recurrence relation*

$$\begin{bmatrix} y_{n+1} \\ \mathbf{Y}_n \\ y_n \\ \mathbf{Y}_{n-1} \end{bmatrix} = \mathbf{R}(z) \begin{bmatrix} y_n \\ \mathbf{Y}_{n-1} \\ y_{n-1} \\ \mathbf{Y}_{n-2} \end{bmatrix},$$

$z = h\lambda$, *where the stability matrix* $\mathbf{R}(z)$ *is*

$$\mathbf{R}(z) = \mathbf{Q}^{-1}(z)\mathbf{M}(z), \tag{3.2}$$

with the matrices $\mathbf{Q}(z)$ *and* $\mathbf{M}(z)$ *defined by*

$$\mathbf{Q}(z) = \begin{bmatrix} 1 & -\psi^T(1) & 0 & 0 \\ 0 & I - z\mathbf{\Lambda} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \mathbf{I} \end{bmatrix},$$

$$\mathbf{M}(z) = \begin{bmatrix} \varphi(1) & \chi^T(1) & 0 & 0 \\ z\mathbf{v} & \mathbf{I} + z(u\gamma^T + \mathbf{\Omega} - \mathbf{\Lambda}) & z(\mathbf{e}\mathbf{b}^T\varphi(\xi) - \mathbf{v}) & z(\mathbf{e}\beta^T - \mathbf{\Omega}) \\ 1 & 0 & 0 & 0 \\ 0 & \mathbf{I} & 0 & 0 \end{bmatrix}.$$

Proof. By applying the method (2.3) to the test equation (3.1) we obtain

$$y_{n+1} = \varphi(1)y_n + \chi^T(1)\mathbf{Y}_{n-1} + \psi^T(1)\mathbf{Y}_n \tag{3.4}$$

where

$$\mathbf{Y}_n = \mathbf{F}_n + z(y_n\mathbf{v} + \mathbf{\Omega}\mathbf{Y}_{n-1} + \mathbf{\Lambda}\mathbf{Y}_n), \tag{3.5}$$

$$\mathbf{F}_n = \mathbf{e} + z\sum_{\nu=0}^{n-1}(\mathbf{b}^T\varphi(\xi)y_\nu + \beta^T\mathbf{Y}_{\nu-1} + \gamma^T\mathbf{Y}_\nu)\mathbf{e}. \tag{3.6}$$

From the expression (3.6) we derive

$$\mathbf{F}_n - \mathbf{F}_{n-1} = z(\mathbf{b}^T\varphi(\xi)y_{n-1} + \beta^T\mathbf{Y}_{n-2} + \gamma^T\mathbf{Y}_{n-1})\mathbf{e}. \tag{3.7}$$

The computation of the difference $\mathbf{Y}_n - \mathbf{Y}_{n-1}$ by substituting the expression (3.5) for both terms $\mathbf{Y}_n$ and $\mathbf{Y}_{n-1}$, and by using (3.7), leads to

$$\begin{aligned}(\mathbf{I}_m - z\mathbf{\Lambda})\,\mathbf{Y}_n = z\mathbf{v}y_n + \big(\mathbf{I} + z(u\gamma^T + \mathbf{\Omega} - \mathbf{\Lambda})\big)\,\mathbf{Y}_{n-1} + \\ + z(\mathbf{e}\mathbf{b}^T\varphi(\xi) - \mathbf{v})y_{n-1} + z(\mathbf{e}\beta^T - \mathbf{\Omega})\mathbf{Y}_{n-2}.\end{aligned}$$

From the last equation and (3.4) the thesis immediately follows. □

We next consider the stability function of the method

$$p(\omega, z) = \det\Big(\lambda\mathbf{I} - \mathbf{R}(z)\Big), \tag{3.8}$$

where $\mathbf{I}$ is the identity matrix of order $2m+2$, and we investigate on the conditions to impose on the collocation abscissas $c_1, ..., c_m$ in order to get A-stable methods: this means that all the roots $\lambda_1, \ldots, \lambda_{2m+2}$ of (3.8) lie in the unit circle for all $z \in \mathbb{C}$ such that $\mathrm{Re}(z) \leq 0$. The investigation, carried out using the Schur criterion (cfr. [22]), has shown the following results for $m = 1$ and $m = 2$.

Theorem 3.2. *Any one-stage collocation method of the type (2.3) is A-stable if and only if $c > 1$.*

Fig. 1 shows the A-stability region in the parameter space (c_1, c_2) for two stage collocation methods (2.3).

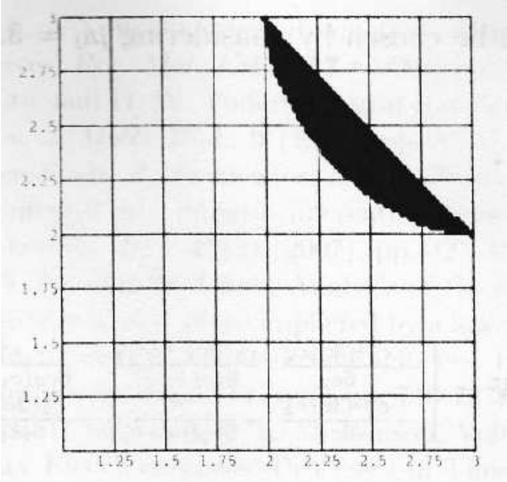

Fig. 1. Region of A-stability in the (c_1, c_2)-plane for two-step methods (2.3) with $m = 2$ and order 4.

4. Examples of methods

We first consider the case $m = 1$. According to Theorem 3.2, for any value of $c > 1$ we obtain $A-$stable methods of order 2. Solving the order conditions (2.7) for $m = 1$, $p = 2$, we obtain

$$\varphi(s) = \frac{s^2 + s(1-2c) + c(c-1)}{c(c-1)}, \quad \chi(s) = \frac{s(c-s)}{c-1}, \quad \psi(s) = \frac{s(1-c+s)}{c}.$$

The weights in (2.4) and (2.5) can be chosen by discretizing the lag-term by the trapezoidal rule and the increment term by the midpoint rule, i.e., $\mu_0 = 1$, $\mu_1 = 3$,

$$\xi = [0, c, 1]^T, \quad \mathbf{D} = c, \quad \mathbf{b} = \left[\frac{1}{2}, 0, \frac{1}{2}\right]^T, \quad \mathbf{W} = c.$$

This leads to a one parameter family of methods of order $p = 2$, depending on the collocation abscissa c.

We next consider the case $m = 2$. Solving the order conditions (2.7) for $m = 2$, $p = 4$, and choosing, according to Fig. 1, $c_1 = \frac{11}{5}$ and $c_2 = \frac{13}{5}$, we obtain

$$\varphi(s) = \frac{\left(126 - 115s + 25s^2\right)\left(66 - 85s + 25s^2\right)}{8316},$$
$$\chi_1(s) = -\frac{s\left(5s-11\right)\left(126 - 115s + 25s^2\right)}{144}, \quad \chi_2(s) = \frac{s\left(5s-14\right)\left(66 - 85s + 25s^2\right)}{54}$$
$$\psi_1(s) = -\frac{s(5s-6)\left(126 - 115s + 25s^2\right)}{66}, \quad \psi_2(s) = \frac{s(5s-9)\left(66 - 85s + 25s^2\right)}{336}$$

The weights in (2.4) and (2.5) can be chosen by considering $\mu_0 = 3$, $\mu_1 = 4$, and

$$\xi = \begin{bmatrix} 0 \\ c_1 \\ c_2 \\ 1 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 0 & c_1 & c_2 \\ 0 & c_1 & c_2 \end{bmatrix},$$

$$\mathbf{b} = \begin{bmatrix} \frac{-1+2c_1+2c_2-6c_1c_2}{12c_1c_2} \\ \frac{1-2c_2}{12c_1(c_1-1)(c_1-c_2)} \\ \frac{2c_1-1}{12c_2(c_2-1)(c_2-c_1)} \\ \frac{-3+4c_1+4c_2-6c_1c_2}{12(c_1-1)(c_2-1)} \end{bmatrix}, \quad \mathbf{W} = \begin{bmatrix} -\frac{c_1^2-3c_1c_2}{6c_2} & \frac{c_1(2c_1-3c_2)}{6(c_1-c_2)} & \frac{c_1^3}{6c_2(c_1-c_2)} \\ -\frac{c_2^2-3c_1c_2}{6c_1} & -\frac{c_1^3}{6c_1(c_1-c_2)} & -\frac{c_2(2c_2-3c_1)}{6(c_1-c_2)} \end{bmatrix},$$

i.e., with $c_1 = \frac{11}{5}$ and $c_2 = \frac{13}{5}$,

$$\xi = \begin{bmatrix} 0 \\ \frac{11}{5} \\ \frac{13}{5} \\ 1 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 0 & \frac{11}{5} & \frac{13}{5} \\ 0 & \frac{11}{5} & \frac{13}{5} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} \frac{233}{616} \\ -\frac{575}{2376} \\ \frac{425}{4536} \\ \frac{499}{648} \end{bmatrix}, \quad \mathbf{W} = \begin{bmatrix} \frac{431}{420} & \frac{22}{9} & -\frac{1331}{1260} \\ \frac{133}{165} & \frac{1372}{495} & -\frac{7}{9} \end{bmatrix}.$$

5. Concluding remarks

We have developed a class of modified two-step collocation methods (2.3) for the numerical solution of VIEs. These methods are of uniform order $p = 2m$ on the whole integration interval. We have discussed their stability properties, deriving $A-$stable methods. Examples of methods have also been provided.

The above methods seem to be promising for further investigations, because of their good properties of accuracy and stability. The uniform order and the continuous approximation to the solution make such methods particulary suitable for a variable stepsize implementation. The implementation issues related to these methods are subject of future work. They include the choice of appropriate starting procedures, estimation of local discretization error and stepsize changing strategies. Our aim is also to extend the results to other functional equations, such as Volterra integro-differential equations.

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