On the numerical treatment of selected oscillatory evolutionary problems

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Abstract. We focus on evolutionary problems whose qualitative behaviour is known a-priori and exploited in order to provide efficient and accurate numerical schemes. For classical numerical methods, depending on constant coefficients, the required computational effort could be quite heavy, due to the necessary employ of very small stepsizes needed to accurately reproduce the qualitative behaviour of the solution. In these situations, it may be convenient to use *special purpose* formulae, i.e. non-polynomially fitted formulae on basis functions adapted to the problem (see [16, 17] and references therein). We show examples of special purpose strategies to solve two families of evolutionary problems exhibiting periodic solutions, i.e. partial differential equations and Volterra integral equations.

Partial differential equations generating periodic wavefronts

Let us consider the following reaction-diffusion problem [15, 18]

$$u_t = u_{xx} + \lambda(r)u - \omega(r)v, \qquad v_t = v_{xx} + \omega(r)u + \lambda(r)v, \tag{1}$$

where $u, v : [0, \infty) \times [0, T] \longrightarrow \mathbb{R}$, $r = \sqrt{u^2 + v^2}$, $\omega(0) > 0$, $\lambda(0) > 0$. It is a nonlinear problem, whose nonlinearity is dictated by the functions $\lambda(r)$ and $\omega(r)$

It is well known (compare [12, 13, 19] and references therein) that such problem generate periodic wavefronts admitting the following parametrization [15],

$$u(x, t) = \widehat{r}\cos(\omega(\widehat{r})t \pm \sqrt{\lambda(\widehat{r})}x), \quad v(x, t) = \widehat{r}\sin(\omega(\widehat{r})t \pm \sqrt{\lambda(\widehat{r})}x), \tag{2}$$

with $\widehat{r} \in \mathbb{R}$ is such that $\lambda(\widehat{r}) > 0$. Though incomputable, the expression (2) a priori clarifies the periodic character in time and space. We provide a spatial discretization of (1), by means of trigonometrically fitted finite differences, as follows [7, 8]. For a given function $u(x, t)$ defined on the rectangular domain $D = [x_0, X] \times [t_0, T] \subset \mathbb{R}^2$, we compute the following three-point finite difference

$$\frac{\partial^2 u}{\partial x^2}(x, t) \approx \frac{1}{h^2}\left(a_0 u(x + h, t) + a_1 u(x, t) + a_2 u(x - h, t)\right), \tag{3}$$

where h is a given spatial stepsize, whose coefficients a_0, a_1 and a_2 are computed in order to make it exact on the functional basis

$$\mathcal{F} = \{1, \sin(\mu x), \cos(\mu x)\}, \tag{4}$$

with $\mu \in \mathbb{R}$, leading to

$$a_0(z) = -\frac{z^2}{2(\cos(z) - 1)}, \quad a_1(z) = \frac{z^2}{\cos(z) - 1}, \quad a_2(z) = -\frac{z^2}{2(\cos(z) - 1)}, \tag{5}$$

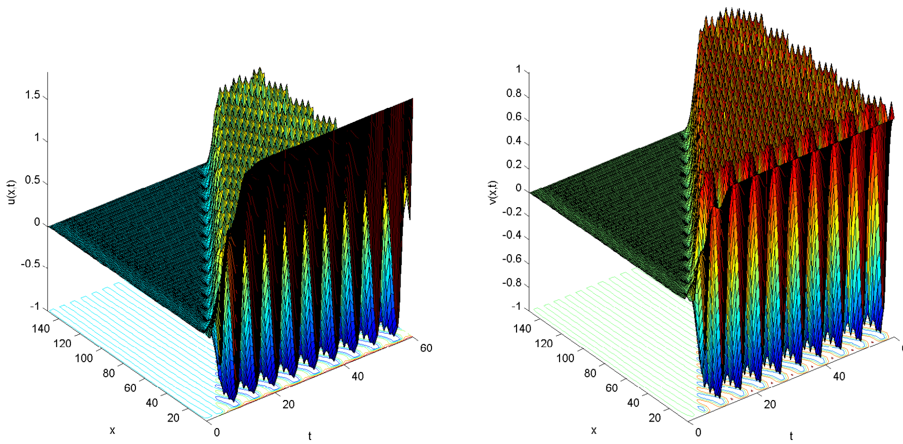

FIGURE 1. Numerical solution of (1), with initial conditions (7), boundary conditions (6), with parameters given by (8). The left figure is the plot of $u(x,t)$, the one on the right is $v(x,t)$. The solution is computed by solving the semi-discretized problem obtained by the three-point trigonometrically fitted finite difference (3), with coefficients (5).

with $z = \mu h$. The chosen fitting space (4) depends on the unknown parameter μ which can be recovered by the parametrization (2) of the wavefront: indeed, at the mesh point (x_i, t_j), the value

$$z_{ij} = \sqrt{\lambda(r_{ij})}h,$$

where $r_{ij} = \sqrt{u_{ij}^2 + v_{ij}^2}$, is assumed as estimation of the parameter. Such an estimate is clearly cheap, since it does not require applying optimization techniques or solving nonlinear systems of equations as in [9, 10, 14] and references therein.

We now present some numerical results obtained by solving the system of PDEs (1) with $\lambda(r)$ and $\omega(r)$ of the form $\lambda(r) = \lambda_0 - r^p$, $\omega(r) = \omega_0 - r^p$, with $\lambda_0, \omega_0, p \in \mathbb{R}^+$.

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial v}{\partial x}(0,t) = 0, \qquad \lim_{x\to+\infty} u(x,t) = \lim_{x\to+\infty} v(x,t) = 0, \tag{6}$$

and the initial conditions

$$u(x,0) = f_0(x), \qquad v(x,0) = g_0(x). \tag{7}$$

We assume

$$\lambda_0 = 1, \quad \omega_0 = 2, \quad p = 1.8, \quad A = 0.1, \quad \xi = 0.8. \tag{8}$$

Figures 1 shows the profile of the solutions originated by applying the trigonometrically fitted spatial semi-discretization with 3 points, i.e through finite differences (3) with coefficients (5), and solving in time with the `ode15s` Matlab routine. Analogously, the solutions of the semi-discretized problem by standard finite difference (3), with coefficients $a_0(z) = 1$, $a_1(z) = -2$, $a_2(z) = 1$, are depicted in Figure 2; also in this case, the `ode15s` time solver is applied. We have involved 50 subinterval in the spatial semi-discretization: hence, the spatial stepsize is $h = 3$. As it is visible from Figure 1, the profile of the obtained solutions is coherent with the expected dynamics, while such a situation is not visible in Figure 2, since an unstable behavior is visible in the results.

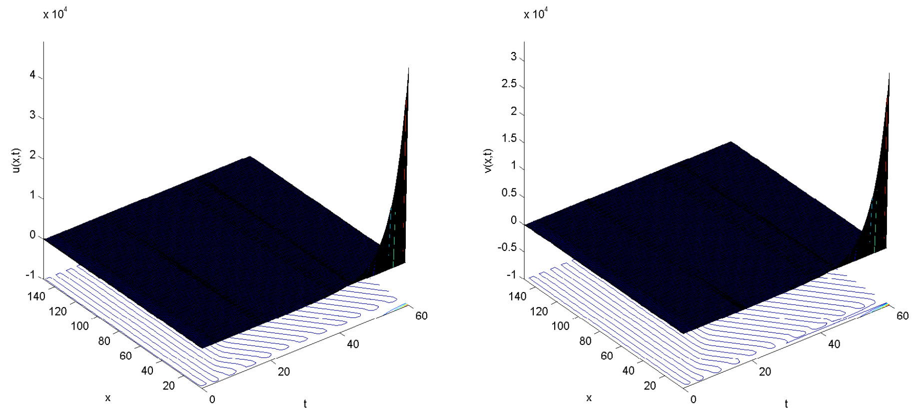

FIGURE 2. Numerical solution of (1), with initial conditions (7), boundary conditions (6), with parameters given by (8). The left figure is the plot of $u(x,t)$, the one on the right is $v(x,t)$. The solution is computed by solving the semi-discretized problem obtained by the standard version of the finite difference (3).

Volterra integral equations with periodic solutions

We consider the Volterra integral equation

$$\begin{aligned} y(x) &= f(x) + \int_{-\infty}^{x} k(x-s)y(s)ds, \;\; x \in [0, x_{end}] \\ y(x) &= \psi(x), \qquad\qquad -\infty < x \leq 0, \end{aligned} \tag{9}$$

with $k \in L^1(\boldsymbol{R}^+)$, f continuous and T-periodic on $[0, x_{end}]$, ψ continuous and bounded on $\boldsymbol{R}^-$. Under suitable hypotheses, (9) has a unique T-periodic solution [1]. In the numerical treatment, standard numerical procedures are not efficient, especially for high frequency values, thus we propose a specially tuned direct quadrature (DQ) method based on exponential fitting (compare also [2, 3, 4, 5]).

Following the exponential fitting theory, we formulate a DQ method which is exact whenever the solution $y(x)$ belongs to the fitting space

$$\mathcal{B}_1 := \{1, x, \sin(\omega x), \cos(\omega x)\}, \tag{10}$$

and $k(x) = \exp(\alpha x)$, $\alpha, \omega \in \mathbb{R}$.
The DQ method we propose is based on the quadrature rule Q, with

$$\int_{X-h}^{X+h} g(x)dx \approx Q[g](X) := h \sum_{k=0}^{1} a_k g(X + \xi_k h) \tag{11}$$

where $X > 0$ and $h > 0$. We impose that such rule is exact on the fitting space

$$\mathcal{B} := \{e^{\alpha x}, xe^{\alpha x}, e^{(\alpha \pm i\omega)x}\}, \tag{12}$$

coming to a nonlinear system of equations in the unknowns weights and nodes. This yields to $a_k = a_k(u,z)$, and $\xi_k = \xi_k(u,z)$, with $u =: \alpha h$, $z := \omega h$. Then it is an easy task to derive the composite quadrature rule based on the formula (11):

$$I[g](X) = \int_a^b g(x)dx \approx Q_m[g] := h \sum_{j=0}^{m-1} \sum_{k=0}^{1} \tilde{a}_k g(t_j + \tilde{\xi}_k h), \tag{13}$$

where $t_j = a + hj$, $j = 0, \ldots, m$, $h = (b-a)/m$, $\tilde{a}_k = a_k/2$, $\tilde{\xi}_k = (1 + \xi_k)/2$.

Given a uniform mesh on $[0, x_{end}]$, $I_h := \{x_n = nh, n = 0, \dots, N\}$, with $h = x_{end}/N$, the DQ method based on the exponentially fitted formula (13) reads

$$y(x_n) \approx f(x_n) + (I\psi)(x_n) + h \sum_{j=0}^{n-1} \sum_{i=1}^{2} \tilde{a}_i k(x_{n-j} - \tilde{\xi}_i h) y(x_j + \tilde{\xi}_i h), \tag{14}$$

$n = 1, ..., N$, where

$$(I\psi)(x_n) = \int_{-\infty}^{0} k(x_n - s)\psi(s) ds,$$

or is a suitable approximation of such integral. To obtain a fully discretization of (9), an approximation of $y(x_j + \tilde{\xi}_i h)$ is needed. Therefore, we introduce an approximation by interpolation function $\mathcal{P}$, on the points

$$(x_{j+l}, y_{j+l}), \quad l = -r_-, \dots, r_+.$$

Two choices are available: the first one is the Lagrange polynomial interpolation, easy but unnatural since we are assuming that the solution is a periodic function. The second one is a mixed-trigonometric interpolation, which is exact on the fitting space (10) by design. In both cases the interpolating function $\mathcal{P}$ can be written as follows

$$\mathcal{P}(x_j + sh) = \sum_{l=-r_-}^{r_+} p_l(s) y_{j+l},$$

where $p_l(s)$ do not depend on x_j but only on r_-, r_+. Once we have approximated the values of the solution $y(x_j + \tilde{\xi}_i h)$ in (14) by the interpolation technique, the fully-discrete method is the following

$$y_n = f(x_n) + (I\psi)(x_n) + h \sum_{j=0}^{n-1} \sum_{i=1}^{2} \tilde{a}_i k(x_{n-j} - \tilde{\xi}_i h) \sum_{l=-r_-}^{r_+} p_l(\tilde{\xi}_i h) y_{j+l},$$

$n = 1, ..., N$. We set $r_+ \leq 1$ to avoid the use of values of the solution in future mesh points. The method is explicit for $r_+ = 0$, and implicit for $r_+ = 1$.

We underline that the proposed method has the same order as a DQ method based on standard 2-nodes Gauss quadrature rule. The advantage of the exponentially fitted DQ method with mixed-trigonometric interpolation, is that the error is smaller when periodic problems are treated and the gain is more relevant for high frequency values.

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