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ON THE STABILITY OF ϑ-METHODS FOR STOCHASTIC VOLTERRA INTEGRAL EQUATIONS

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Abstract. The paper is focused on the analysis of stability properties of a family of numerical methods designed for the numerical solution of stochastic Volterra integral equations. Stability properties are provided with respect to the basic test equation, as well as to the convolution test equation. For each equation, stability properties are intended both in the mean-square and in the asymptotic sense. Stability regions are also provided for a selection of methods. Numerical experiments confirming the theoretical study are also given.

1. **Introduction.** Stochastic Volterra integral Equations (SVIEs) are equations of the form

$$X_t = X_0 + \int_0^t a(t,s,X_s)ds + \int_0^t b(t,s,X_s)dW_s, \quad t \in [0,T], \tag{1}$$

where a and b are measurable functions and the initial condition X_0 is a random variable. The second integral in the right hand side is an Itô integral, which has to be taken with respect to the Brownian motion W_s. The solution X_t is a random variable for each t.

Initial contributions regarding the numerical solution of SVIEs have been given in [16, 17, 18], where the authors extended classical methods for stochastic ordinary differential equations (i.e. Euler Maruyama and Milstein methods; see [10, 12] and references therein) to the integral case. However, the analysis mainly regarded their accuracy properties and, in the existing literature on the topic, stability issue have not yet been provided. Our aim is to develop a stability analysis of numerical methods for SVIEs, by exploring a more general family of methods for (1), i.e. that of stochastic ϑ-methods (see [8, 11] for insights on ϑ-methods for stochastic differential and integro-differential equations).

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Similarly as in the deterministic case (see [1, 4, 5, 6] and references therein), we first analyze the stability properties of stochastic ϑ-methods with respect to the following basic linear test equation,

$$X_t = X_0 + \int_0^t \lambda X_s ds + \int_0^t \mu X_s dW_s, \quad t \in [0, T], \tag{2}$$

with $\lambda, \mu \in \mathbb{R}$. Moreover we will also consider the convolution test equation,

$$X_t = X_0 + \int_0^t (\lambda + \sigma(t-s)) X_s ds + \int_0^t \mu X_s dW_s, \quad t \in [0, T], \tag{3}$$

with $\lambda, \mu, \sigma \in \mathbb{R}$. In both cases, we provide stability issues both in the mean-square and in the asymptotic sense (for the analysis of stability of numerical methods for SDEs, see [3, 7, 8, 13, 14] and references therein), as explained in details in the remainder of the manuscript. The paper is organized as follows: Section 2 presents the family of stochastic ϑ-methods for SVIEs (1); stability issues with respect to the basic test equation (2) are given in Section 3, while Section 4 is devoted to analyzing stability properties with respect to the convolution test equation (3). Some numerical experiments confirming the theoretical analysis are given in Section 5 and, finally, concluding remarks are provided in Section 6.

2. ϑ-methods for SVIEs.

As usual in the context of Volterra integral equations (see [1, 4, 5, 6] and references therein), we introduce the set of grid points

$$\mathcal{I}_h = \{t_n = nh, \quad n = 0, ..., N, \ Nh = T\}$$

equidistantly spaced, being h the chosen fixed stepsize. By evaluating the SVIE (1) in the generic mesh point t_n, we have

$$X_{t_n} = X_0 + \int_0^{t_n} a(t_n, s, X_s)\, ds + \int_0^{t_n} b(t_n, s, X_s)\, dW_s.$$

Let $Y_0 = X_0$. We introduce the stochastic ϑ-method, having the form

$$Y_n = Y_0 + h \sum_{i=0}^{n-1} (\vartheta a(t_n, t_{i+1}, Y_{i+1}) + (1-\vartheta) a(t_n, t_i, Y_i)) + \sum_{i=0}^{n-1} b(t_n, t_i, Y_i) \Delta W_i, \tag{4}$$

where $h = t_{i+1} - t_i$ e $\Delta W_i = W_{i+1} - W_i$. Taking into account that the Wiener increments ΔW_i can be replaced by the scaled random variables $\sqrt{h} V_i$, where V_i is a standard Gaussian random variable, i.e. it is $\mathcal{N}(0, 1)$-distributed, the method assumes the form

$$Y_n = Y_0 + h \sum_{i=0}^{n-1} (\vartheta a(t_n, t_{i+1}, Y_{i+1}) + (1-\vartheta) a(t_n, t_i, Y_i)) + \sqrt{h} \sum_{i=0}^{n-1} b(t_n, t_i, Y_i) V_i. \tag{5}$$

The following theorem exhibits the convergence order of (5), under the hypothesis (6), that also guarantee the existence and uniqueness of the solution of (1) (see [15, 18] and references therein). The proof of this theorem follows analogously as in [16], which corresponds to the case $\vartheta = 0$.

Theorem 2.1. *Assuming that the coefficients a and b of (1) satisfy*

$$
\begin{aligned}
|a(t,s,x) - a(t,s,y)| &\leq K_1(t,s)|x-y|, \\
|b(t,s,x) - b(t,s,y)| &\leq K_2(t,s)|x-y|, \\
|a(t_1,s,x) - a(t_2,s,x)|^2 &\leq K_3(t_1,t_2,s)(1+|x|^2)|t_1-t_2|,
\end{aligned}
\tag{6}
$$

$$|b(t_1,s,x) - b(t_1,s,x)|^2 \leq K_4(t_1,t_2,s)(1+|x|^2)|t_1-t_2|,$$

with $K > 0$, for all $s \leq t \in [0,T]$ and $x \in \mathbb{R}$. Then, the stochastic ϑ-method (5) is convergent of order $1/2$, i.e. there exist a constant C such that

$$\mathbb{E}|X_n - Y_n|^2 \leq Ch. \tag{7}$$

In analogy with stochastic differential case [3, 12], the order of convergence of the stochastic ϑ-method can be improved [17] by adding further terms in the numerical approximation, leading to

$$
\begin{aligned}
Y_n = Y_0 &+ h\sum_{i=0}^{n-1}(\vartheta a(t_n,t_{i+1},Y_{i+1}) + (1-\vartheta)a(t_n,t_i,Y_i)) + \sum_{i=0}^{n-1} b(t_n,t_i,Y_i)\Delta W_i \\
&+ \sum_{i=0}^{n-1}\frac{\partial a}{\partial x}(t_n,t_i,Y_i)b(t_i,t_i,Y_i)\int_{t_i}^{t_{i+1}}\int_{t_i}^{s} dW_u\, ds \\
&+ \sum_{i=0}^{n-1}\frac{\partial b}{\partial x}(t_n,t_i,Y_i)b(t_i,t_i,Y_i)\int_{t_i}^{t_{i+1}}\int_{t_i}^{s} dW_u\, dW_s.
\end{aligned}
\tag{8}
$$

where $Y_0 = X_0$, and $\frac{\partial}{\partial x}$ denotes the partial derivative with the respect to the second argument.

As highlighted in [3, 13], the values of increments and the above double integrals can be obtained as sample values of normal random variables using the transformations

$$\Delta W_i = V_{i,1}\sqrt{h},$$

$$\int_{t_i}^{t_{i+1}}\int_{t_i}^{s} dW_u\, ds = \frac{1}{2}\left(V_{i,1} + \frac{V_{i,2}}{\sqrt{3}}\right)h\sqrt{h}, \qquad \int_{t_i}^{t_{i+1}}\int_{t_i}^{s} dW_u\, dW_s = \frac{1}{2}\left(V_{i,1}^2 - 1\right)h,$$

where $V_{i,1}$ and $V_{i,2}$ are mutually independent $\mathcal{N}(0,1)$ random variables. Therefore, the improved stochastic ϑ-method assumes the form

$$
\begin{aligned}
Y_n = Y_0 &+ h\sum_{i=0}^{n-1}(\vartheta a(t_n,t_{i+1},Y_{i+1}) + (1-\vartheta)a(t_n,t_i,Y_i)) \\
&+ \sqrt{h}\sum_{i=0}^{n-1} b(t_n,t_i,Y_i)V_{i,1} \\
&+ \frac{1}{2}h\sqrt{h}\sum_{i=0}^{n-1}\frac{\partial a}{\partial x}(t_n,t_i,Y_i)b(t_i,t_i,Y_i)\left(V_{i,1} + \frac{V_{i,2}}{\sqrt{3}}\right) \\
&+ \frac{1}{2}h\sum_{i=0}^{n-1}\frac{\partial b}{\partial x}(t_n,t_i,Y_i)b(t_i,t_i,Y_i)\left(V_{i,1}^2 - 1\right).
\end{aligned}
\tag{9}
$$

Similarly as in [17], suitably approximating the partial derivatives in (9) leads to the following derivative free method

$$
\begin{aligned}
Y_n &= Y_0 + h\sum_{i=0}^{n-1}\left(\vartheta a(t_n,t_{i+1},Y_{i+1}) + (1-\vartheta)a(t_n,t_i,Y_i)\right) + \sqrt{h}\sum_{i=0}^{n-1} b(t_n,t_i,Y_i)V_{i,1}\\
&+ \frac{h}{2}\sum_{i=0}^{n-1}\left(a(t_n,t_i,Y_i + a(t_i,t_i,Y_i)h + b(t_i,t_i,Y_i)\sqrt{h}) - a(t_n,t_i,Y_i)\right)\left(V_{i,1} + \frac{V_{i,2}}{\sqrt{3}}\right)\\
&+ \frac{\sqrt{h}}{2}\sum_{i=0}^{n-1}\left(b(t_n,t_i,Y_i + a(t_i,t_i,Y_i)h + b(t_i,t_i,Y_i)\sqrt{h}) - b(t_n,t_i,Y_i)\right)\left(V_{i,1}^2 - 1\right).
\end{aligned}
\tag{10}
$$

The following theorem exhibits the convergence order of (9)and (10) and its proof follows analogously as in [17], which corresponds to the case $\vartheta = 0$.

Theorem 2.2. *Under the same hypothesis of Theorem 1 and assuming that the coefficients a and b in (1) and their derivatives up to order 3 are bounded, the improved stochastic ϑ-methods (9) and (10) have order 1, i.e.*

$$\mathbb{E}(|X_n - Y_n|^2) \leq Kh^2. \tag{11}$$

3. Stability analysis with respect to the basic test equation.

We now study the stability properties with respect to the basic test equation (2). We observe that this equation is equivalent to the linear test equation for SDEs

$$dX_t = \lambda X_t dt + \mu X_t dW_t$$

and, as a consequence, the following well-known conditions [10] for the mean-square and asymptotic stability of the exact solution respectively occur

$$\lim_{t\to\infty}\mathbb{E}|X(t)|^2 = 0 \quad \Leftrightarrow \quad \lambda + \frac{1}{2}\mu^2 < 0, \tag{12}$$

$$\lim_{t\to\infty}|X(t)| = 0 \;\text{ w.p.1} \quad \Leftrightarrow \quad \lambda - \frac{1}{2}\mu^2 < 0. \tag{13}$$

Theorem 3.1. *Let $x = h\lambda$ and $y = h\mu^2$. The recurrence relation for the stochastic ϑ-methods (5), (9) and (10) with respect to the basic test equation (2) assumes the form*

$$Y_{n+1} = (\alpha + \beta V_{n,1} + \gamma V_{n,1}^2 + \delta Z_n)Y_n, \tag{14}$$

with

(i) $\alpha = \dfrac{1+(1-\vartheta)x}{1-\vartheta x}$, $\beta = \dfrac{\sqrt{y}}{1-\vartheta x}$, $\gamma = 0$, $\delta = 0$ *for method* (5),

(ii) $\alpha = \dfrac{1+(1-\vartheta)x - \frac{1}{2}y}{1-\vartheta x}$, $\beta = \dfrac{\sqrt{y}}{1-\vartheta x}$, $\gamma = \dfrac{y}{2(1-\vartheta x)}$, $\delta = \dfrac{x\sqrt{y}}{1-\vartheta x}$ *for method* (9),

(iii) $\alpha = \dfrac{1+(1-\vartheta)x - \frac{1}{2}\left(x\sqrt{y}+y\right)}{1-\vartheta x}$, $\beta = \dfrac{\sqrt{y}}{1-\vartheta x}$, $\gamma = \dfrac{x\sqrt{y}+y}{2(1-\vartheta x)}$, $\delta = \dfrac{x\sqrt{y}+x^2}{1-\vartheta x}$ *for method* (10),

and $Z_n = \frac{1}{2}\left(V_{n,1} + \frac{V_{n,2}}{\sqrt{3}}\right)$.

Proof. (*i*) Applying (5) to the test equation (2), we obtain

$$Y_{n+1} = Y_0 + x \sum_{i=0}^{n} (\vartheta Y_{i+1} + (1-\vartheta)Y_i) + \sqrt{y} \sum_{i=0}^{n} Y_i V_i.$$

Hence, by isolating the term with $i = n$ in both sums in the right-hand side, we have

$$Y_{n+1} = Y_n + x(\vartheta Y_{n+1} + (1-\vartheta)Y_n) + \sqrt{y} Y_n V_n,$$

that gives the thesis by denoting $V_n = V_{n,1}$.

(*ii*) Applying (9) to the test equation (2), we obtain

$$Y_{n+1} = Y_0 + x \sum_{i=0}^{n} (\vartheta Y_{i+1} + (1-\vartheta)Y_i) + \sqrt{y} \sum_{i=0}^{n} Y_i V_{i,1}$$
$$+ \sum_{i=0}^{n} x\sqrt{y} Z_i Y_i + \frac{1}{2} y \sum_{i=0}^{n} Y_i (V_{i,1}^2 - 1).$$

The result then follows by arguments similar to those of case (*i*). The proof of case (*iii*) proceeds analogously.

□

Theorem 3.2. *The stochastic ϑ-methods (5), (9), (10) are mean-square stable with respect to the basic test equation (2) if and only if*

$$\left| \alpha^2 + \beta^2 + 3\gamma^2 + \frac{\delta^2}{3} + 2\alpha\gamma + \beta\delta \right| < 1,$$

where α, β, γ and δ are given in Theorem 3.1.

Proof. By (14) we obtain, passing to the expectations,

$$\mathbb{E}|Y_{n+1}|^2$$
$$= \left(\alpha^2 + \beta^2 \mathbb{E}|V_{n,1}|^2 + \gamma^2 \mathbb{E}|V_{n,1}|^4 + \delta^2 \mathbb{E}|Z_n|^2 + 2\alpha\gamma \mathbb{E}|V_{n,1}|^2 + \beta\delta \mathbb{E} V_{n,1}^2\right) \mathbb{E}|Y_n|^2,$$

taking into account that $\mathbb{E}V_{n,1} = \mathbb{E}V_{n,1}^3 = \mathbb{E}V_{n,2} = \mathbb{E}Z_n = 0$. Since $\mathbb{E}|V_{n,1}|^2 = 1$, $\mathbb{E}|V_{n,1}|^4 = 3$, $\mathbb{E}|Z_n|^2 = 1/3$, the result follows. □

Theorem 3.3. *The stochastic ϑ-methods (5), (9), (10) are asymptotically stable if and only if*

$$\mathbb{E}(\log \left|\alpha + \beta V_{n,1} + \gamma V_{n,1}^2 + \delta Z_n\right|) < 0,$$

where α, β, γ and δ are given in Theorem 3.1.

Proof. By (14), we obtain

$$Y_n = \left(\prod_{i=0}^{n-1} Q_i\right) Y_0,$$

with $Q_i = \alpha + \beta V_{i,1} + \gamma V_{i,1}^2 + \delta Z_i$. Then, the thesis follows from Lemma 5.1 in [8]. □

Remark 1. We observe that the recurrence relation (14) for the stochastic ϑ-method (5) corresponds to the recurrence relation of the Euler-Maruyama version of the stochastic ϑ-method for SDEs in [2, 8]. Moreover, if we remove from (9) the sum involving the derivative $\frac{\partial a}{\partial x}$, i.e. if we remove the double integral in $dW_u ds$ from (8), then it assumes the form

$$Y_n = Y_0 + h\sum_{i=0}^{n-1}(\vartheta a(t_n, t_{i+1}, Y_{i+1}) + (1-\vartheta)a(t_n, t_i, Y_i)) + \sqrt{h}\sum_{i=0}^{n-1} b(t_n, t_i, Y_i)V_{i,1}$$
$$+ \frac{1}{2}h\sum_{i=0}^{n-1}\frac{\partial b}{\partial x}(t_n, t_i, Y_i)b(t_i, t_i, Y_i)\left(V_{i,1}^2 - 1\right). \tag{15}$$

and the related recurrence relation (14) corresponds to the recurrence relation of the Milstein version of the stochastic ϑ-method for SDEs in [2], with parameters

$$\alpha = \frac{1 + (1-\vartheta)x - \frac{1}{2}y}{1 - \vartheta x}, \ \beta = \frac{\sqrt{y}}{1-\vartheta x}, \ \gamma = \frac{y}{2(1-\vartheta x)}, \ \delta = 0.$$

Figures 1 and 2 respectively show the regions of mean-square and asymptotic stability of (5), (9), (10) and (15) with respect to the basic test equation (2). As visible from these figures, the rectangular and improved rectangular methods introduced in [16, 17], i.e. those corresponding to $\vartheta = 0$ in (5), (9) and (10), have only bounded stability regions. Methods (5), (9), (10) and (15) here introduced can achieve unbounded stability regions with suitable choices of the parameter $\vartheta \geq 1/2$. We also observe that the numerical differentiation leading to the derivative free method (10) has a negative effect on the stability regions.

Moreover, Figure 3 depicts mean-square stability regions for values of $\vartheta > 1$. As visible from the figure, in this case some methods are A-stable. This issue is coherent with the evidence highlighted in [9] for stochastic ϑ-methods in the framework of SDEs.

4. **Stability analysis with respect to the convolution test equation.** We now analyze the stability properties with respect to the convolution test equation (3). To this purpose, the following result first providing recurrence relations is useful.

Theorem 4.1. *Let $x = h\lambda$, $y = h\mu^2$ and $z = h^2\sigma$. The recurrence relation for the stochastic ϑ-methods (5), (9) and (10) with respect to the convolution test equation (3) assumes the form*

$$(1-\vartheta x)Y_{n+2} = (2+(1-2\vartheta)x+z+A_{n+1}+B_{n+1})Y_{n+1} - (1+(1-\vartheta)x+A_n)Y_n, \tag{16}$$

with

$$(1-\vartheta x)Y_1 = (1 + (1-\vartheta)x + (1-\vartheta)z + A_0 + B_0)Y_0$$

and

$$A_n = \sqrt{y}V_{n,1} + \zeta(V_{n,1}^2 - 1) + \eta Z_n, \qquad B_n = \psi Z_n,$$

where

(i) $\zeta = \eta = \psi = 0$ *for method* (5),

(ii) $\zeta = \frac{1}{2}y$, $\eta = x\sqrt{y}$, $\psi = z\sqrt{y}$ *for method* (9),

(iii) $\zeta = \frac{1}{2}(x\sqrt{y} + y)$, $\eta = x(x + \sqrt{y})$, $\psi = z(x + \sqrt{y})$ *for method* (10),

(iv) $\zeta = \frac{1}{2}y$, $\eta = 0$, $\psi = z\sqrt{y}$ *for method* (15),

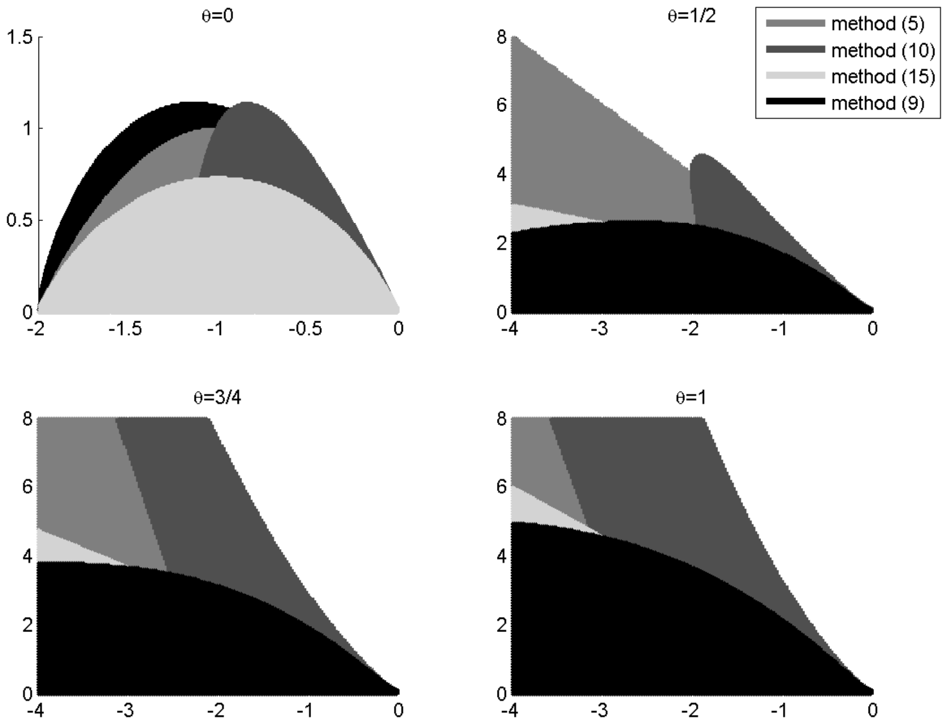

FIGURE 1. Mean-square stability regions in the (x, y)-plane with respect to the basic test equation (2).

being $Z_n = \frac{1}{2}\left(V_{n,1} + \frac{V_{n,2}}{\sqrt{3}}\right)$.

Proof. Applying the stochastic ϑ-methods (5), (9) and (10) to the convolution test equation (3) we obtain

$$Y_{n+1} = Y_0 + \sum_{i=0}^{n} \vartheta(x+z(n-i))Y_{i+1} + \sum_{i=0}^{n}\Big((1-\vartheta)x + A_i + ((1-\vartheta)z + B_i)(n+1-i)\Big)Y_i,$$

with A_i and B_i given by (i), (ii) and (iii), respectively. This is equivalent to

$$(1-\vartheta x)Y_{n+1} = \big(1+(1-\vartheta)x+(1-\vartheta)z+A_n+B_n\big)Y_n + \sum_{i=0}^{n-1}\big(\vartheta z Y_{i+1} + ((1-\vartheta)z+B_i)Y_i\big).$$

The thesis follows by substracting $(1-\vartheta x)Y_{n+1}$ from $(1-\vartheta x)Y_{n+2}$ and suitably rearranging the involved terms. $\square$

Theorem 4.2. *The stochastic ϑ-methods* (5), (9), (10) *and* (15) *are mean-square with respect to the convolution test equation* (3) *if the spectral radius $\rho(K)$ of matrix*

$$K = \begin{bmatrix} 0 & 0 & 1 \\ -\frac{\mathbb{E}(A_n(A_n+B_n))}{(1-\vartheta x)^2} & -\frac{\nu}{1-\vartheta x} & \frac{\mu}{1-\vartheta x} \\ \mathbb{E}(\beta_n) - \frac{2\mu\mathbb{E}(A_n(A_n+B_n))}{(1-\vartheta x)^3} & -\frac{2\nu\mu}{(1-\vartheta x)^2} & \mathbb{E}(\alpha_n) \end{bmatrix} \tag{17}$$

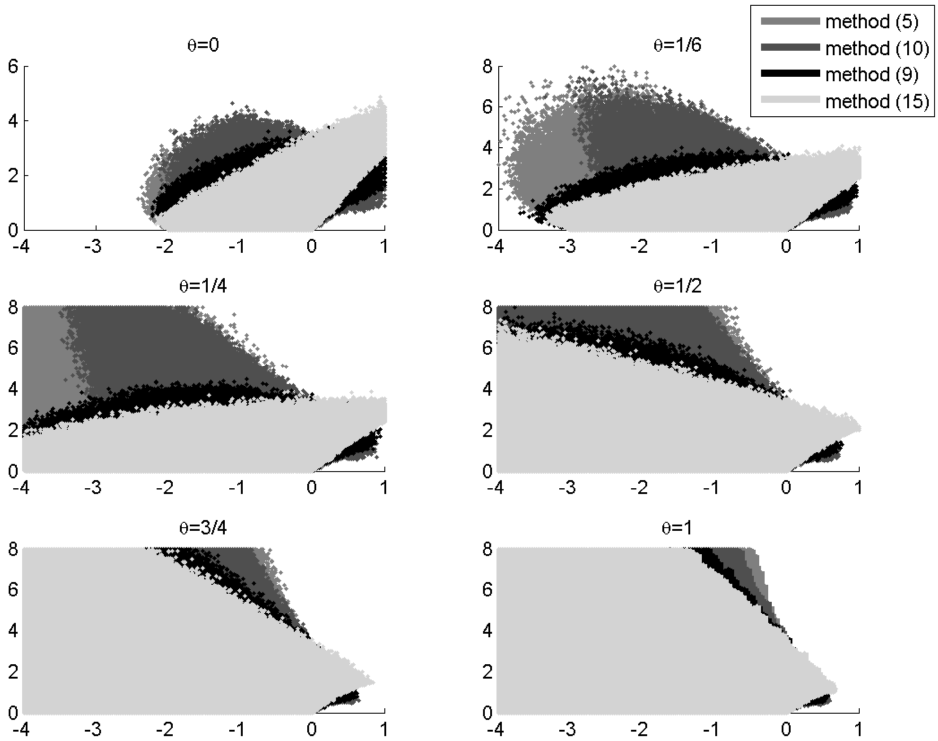

FIGURE 2. Asymptotic stability regions in the (x, y)-plane with respect to the basic test equation (2).

is less than 1, where

$$\mu = 2 + (1 - 2\vartheta)x + z, \quad \nu = 1 + (1 - \vartheta)x \tag{18}$$

and

$$\alpha_n = \left(\frac{\mu + A_{n+1} + B_{n+1}}{1 - \vartheta x}\right)^2, \quad \beta_n = \left(\frac{\nu + A_n}{1 - \vartheta x}\right)^2.$$

Proof. With above notation, recurrence relation (16) becomes

$$(1 - \vartheta x)Y_{n+2} = (\mu + A_{n+1} + B_{n+1})Y_{n+1} - (\nu + A_n)Y_n. \tag{19}$$

Squaring this relation, passing to expectations and employing the relation

$$(1 - \vartheta x)\mathbb{E}(A_n Y_n Y_{n+1}) = \mathbb{E}(A_n(A_n + B_n))\mathbb{E}(Y_n^2)$$

leads to

$$\mathbb{E}(Y_{n+2}^2) = \left(\mathbb{E}(\beta_n) - \frac{2(\mu + \mathbb{E}(A_{n+1}) + \mathbb{E}(B_{n+1}))\mathbb{E}(A_n(A_n + B_n))}{(1 - \vartheta x)^3}\right)\mathbb{E}(Y_n^2)$$
$$- \frac{2\nu(\mu + \mathbb{E}(A_{n+1}) + \mathbb{E}(B_{n+1}))}{(1 - \vartheta x)^2}\mathbb{E}(Y_n Y_{n+1}) + \mathbb{E}(\alpha_n)\mathbb{E}(Y_{n+1}^2).$$

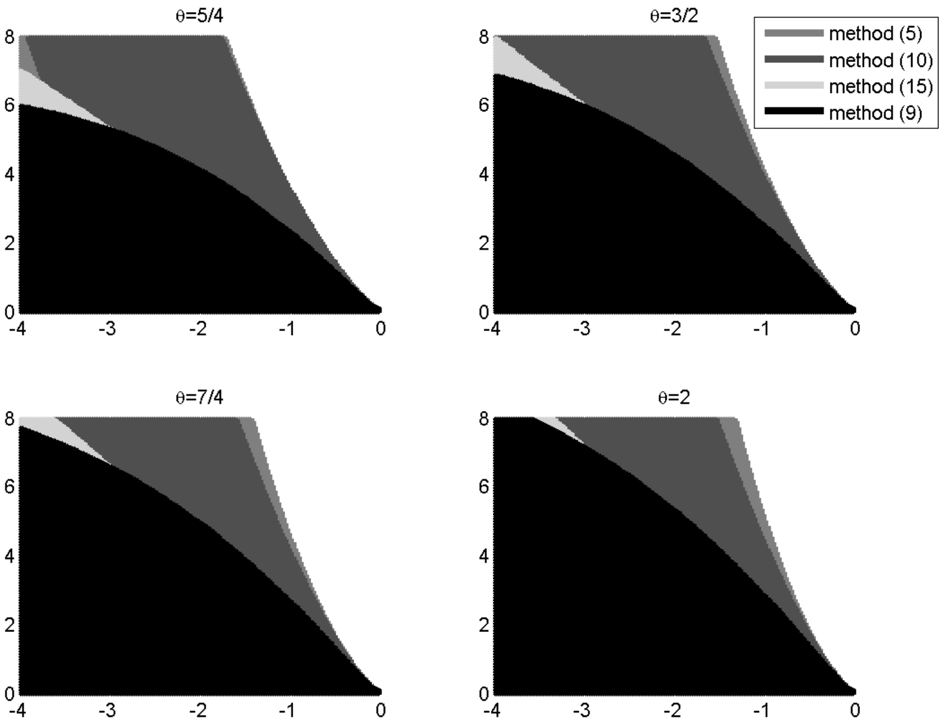

FIGURE 3. Mean-square stability regions in the (x, y)-plane with respect to the basic test equation (2) for values of $\vartheta \geq 1$.

Multiplying Equation (19) by Y_{n+1}, passing to expectations leads to

$$\mathbb{E}(Y_{n+1}Y_{n+2}) = -\frac{\mathbb{E}(A_n(A_n+B_n))}{(1-\vartheta x)^2}\mathbb{E}(Y_n^2) - \frac{\nu}{1-\vartheta x}\mathbb{E}(Y_nY_{n+1}) + \frac{\mu+\mathbb{E}(A_{n+1})+\mathbb{E}(B_{n+1})}{1-\vartheta x}\mathbb{E}(Y_{n+1}^2).$$

Since from (i)–(iv) of Theorem 4.1 follows that $\mathbb{E}(A_{n+1}) = \mathbb{E}(B_{n+1}) = 0$, we obtain

$$\begin{bmatrix} \mathbb{E}(Y_{n+1}^2) \\ \mathbb{E}(Y_{n+1}Y_{n+2}) \\ \mathbb{E}(Y_{n+2}^2) \end{bmatrix} = K \begin{bmatrix} \mathbb{E}(Y_n^2) \\ \mathbb{E}(Y_nY_{n+1}) \\ \mathbb{E}(Y_{n+1}^2) \end{bmatrix},$$

with K given by (17). □

We observe that, by taking into account (i)–(iv) in Theorem 4.1, expected values in (17) can be computed as follows

$$\mathbb{E}(A_n(A_n+B_n)) = y + 2\zeta^2 + \frac{1}{3}\eta^2 + \sqrt{y}\eta + \frac{1}{2}\sqrt{y}\psi + \frac{1}{3}\eta\psi,$$

$$\mathbb{E}(\alpha_n) = \frac{\mu^2 + y + 2\zeta^2 + \frac{1}{3}\eta^2 + \sqrt{y}\eta + \frac{1}{3}\psi^2 + \mu\sqrt{y}\psi + \frac{2}{3}\mu\eta\psi}{(1-\vartheta x)^2},$$

$$\mathbb{E}(\beta_n) = \frac{\nu^2 + y + 2\zeta^2 + \frac{1}{3}\eta^2 + \sqrt{y}\eta}{(1-\vartheta x)^2},$$

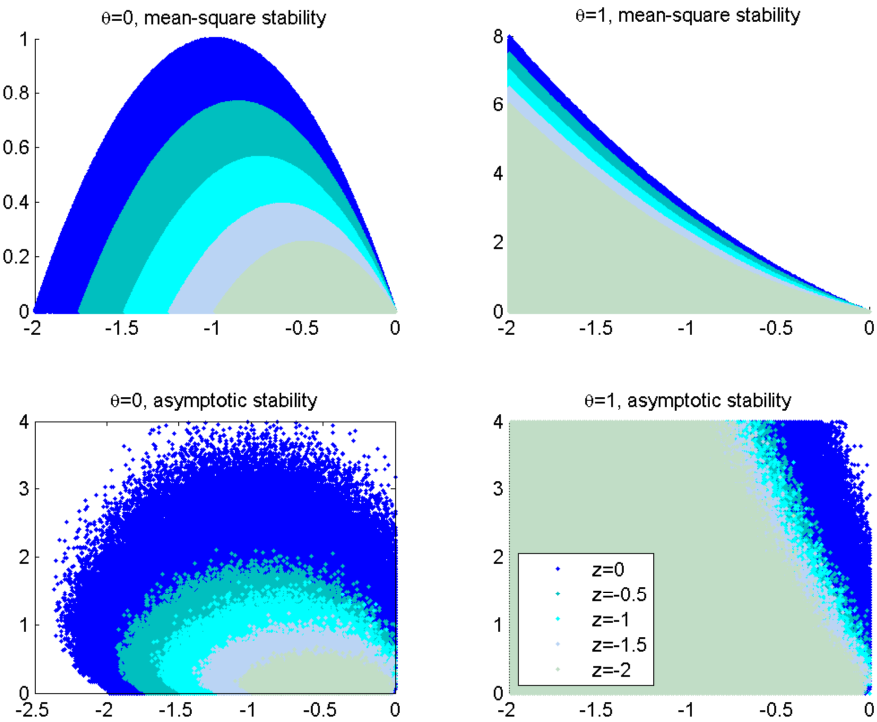

FIGURE 4. Mean-square and asymptotic stability regions in the (x, y)-plane with respect to the convolution test equation (3) for the stochastic ϑ-method (5) for several choices of ϑ and z.

with μ and ν given by (18).

Figures 4 and 5 show the mean-square and asymptotic stability regions of above methods with respect to the convolution test equation (3).

5. **Numerical tests.** We now present a selection of numerical experiments confirming the theoretical expectations regarding the stability properties of methods (5), (9), (10) and (15) presented in the previous sections.

We first consider the basic test equation (2) with $\lambda = -8$ and $\mu = 2\sqrt{2}$ and apply methods (5), (9), (10) and (15) with $\vartheta = 1/2$. The mean-squares of the obtained numerical solutions over 1000 realizations are depicted in Fig. (6), where it is visible that the choice of the stepsize has a direct influence on the mean-square stability properties. This is coherent with the stepsize restrictions shown in Fig. 1: indeed, for $h = 1/2$, the corresponding point $(x, y) = (-4, 4)$ lies inside the stability region only of method (5), while for $h = 1/8$, the corresponding point $(x, y) = (-1, 1)$ lies inside the stability region of all methods. The asymptotic stability analysis, leading to the stability regions in Fig. 2, is now confirmed by the results in Fig. 7: also in this case, the choice of the stepsize, coherent with Fig. 2, leads to the expected stable and unstable behaviours.

We next consider the convolution test equation (3) with $\lambda = -4$, $\mu = 2\sqrt{2}$ and $\sigma = -8$ and apply methods (5), (9), (10) and (15) with $\vartheta = 1$. In correspondence of $h = 1/2$, the point $(x, y, z) = (-2, 4, -2)$ is identified in Fig. 5: as visible in the figure, methods (9) and (15) are unstable, while methods (5) and (10) are stable. Such a behaviour is coherent with the numerical results obtained in Fig.

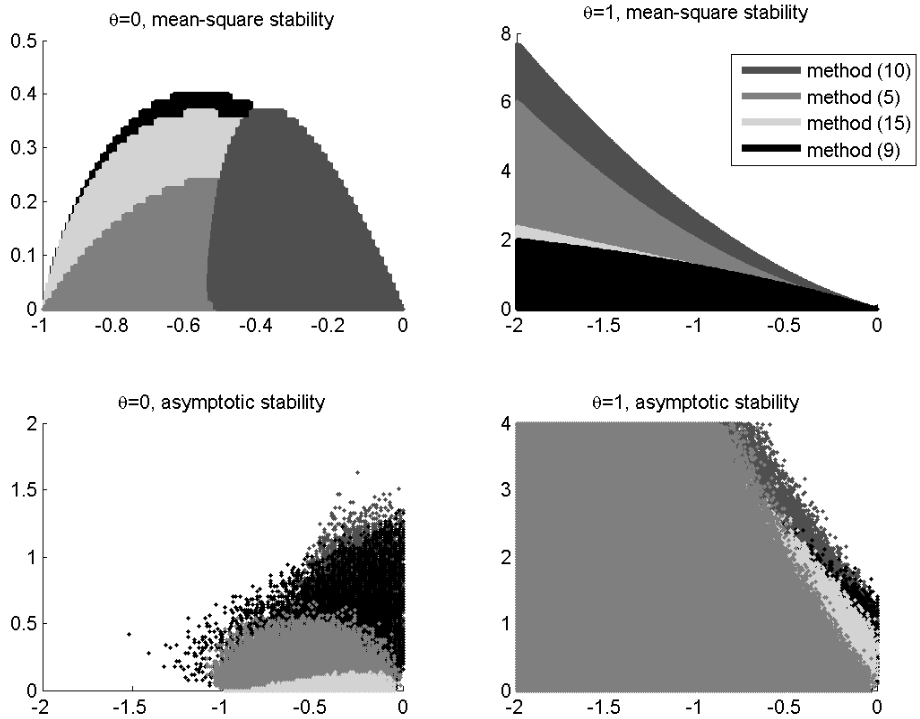

FIGURE 5. Mean-square and asymptotic stability regions in the (x, y)-plane with respect to the convolution test equation (3) for $z = -2$ and several choices of ϑ.

8 (top), where the mean-squares of the obtained numerical solutions over 1000 realizations are drawn. By considering instead the convolution test equation (3) with $\lambda = -4$, $\mu = 2\sqrt{5}/5$ and $\sigma = -128$, in correspondence of $h = 1/8$, the point $(x, y, z) = (-1/2, 1/10, -2)$ lies inside the stability region of all methods for $\vartheta = 1$. This stable behaviour is also visible in Fig. 8 (bottom).

The asymptotic stability analysis, leading to the stability regions in Fig. 5, is confirmed by the results in Fig. 9: also in this case, the choice of the parameters specified in Fig. 9, corresponding to $(x, y, z) = (-1/2, 3, -2)$ (top) and $(x, y, z) = (-1/8, 3/4, -2)$ (bottom), leads to the expected unstable and stable behaviours, respectively.

6. **Conclusions.** In this paper we have analyzed mean-square and asymptotic stability properties of a selection of ϑ-methods, i.e. (5), (9), (10) and (15) for the numerical solution of Volterra stochastic integral equations (1). The analysis has highlighted mean-square and asymptotic stability properties, which are also confirmed in the numerical experiments. Further developments of this research will regard stability and accuracy analysis of wider families of methods, such as Runge-Kutta methods for SVIEs (1).

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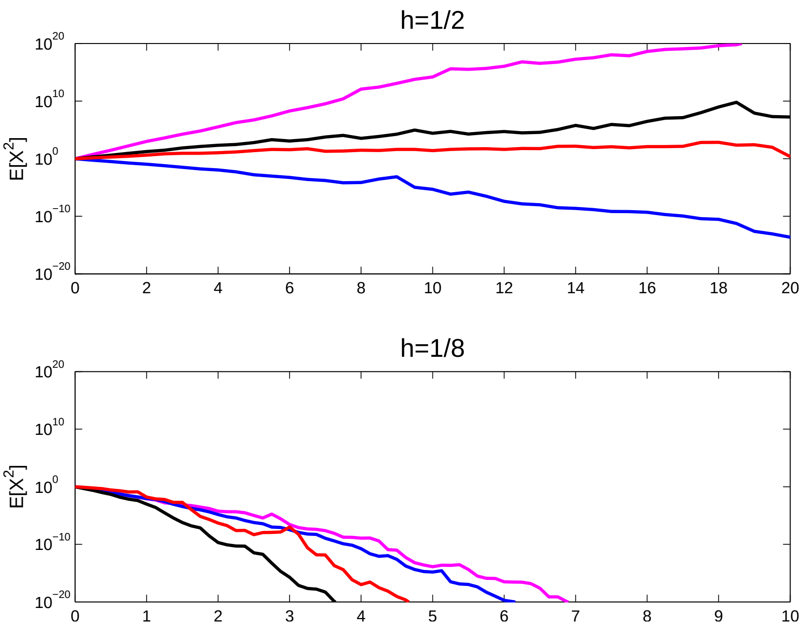

FIGURE 6. Mean-square of the numerical solution of problem (2), with $\lambda = -8$ and $\mu = 2\sqrt{2}$, obtained by applying methods (5) (blue), (9) (black), (10) (magenta) and (15) (red) with $\vartheta = 1/2$.

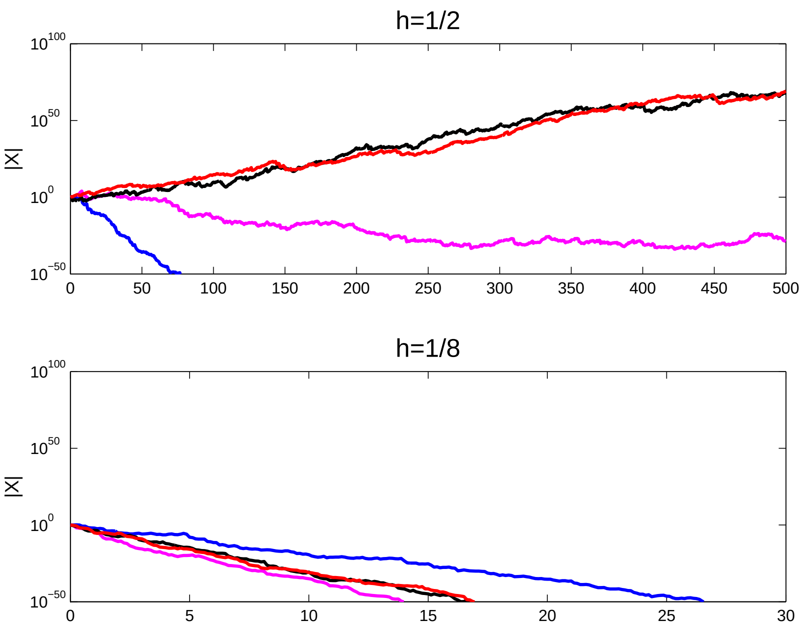

FIGURE 7. Absolute value of the numerical solution of problem (2), with $\lambda = -8$ and $\mu = 4$, obtained by applying methods (5) (blue), (9) (black), (10) (magenta) and (15) (red) with $\vartheta = 1/2$.

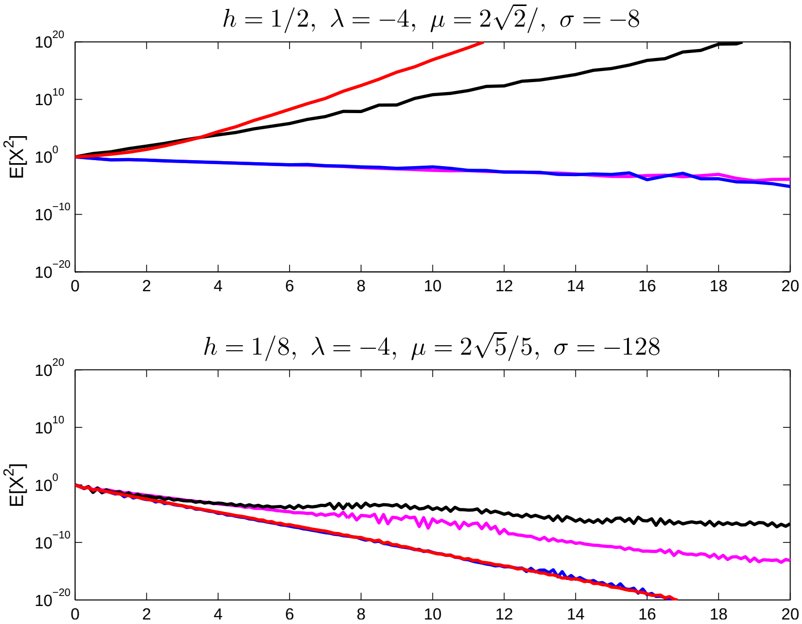

FIGURE 8. Mean-square of the numerical solution of problem (3), with $\lambda = -4$, $\mu = 2\sqrt{5}/5$ and $\sigma = -2/h^2$, obtained by applying methods (5) (blue), (9) (black), (10) (magenta) and (15) (red) with $\vartheta = 1$.

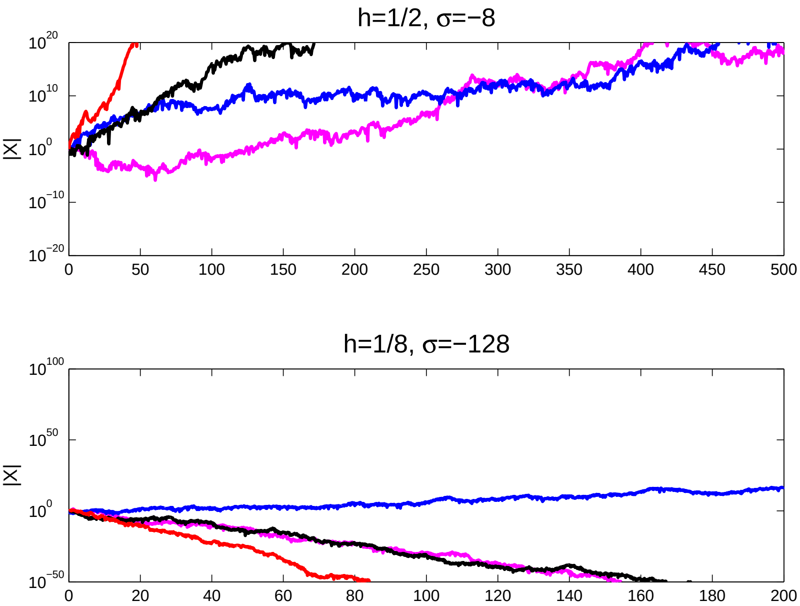

FIGURE 9. Absolute value of the numerical solution of problem (3), with $\lambda = -1$, $\mu = \sqrt{6}$ and $\sigma = -2/h^2$, obtained by applying methods (5) (blue), (9) (black), (10) (magenta) and (15) (red) with $\vartheta = 1$.

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