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Adapted explicit two-step peer methods

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Abstract. In this paper, we present a general class of exponentially fitted two-step peer methods for the numerical integration of ordinary differential equations. The numerical scheme is constructed in order to exploit a-priori known information about the qualitative behaviour of the solution by adapting peer methods already known in literature. Examples of methods with 2 and 3 stages are provided. The effectiveness of this problem-oriented approach is shown through some numerical tests on well-known problems.

Keywords. Peer methods, exponential fitting.

1 Introduction

The presented work aims to solve initial value problems in ordinary differential systems

$$y'(t) = f(t, y(t)), \qquad y(t_0) = y_0 \in \mathbb{R}^d, \qquad t \in [t_0, T], \tag{1.1}$$

where $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ is smooth enough to guarantee the existence and the uniqueness of the solution. In particular, we propose an adapted numerical integration which exploits a-priori known information about the behaviour of the exact solution.

General purpose formulae are developed in order to be exact (within round-off error) on polynomials up to a certain degree. However, when the exact solution has a particular behaviour known in advance (e.g. oscillatory, periodic, exponentially decaying), the resulting methods could require a very small stepsize in order to preserve this property. For this reason, *special purpose* formulae may be more convenient because they are constructed in order to be exact on functions other than polynomials, following the well-known exponential fitting strategy [21]. The basis functions are normally supposed to belong to a finite-dimensional space $\mathcal{F}_q = \{\phi_0(t), \phi_1(t), \ldots, \phi_q(t)\}$ called fitting space and are selected according to the a-priori known information concerning the behaviour of the exact solution. As a result, the coefficients of the corresponding methods are no longer constant as in the classic case, but rely on parameters characterizing the exact solution, whose values are generally unknown. Hence, the exponential fitting technique requires

the choice of a suitable fitting space and a proper estimation or computation of the afore-mentioned parameters.
Following the general idea shown in [21], the adaptation of already existing schemes has led to exponentially fitted methods for a wide range of problems such as interpolation, numerical differentiation and quadrature [6–8, 19, 20, 22, 23, 36], numerical solution of integral equations [4, 5], partial differential equations [12–15] and ordinary differential equations [1, 10, 11, 20, 21, 37]. Adapted Runge-Kutta methods have been constructed in [9, 14, 16, 17, 20, 21, 26, 27, 33, 34]. In [26] it has been shown that for any fitting space $\mathcal{F}_q$ of smooth linearly independent real functions there exists a q-stage Runge-Kutta method fitted to $\mathcal{F}_q$. However, the stage order of a Runge-Kutta method extremely influences the highest dimension that can be achieved by the fitting space, especially in case of explicit Runge-Kutta methods, whose stage order is 1. For instance, in [35] an explicit four stage RK method has been constructed on a fitting space having the maximum dimension equal to 3. By contrast, linear multistep methods do not impose such a strong dimensional limit, as shown in [17]. Indeed, a k-step method can be fitted on a $k+1$-dimensional fitting space.
Therefore, it appears worthwhile to combine the advantages of Runge-Kutta and linear multistep methods with the afore-mentioned benefits related to exponential fitting: this idea gives birth to exponentially fitted peer methods.
Peer methods are characterized by several stages like Runge-Kutta ones but all of these stages exhibit the same properties, especially in terms of accuracy and stability (see [3,24,31,38] and references therein). Combining the benefits of Runge-Kutta and linear multistep methods, they achieve good stability features and overcome the crucial issue of order reduction within the integration of highly stiff systems [30]. Moreover, two-step peer methods are very suitable for a parallel implementation because the actual stages rely only on the previous ones [2,28,30,32].
In [3,25], it has been shown that it is possible to construct explicit two-step s-stage peer methods adapted on a fitting space of high dimension $2s$. In particular, the authors have derived explicit peer methods having 2 and 3 stages and tuned on trigonometric bases. In this paper, we develop a general class of exponentially fitted two-step explicit peer methods having order s, by employing the six-step procedure presented in [19, 21].
In summary, Section 2 is devoted to the construction of explicit exponentially fitted peer methods adapted to a general fitting space. The results of this section are used to develop peer methods with 2 and 3 stages in Section 3. Finally, we present some numerical tests on realistic problems in Section 4 and we discuss some conclusions in Section 5.

2 Explicit two-step peer methods

For the construction of explicit s-stage two-step peer methods, we consider a set of admissible fixed nodes c_i for $i = 1, \dots, s$ such that

$$|c_i - c_j| \neq 0, \qquad \forall i \neq j, \tag{2.2}$$

and we suppose that for any stepsize $h > 0$ there exists a starting procedure to approximate the solution in the internal grid points $t_{0\,i} = t_0 + c_i\,h,\ i = 1, \dots, s$. An s-stage two-step peer method with fixed stepsize h has the following expression

$$Y_{ni} = \sum_{j=1}^{s} b_{ij}\ Y_{n-1\,j} + h \sum_{j=1}^{s} a_{ij}\ f(t_{n-1\,j}, Y_{n-1\,j}) + h \sum_{j=1}^{i-1} r_{ij}\ f(t_{nj}, Y_{nj}), \tag{2.3}$$

$i = 1, \dots, s$, $n = 1, \dots, N$, where

$$Y_{ni} \approx y(t_{ni}), \quad t_{ni} = t_n + c_i\,h, \qquad i = 1, \dots, s.$$

We recall that no extraordinary numerical solution with different properties is computed, but we simply assume that $c_s = 1$, so Y_{ns} is the approximation of the solution at grid point t_{n+1}. The other nodes are chosen such that $c_i < 1$ for $i = 1, \dots, s-1$.
In this treatise, we suppose for simplicity that problem (1.1) is scalar and we employ the following notation:

$$Y_n = [Y_{ni}]_{i=1}^{s}\,,\ F(Y_n) = [f(t_{ni}, Y_{ni})]_{i=1}^{s}\,,$$

$$A = [a_{ij}]_{i,j=1}^{s}\,,\ B = [b_{ij}]_{i,j=1}^{s}\,,\ R = [r_{ij}]_{i,j=1}^{s}\,,$$

where A and B are full matrices and R is a stricly lower triangular matrix. Hence, the method (2.3) can be rewritten in a more compact form

$$Y_n = B\,Y_{n-1} + h\,A\,F(Y_{n-1}) + h\,R\,F(Y_n). \tag{2.4}$$

The matrices of coefficients A, B and R are constructed in order to achieve high order (uniformly for all components Y_{ni}) and good stability properties. We recall that the method (2.3) has order of consistency p if $\Delta_{ni} = \mathcal{O}(h^{p+1})$ for $i = 1, \dots, s$, where Δ_{ni} is the residual obtained by inserting the exact solution in the numerical scheme (2.3), i.e. [38]

$$\Delta_{ni} = y(t_{n\,i}) - \sum_{j=1}^{s} b_{ij}\ y(t_{n-1\,j}) - h \sum_{j=1}^{s} a_{ij}\ y'(t_{n-1\,j}) - h \sum_{j=1}^{i-1} r_{ij}\ y'(t_{nj}). \tag{2.5}$$

Schmitt and Weiner in [29] have related this property to the following condition

$$\begin{aligned} \mathrm{AB}(q) = c_i^m - \sum_{j=1}^{s} b_{ij}\,(c_j-1)^m - m\sum_{j=1}^{s} a_{ij}\,(c_j-1)^{m-1} \\ - m\sum_{j=1}^{i-1} r_{ij}\,c_j^{m-1} = 0, \quad m=0,\dots,q-1,\ i=1,\dots,s, \end{aligned} \tag{2.6}$$

as follows:

Theorem 2.1. *If* $\mathrm{AB}(p+1)$ *is verified, the explicit* s*-stage peer method* (2.3) *has order of consistency* p*.*

As a consequence, the peer method (2.3) has order $p \geq s-1$ if

$$B\,\mathbf{1} = \mathbf{1}, \tag{2.7a}$$

$$A = \left(CV_0D^{-1} - RV_0 - \frac{1}{s}\beta\,e_s\right)V_1^{-1} - B\,(C-\mathbb{I})V_1D^{-1}V_1^{-1}, \tag{2.7b}$$

where $\mathbf{1} = [1,1,\dots,1]^T$, $C = \mathrm{diag}(c_1,\dots,c_s)$, $D = \mathrm{diag}(1,\dots,s)$, $e_s = [0,0,\dots,1]$, $\beta \in \mathbb{R}^s$ is an arbitrary vector and

$$V_0 = \begin{bmatrix} 1 & c_1 & \dots & c_1^{s-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & c_s & \dots & c_s^{s-1} \end{bmatrix}, \quad V_1 = \begin{bmatrix} 1 & (c_1-1) & \dots & (c_1-1)^{s-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & (c_s-1) & \dots & (c_s-1)^{s-1} \end{bmatrix}.$$

The method has order equal to $p = s$ if $\beta = 0$.

2.1 Exponentially fitted explicit peer methods

In this work, we aim to construct a general class of exponentially fitted explicit two-step peer methods with a fixed stepsize h, which are particularly suitable for problems with hyperbolic or trigonometric solutions. For this purpose, we consider the fitting space

$$\mathcal{F} = \left\{1, t, t^2, \dots, t^K, e^{\pm\mu t}, t\,e^{\pm\mu t}, t^2e^{\pm\mu t}, \dots, t^P e^{\pm\mu t}\right\}, \tag{2.8}$$

where μ is a parameter characterizing the exact solution and it is real or complex if the exact solution belongs to the space spanned by hyperbolic functions or trigonometric functions, respectively. Moreover, we assume that $K = -1$ if there are not classic components and $P = -1$ if there are not exponentially fitting ones. We next associate to the numerical scheme (2.3) the linear operator defined as difference between the exact solution and the numerical solution computed by the method, as follows:

$$\begin{aligned}\mathcal{L}_i[h,\mathbf{w}]\, y(t) = y(t+c_i\, h) - \sum_{j=1}^{s} b_{ij}\, y(t+(c_j-1)\, h) \\ - h\sum_{j=1}^{s} a_{ij}\, y'(t+(c_j-1)\, h) - h\sum_{j=1}^{i-1} r_{ij}\, y'(t+c_j\, h),\end{aligned} \tag{2.9}$$

$i = 1, \dots, s$, where $\mathbf{w}$ contains the coefficients of the method, i. e. the values of the matrices A, B and R. We recall that for a given set of admissible nodes, the method (2.3) is adapted to the fitting space $\mathcal{F}$ if it is exact (within round-off error) on the functions belonged to $\mathcal{F}$, which is equivalent to annihilating the difference operator (2.9) on these basis functions. The resulting system has the coefficients of the method as unknowns, due to the dependence of the difference operator on them. These basic concepts have given raise to the six-step algorithm presented in [19,21] which we apply in the following for the construction of desidered adapted peer methods.

Step 1

We first want to impose that the numerical scheme (2.3) is exact on classic components t^m of the fitting space $\mathcal{F}$ (2.8). As we have already remarked, this is equivalent to annihilating the linear difference operator (2.9) on t^m. Hence, taking into account the invariance for translations of the difference operator due to its linearity, we construct the classic moments for a generic $m > 0$

$$\mathcal{L}_{i\,m}(h,\mathbf{w}) = \mathcal{L}_i[h,\mathbf{w}]t^m|_{t=0}, \quad i = 1, \dots, s,$$

and the corresponding dimensionless classic moments

$$\begin{aligned}\mathcal{L}^*_{i\,m}(h,\mathbf{w}) &= \frac{\mathcal{L}_{i\,m}(h,\mathbf{w})}{h^m} \\ &= c_i^m - \sum_{j=1}^{s} b_{ij}\,(c_j-1)^m - m\sum_{j=1}^{s} a_{ij}\,(c_j-1)^{m-1} - m\sum_{j=1}^{i-1} r_{ij}\, c_j^{m-1}, \\ &\text{for } \ i = 1, \dots, s.\end{aligned} \tag{2.10}$$

Step 2

We identify the maximum value M that ensures the compatibility of the system

$$\mathcal{L}^*_{i\,m}(h,\mathbf{w}) = 0, \quad i = 1, \dots, s, \quad m = 0, 1, \dots, M-1. \tag{2.11}$$

This system is equivalent to annihilating the difference operator (2.9) on polynomials with a degree less or equal to $M-1$ and corresponds to the condition AB(M)

$$\mathrm{AB}(M) = c_i^m - \sum_{j=1}^{s} b_{ij}\,(c_j-1)^m - m\sum_{j=1}^{s} a_{ij}\,(c_j-1)^{m-1} - m\sum_{j=1}^{i-1} r_{ij}\,c_j^{m-1} = 0,$$
$$i=1,\dots,s, \quad m=0,\dots,M-1. \tag{2.12}$$

Therefore, we may construct an s-order peer method if $M = s+1$, due to the Theorem 2.1. In this case, the peer method (2.3) is exact on polynomials up to degree equal to s.

Step 3

We next aim to impose the exactness of the peer method (2.3) on the basis functions $t^m e^{\pm\mu t}$ in the hybrid fitting space (2.8). This corresponds to annihilating the difference operator (2.9) on the functions $t^m e^{\pm\mu t}$, which in turn is equivalent to annihilating in $\pm z$ the so-called dimensionless μ-moments of order m on $\mathcal{L}$

$$E^*_{i\,m}(z,\mathbf{w}) = \frac{1}{h^m}\mathcal{L}_i[h,\mathbf{w}]t^m e^{\mu t}|_{t=0}, \quad i=1,\dots,s, \tag{2.13}$$

where $z=\mu h$. We next observe that the system

$$E^*_{i\,m}(\pm z,\mathbf{w}) = 0, \quad m=0,1,\dots,P, \quad i=1,\dots,s,$$

is equivalent to the system

$$G_i^{\pm(m)}(Z,\mathbf{w}) = 0, \quad m=0,1,\dots,P, \quad i=1,\dots,s,$$

where $Z=z^2$ and $G_i^{\pm(m)}(Z,\mathbf{w})$ are defined at each stage i as

$$G_i^{+}(Z,\mathbf{w}) = \frac{E^*_{i0}(z,w)+E^*_{i0}(-z,w)}{2}, \; G_i^{-}(Z,\mathbf{w}) = \frac{E^*_{i0}(z,w)-E^*_{i0}(-z,w)}{2z}, \tag{2.14}$$

for $m=0$ and as the related derivatives for $m>0$. In the following theorem, we present an explicit expression for the μ-moments of order $m=0$ on $\mathcal{L}$.

Theorem 2.2. *The μ-moments of order $m=0$ on $\mathcal{L}$ assume for $i=1,\dots,s$ the following form:*

$$E^*_{i0}(z,\mathbf{w}) = e^{zc_i} - \sum_{j=1}^{s} b_{ij}\,e^{z\,(c_j-1)} - z\sum_{j=1}^{s} a_{ij}\,e^{z\,(c_j-1)} - z\sum_{j=1}^{i-1} r_{ij}\,e^{z\,c_j}. \tag{2.15}$$

Proof. Applying the definition (2.13) of μ-moments on $\mathcal{L}$ with $m = 0$ and $i = 1, \ldots, s$, we obtain

$$\begin{aligned} E^*_{i0}(z, \mathbf{w}) &= \mathcal{L}_i[h, \mathbf{w}] e^{\mu t}|_{t=0} \\ &= e^{\mu\,(t+c_i h)} - \sum_{j=1}^{s} b_{ij}\; e^{\mu\,(t+(c_j-1)\,h)} - h \sum_{j=1}^{s} a_{ij}\; \mu\, e^{\mu\,(t+(c_j-1)\,h)} \\ &\quad - h \sum_{j=1}^{i-1} r_{ij}\; \mu\, e^{\mu\,(t+c_j h)} \Big|_{t=0}, \end{aligned}$$

which leads to the thesis by replacing $z = \mu h$ and $t = 0$. □

For a simpler construction of G-functions, we employ the η-functions defined in [21] as follows:

- if Z is real

$$\eta_{-1}(Z) = \frac{e^{\sqrt{Z}} + e^{-\sqrt{Z}}}{2} = \begin{cases} \cos\sqrt{|Z|} & \text{if } Z < 0, \\ \cosh\sqrt{Z} & \text{if } Z \geq 0, \end{cases}$$

$$\eta_0(Z) = \begin{cases} \dfrac{e^{\sqrt{Z}} - e^{-\sqrt{Z}}}{2\sqrt{Z}} & \text{if } Z \neq 0, \\ 1 & \text{if } Z = 0, \end{cases} = \begin{cases} \dfrac{\sin(\sqrt{|Z|})}{\sqrt{|Z|}} & \text{if } Z < 0, \\ 1 & \text{if } Z = 0, \\ \dfrac{\sinh(\sqrt{Z})}{\sqrt{Z}} & \text{if } Z > 0, \end{cases}$$

- if Z is not real

$$\eta_{-1}(Z) = \cos\left(\mathrm{i}\sqrt{Z}\right), \quad \eta_0(Z) = \begin{cases} \dfrac{\sin\left(\mathrm{i}\sqrt{Z}\right)}{\mathrm{i}\sqrt{Z}} & \text{if } Z \neq 0, \\ 1 & \text{if } Z = 0. \end{cases}$$

In both cases, we define $\eta_\sigma(Z)$ for $\sigma > 0$, as follows:

$$\eta_\sigma(Z) = \begin{cases} \dfrac{\eta_{\sigma-2}(Z) - (2\sigma - 1)\eta_{\sigma-1}(Z)}{Z} & \text{if } Z \neq 0, \\ \dfrac{2^\sigma \sigma!}{(2\sigma+1)!} & \text{if } Z = 0. \end{cases}$$

We recall that their derivatives verify the condition [21]

$$\eta'_\sigma(Z) = \frac{1}{2}\eta_{\sigma+1}(Z), \quad \sigma = -1, 0, 1, \ldots \tag{2.16}$$

In the following theorem, we express the G-functions and their derivatives in terms of the $\eta-$functions.

Theorem 2.3. *The G-functions and their derivatives assume the following expres-*

sions for $i = 1, \dots, s$*:*

$$G_i^{+}(Z,\mathbf{w}) = \eta_{-1}\left(c_i^2 Z\right) - \sum_{j=1}^{s} b_{ij}\ \eta_{-1}\left((c_j-1)^2 Z\right) - Z\sum_{j=1}^{i-1} r_{ij}\ c_j\ \eta_0\left(c_j^2 Z\right) - Z\sum_{j=1}^{s} a_{ij}\ (c_j-1)\ \eta_0\left((c_j-1)^2 Z\right), \tag{2.17a}$$

$$G_i^{-}(Z,\mathbf{w}) = c_i\ \eta_0\left(c_i^2 Z\right) - \sum_{j=1}^{s} b_{ij}\ (c_j-1)\ \eta_0\left((c_j-1)^2 Z\right) - \sum_{j=1}^{s} a_{ij}\ \eta_{-1}\left((c_j-1)^2 Z\right) - \sum_{j=1}^{i-1} r_{ij}\ \eta_{-1}\left(c_j^2 Z\right), \tag{2.17b}$$

$$\begin{aligned} G_i^{+(m)}(Z,\mathbf{w}) = {} & \frac{c_i^{2m}}{2^m}\eta_{m-1}\left(c_i^2 Z\right) - \sum_{j=1}^{s} b_{ij}\ \frac{(c_j-1)^{2m}}{2^m}\eta_{m-1}\left((c_j-1)^2 Z\right) \\ & - \sum_{j=1}^{s} a_{ij}\ \left(\frac{m\,(c_j-1)^{2m-1}}{2^{m-1}}\eta_{m-1}\left((c_j-1)^2 Z\right) + \frac{(c_j-1)^{2m+1}}{2^m}\ Z\ \eta_m\left((c_j-1)^2 Z\right)\right) \\ & - \sum_{j=1}^{i-1} r_{ij}\ \left(\frac{m\,c_j^{2m-1}}{2^{m-1}}\eta_{m-1}\left(c_j^2 Z\right) + \frac{c_j^{2m+1}}{2^m} Z\eta_m\left(c_j^2 Z\right)\right), \end{aligned} \tag{2.17c}$$

$$\begin{aligned} G_i^{-(m)}(Z,\mathbf{w}) = {} & \frac{c_i^{2m+1}}{2^m}\eta_m\left(c_i^2 Z\right) - \sum_{j=1}^{s} b_{ij}\ \frac{(c_j-1)^{2m+1}}{2^m}\eta_m\left((c_j-1)^2 Z\right) \\ & - \sum_{j=1}^{s} a_{ij}\ \frac{(c_j-1)^{2m}}{2^m}\eta_{m-1}\left((c_j-1)^2 Z\right) \\ & - \sum_{j=1}^{i-1} r_{ij}\ \frac{c_j^{2m}}{2^m}\eta_{m-1}\left(c_j^2 Z\right), \end{aligned} \tag{2.17d}$$

for $m = 1, \dots, P$.

Proof. From the defintion (2.14) of the G-functions and the expression of the μ-moments E^*_{i0} obtained in Theorem 2.2, we have

$$\begin{aligned} G_i^+(Z,\mathbf{w}) &= \frac{1}{2}\left(e^{z\,c_i} + e^{-z\,c_i}\right) - \frac{1}{2}\sum_{j=1}^{s} b_{ij}\ \left(e^{z\,(c_j-1)} + e^{-z\,(c_j-1)}\right) \\ &-\frac{z}{2}\sum_{j=1}^{s} a_{ij}\ \left(e^{z\,(c_j-1)} - e^{-z\,(c_j-1)}\right) - \frac{z}{2}\sum_{j=1}^{i-1} r_{ij}\ \left(e^{z\,c_j} - e^{-z\,c_j}\right), \end{aligned} \tag{2.18}$$

which leads to Equation (2.17a) by evaluating functions η_{-1} and η_0 in $c_i^2 Z$ and $(c_j-1)^2 Z$, where $Z = z^2$.
Equation (2.17b) can be proved in a similar way.
We next derive function G_i^+ (2.17a) taking into account the relation (2.16) among the derivatives of $\eta-$functions, obtaining

$$\begin{aligned} G_i^{+(1)}(Z,w) = \frac{c_i^2}{2}\eta_0\left(c_i^2 Z\right) &- \sum_{j=1}^{s} b_{ij}\ \frac{(c_j-1)^2}{2}\eta_0\left((c_j-1)^2 Z\right) \\ &- \sum_{j=1}^{s} a_{ij}\ \left((c_j-1)\,\eta_0\left((c_j-1)^2 Z\right) + \frac{(c_j-1)^3}{2} Z\ \eta_1\left((c_j-1)^2 Z\right)\right) \\ &- \sum_{j=1}^{i-1} r_{ij}\ \left(c_j\,\eta_0\left(c_j^2 Z\right) + \frac{c_j^3}{2} Z\ \eta_1\left(c_j^2 Z\right)\right), \end{aligned}$$

so Equation (2.17c) is proved for $m = 1$. We get Equation (2.17c) for $m > 1$ by induction.
On the other hand, the first derivative of G_i^- (2.17b) is

$$\begin{aligned} G_i^{-(1)}(Z,w) = \frac{c_i^3}{2}\ \eta_1\left(c_i^2 Z\right) &- \sum_{j=1}^{s} b_{ij}\ \frac{(c_j-1)^3}{2}\ \eta_1\left((c_j-1)^2 Z\right) \\ &- \sum_{j=1}^{s} a_{ij}\ \frac{(c_j-1)^2}{2}\eta_0\left((c_j-1)^2 Z\right) - \sum_{j=1}^{i-1} r_{ij}\ \frac{c_j^2}{2}\ \eta_0\left(c_j^2 Z\right), \end{aligned} \tag{2.19}$$

which leads to Equation (2.17d) for $m > 1$ by induction. □

Step 4
We identify the possible expressions for the fitting space (2.8) taking into account that $M = s+1$ and the self-consistency condition

$$K + 2P = M - 3 \tag{2.20}$$

has to be verified. We observe that the number of stages s and the dimension M of the system (2.11) have different parities, so the the number $K+1 = s-1-2P$ of classic functions in the fitting space is odd or even, if s is even or odd, respectively. For simplicity, we choose

- $K = 0$ if s is even, so the fitting space is

$$\mathcal{F} = \left\{1, e^{\pm\mu t}, t\, e^{\pm\mu t}, t^2\, e^{\pm\mu t}, \ldots, t^P\, e^{\pm\mu t}\right\}; \tag{2.21}$$

- $K = -1$ if s is odd, so the fitting space is

$$\mathcal{F} = \left\{e^{\pm\mu t}, t\, e^{\pm\mu t}, t^2\, e^{\pm\mu t}, \ldots, t^P\, e^{\pm\mu t}\right\}. \tag{2.22}$$

Step 5
We obtain the coefficients of the exponentially fitted peer method by solving the system:

$$\mathcal{L}^*_{i\,m}(h, \mathbf{w}) = 0, \quad i = 1, \ldots, s, \quad m = 0, \ldots, K, \tag{2.23a}$$

$$G_i^{\pm(m)}(Z, \mathbf{w}) = 0, \quad i = 1, \ldots, s, \quad m = 0, \ldots, P. \tag{2.23b}$$

For the afore-mentioned fitting spaces (2.21)-(2.22), this system becomes:

- if s is even

$$\mathcal{L}^*_{i0}(h, \mathbf{w}) = 0, \quad i = 1, \ldots, s \tag{2.24a}$$

$$G_i^{\pm(m)}(Z, \mathbf{w}) = 0, \quad i = 1, \ldots, s, \quad m = 0, \ldots, P, \tag{2.24b}$$

  where $P = \frac{s}{2} - 1$ due to the self-consistency condition (2.20);

- if s is odd

$$G_i^{\pm(m)}(Z, \mathbf{w}) = 0, \quad i = 1, \ldots, s, \quad m = 0, \ldots, P, \tag{2.25}$$

  where $P = \frac{s-1}{2}$ due to the self-consistency condition (2.20).

In the following theorem, we recast such systems in order to obtain the coefficients of exponentially fitted peer methods in an easy way.

Theorem 2.4. *System* (2.24a) *is equivalent to*

$$B\,\mathbf{1} = \mathbf{1} \tag{2.26}$$

and systems (2.24b) *and* (2.25) *are equivalent to*

$$\mathcal{D}(c, C) - B\,\mathcal{D}(c - \mathbf{1}, \hat{C}) - A\,\mathcal{E}(c - \mathbf{1}, \hat{C}) - R\,\mathcal{E}(c, C) = 0, \tag{2.27}$$

where $c = [c_1, \ldots, c_s]^T$, $C = diag(c_1, \ldots, c_s)$, $\hat{C} = diag(c_1 - 1, \ldots, c_s - 1)$, $\mathbf{1} = [1, 1, \ldots, 1]^T$, *each column* j *of the matrices* $\mathcal{D}(v, W) \in \mathcal{M}^{s \times 2(P+1)}$ *and* $\mathcal{E}(v, W) \in \mathcal{M}^{s \times 2(P+1)}$ *depends on the generic vector* v *and the generic diagonal matrix* W*, as follows:*

$$\begin{aligned}
(\mathcal{D}(v, W))^j &= \frac{W^{j-1}\theta_{\frac{j-2-\delta}{2}, v}}{2^{\frac{j-2+\delta}{2}}}, \\
(\mathcal{E}(v, W))^j &= \frac{(j-1)^\delta\, W^{j-2}\,\theta_{\frac{j-4+\delta}{2}, v}}{2^{\frac{j-2+\delta}{2}}} + \frac{\delta\, W^j\, Z\,\theta_{\frac{j-1}{2}, v}}{2^{\frac{j-1}{2}}},
\end{aligned} \tag{2.28}$$

with

$$\delta = \begin{cases} 0 & \text{if } j \text{ even}, \\ 1 & \text{if } j \text{ odd}, \end{cases} \qquad P = \begin{cases} (s-2)/2 & \text{if } j \text{ even}, \\ (s-1)/2 & \text{if } j \text{ odd}, \end{cases}$$

and the vector $\theta_{\sigma,v}$ *is defined by evaluating the function* η_σ *on* $v_i^2\,Z$ *where* v_i *are the component of a generic vector* v*, as follows:*

$$\theta_{\sigma,v} = \left[\eta_\sigma\,(v_1^2\,Z), \ldots, \eta_\sigma\,(v_s^2\,Z)\right]. \tag{2.29}$$

Proof. Annihilating the dimensionless classic moments of order $m = 0$ in (2.24a) is equivalent to solving the system

$$\mathcal{L}_{i0}^*(h, w) = 1 - \sum_{j=1}^{s} b_{ij} \;= 0, \quad i = 1, \ldots, s,$$

which can be recasted in a matrix form as follows

$$\mathbf{1} - B\,\mathbf{1} = \mathbf{0}, \qquad \mathbf{0} = (0, 0, \ldots, 0)^T.$$

Therefore, Equation (2.26) holds.
System (2.24b) (or, equivalently, (2.25)) for G_i^+ assumes the following expression for $i = 1, \ldots, s$:

$$\begin{aligned} G_i^+(Z,w) &= \eta_{-1}\left(c_i^2 Z\right) - \sum_{j=1}^{s} b_{ij}\,\eta_{-1}\left((c_j-1)^2 Z\right) \\ &- Z\sum_{j=1}^{s} a_{ij}\,(c_j-1)\,\eta_0\left((c_j-1)^2 Z\right) - Z\sum_{j=1}^{i-1} r_{ij}\,c_j\,\eta_0\left(c_j^2 Z\right) = 0, \end{aligned} \tag{2.30}$$

which can be written in a compact form

$$\theta_{-1,\,c} - B\,\theta_{-1,\,c-\mathbf{1}} - Z\,A\,(\,\hat{C}\,\theta_{0,\,c-\mathbf{1}}\,) - Z\,R\,(C\,\theta_{0,\,c}) = \mathbf{0}, \tag{2.31}$$

where $c = [c_1, \ldots, c_s]^T$, $C = \text{diag}(c_1, \ldots, c_s)$, $\hat{C} = diag(c_1 - 1, \ldots, c_s - 1)$ and the vector $\theta_{\sigma,v}$ is defined by evaluating the function η_σ on $v_i^2\,Z$ where v_i are the component of a generic vector v as in (2.29).
On the other hand, system (2.24b) for G_i^- can be recasted as

$$C\,\theta_{0,\,c} - B\,(\,\hat{C}\,\theta_{0,\,c-\mathbf{1}}\,) - A\,\theta_{-1,\,c-\mathbf{1}} - R\,\theta_{-1,\,c} = \mathbf{0}. \tag{2.32}$$

In a similar way, systems (2.24b) for $G_i^{+(m)}$ and $G_i^{-(m)}$ with $m = 1, \ldots, P$ are respectively equivalent to

$$\begin{aligned} \frac{1}{2^m}\,C^{2m}\,\theta_{m-1,\,c} - B\,&\left(\frac{1}{2^m}\hat{C}^{2m}\,\theta_{m-1,\,c-\mathbf{1}}\right) \\ &- A\,\left(\frac{m}{2^{m-1}}\hat{C}^{2m-1}\,\theta_{m-1,\,c-\mathbf{1}} + \frac{Z}{2^m}\hat{C}^{2m+1}\theta_{m,c-1}\right) \\ &- R\left(\frac{m}{2^{m-1}}C^{2m-1}\,\theta_{m-1,\,c} + \frac{Z}{2^m}C^{2m+1}Z\theta_{m,\,c}\right) = \mathbf{0}, \end{aligned} \tag{2.33a}$$

$$\frac{1}{2^m}\left(C^{2m+1}\theta_{m,c} - B\left(\hat{C}^{2m+1}\theta_{m,c-\mathbf{1}}\right) - A\left(\hat{C}^{2m}\theta_{m-1,c-\mathbf{1}}\right) - R\,C^{2m}\theta_{m-1,c}\right) = \mathbf{0}. \tag{2.33b}$$

We next construct the matrix $\mathcal{D}(c, C) \in \mathcal{M}^{s\,\times\,2(P+1)}$ such that its first and second columns correspond to the first vectors of the systems (2.31) and (2.32), respectively. Then the other columns are the first vectors of the system (2.33a) and (2.33b), alternatively:

$$\mathcal{D}(c, C) =$$

$$\left[\theta_{-1,c} \quad C\,\theta_{0,c} \quad \frac{1}{2}\,C^2\,\theta_{0,\,c} \quad \frac{1}{2}C^3\,\theta_{1,\,c} \quad \cdots \quad \frac{1}{2^P}\,C^{2P}\,\theta_{P-1,\,c} \quad \frac{1}{2^P}C^{2P+1}\,\theta_{P,\,c}\right],$$

which can be written in extensive form as follows:

$$\mathcal{D}(c,C) = \begin{bmatrix} \eta_{-1}\left(c_1^2 Z\right) & c_1\,\eta_0\left(c_1^2 Z\right) & \dots & \dfrac{1}{2^P}\,c_1^{2P}\,\eta_{P-1}\left(c_1^2\,Z\right) & \dfrac{1}{2^P}c_1^{2P+1}\,\eta_P\left(c_1^2\,Z\right) \\ \vdots & \vdots & & \vdots & \vdots \\ \eta_{-1}\left(c_s^2 Z\right) & c_s\,\eta_0\left(c_s^2 Z\right) & \dots & \dfrac{1}{2^P}\,c_s^{2P}\,\eta_{P-1}\left(c_s^2\,Z\right) & \dfrac{1}{2^P}c_s^{2P+1}\,\eta_P\left(c_s^2\,Z\right) \end{bmatrix}.$$

Similarly, we construct the matrices $\mathcal{D}(c-\mathbf{1},\hat{C}), \mathcal{E}(c-\mathbf{1},\hat{C}), \mathcal{E}(c,C) \in \mathcal{M}^{s\times 2(P+1)}$ by considering the vectors multiplying B, A and R in system (2.31)-(2.33), respectively. In this way, system (2.31)-(2.33) is equivalent to (2.27). □

In case of s even, $P=\frac{s}{2}-1$, so $\mathcal{D}(v,N)$ and $\mathcal{E}(v,N)$ are square matrices. Moreover, if $\mathcal{E}(c-\mathbf{1},\hat{C})$ is invertible, we can compute the matrix A, as follows:

$$A = \left(\mathcal{D}(c,C) - B\,\mathcal{D}(c-\mathbf{1},\hat{C}) - R\,\mathcal{E}(c,C)\right)\mathcal{E}(c-\mathbf{1},\hat{C})^{-1}. \qquad (2.34)$$

Therefore, the following theorem holds:

Theorem 2.5. *In case of even number of stages s, a peer method* (2.3) *is exact on the basis functions*

$$\mathcal{F} = \left\{1, e^{\pm\mu t}, t\,e^{\pm\mu t}, t^2\,e^{\pm\mu t}, \dots, t^P\,e^{\pm\mu t}\right\},$$

if its matrices of coefficients verify the following conditions

$$B\,\mathbf{1} = \mathbf{1} \qquad (2.35a)$$

$$A = \left(\mathcal{D}(c,C) - B\,\mathcal{D}(c-\mathbf{1},\hat{C}) - R\,\mathcal{E}(c,C)\right)\mathcal{E}(c-\mathbf{1},\hat{C})^{-1}, \qquad (2.35b)$$

provided that the matrix $\mathcal{E}(c-\mathbf{1},\hat{C})$ is invertible.

In case of s odd, $P=\frac{s-1}{2}$ and $\mathcal{D}(v,N)$ and $\mathcal{E}(v,N)$ have dimensions $s\times(s+1)$. Therefore, for ease of presentation, we recast system (2.27) as follows:

$$\theta_{-1,c} - B\,\theta_{-1,\,c-1} - A\left(Z\,\hat{C}\,\theta_{0,\,c-\mathbf{1}}\right) - R\left(Z\,C\,\theta_{0,\,c}\right) = \mathbf{0}, \qquad (2.36a)$$

$$F_1 - B\,F_2 - A\,F_3 - R\,F_4 = 0, \qquad (2.36b)$$

where F_i for $i=1,2,3,4$ are square matrices obtained by deleting the first column from $\mathcal{D}(c,C)$, $\mathcal{D}(c-\mathbf{1},\hat{C})$, $\mathcal{E}(c-\mathbf{1},\hat{C})$, $\mathcal{E}(c,C)$, respectively.
For instance, we can choose the following matrix B:

$$B = H_1 - A\,H_2 - R\,H_3, \qquad (2.37)$$

where

$$H_1 = (0 \,|\, \theta_{-1,c}), \quad H_2 = (0 \,|\, Z\,\hat{C}\,\theta_{0,c-\mathbf{1}}), \quad H_3 = (0 \,|\, Z\,C\,\theta_{0,c}) \in \mathcal{M}^{s\times s}.$$

Employing, as in [38], the nodes

$$c_i = \frac{i-1}{s-1}, \quad i = 1,\dots,s,$$

the matrix B (2.37) verifies condition (2.36a). Thus, replacing the expression of the matrix B in (2.36b), we obtain

$$A = (F_1 - H_1F_2 - R(F_4 - H_3F_2))(F_3 - H_2F_2)^{-1}. \tag{2.38}$$

Theorem 2.6. *In case of odd number of stages s, a peer method* (2.3) *having the following matrices of coefficients*

$$B = H_1 - A\,H_2 - R\,H_3, \tag{2.39a}$$

$$A = (F_1 - H_1F_2 - R(F_4 - H_3F_2))(F_3 - H_2F_2)^{-1}, \tag{2.39b}$$

is exact on the basis functions

$$\mathcal{F} = \left\{e^{\pm\mu t}, t\,e^{\pm\mu t}, t^2\,e^{\pm\mu t}, \dots, t^P\,e^{\pm\mu t}\right\},$$

provided that the matrix $F_3 - H_2F_2$ is invertible,

$$H_1 = (0 \,|\, \theta_{-1,c}), \quad H_2 = (0 \,|\, Z\,\hat{C}\,\theta_{0,c-\mathbf{1}}), \quad H_3 = (0 \,|\, Z\,C\,\theta_{0,c}) \in \mathcal{M}^{s\times s}$$

and F_i for $i = 1,2,3,4$ are square matrices obtained by deleting the first column from $\mathcal{D}(c, C)$, $\mathcal{D}(c-\mathbf{1}, \hat{C})$, $\mathcal{E}(c-\mathbf{1}, \hat{C})$, $\mathcal{E}(c, C)$, respectively.

We remark that in both cases some entries of matrix B and all entries of matrix R are free parameters for the exponentially fitted peer method.

Step 6

We compute the leading term of the local truncation error at each stage, as follows:

$$(lte_{ef})_i = (-1)^{P+1}h^{s+1}\frac{\mathcal{L}^*_{i,K+1}(h,\mathbf{w})}{(K+1)!\,Z^{P+1}}\,D^{K+1}(D^2-\mu^2)^{P+1}y(t), \quad i = 1,\dots,s, \tag{2.40}$$

where we denote D the derivative with respect to time.
As before, we choose for simplicity $K = 0$ and $K = -1$ for s even or odd, respectively. In these cases, the afore-mentioned leading term assumes the following expressions:

- if s is even

$$(lte_{ef})_i = \frac{(-1)^{\frac{s}{2}}\ h^{s+1}}{Z^{\frac{s}{2}}} \left(c_i - \sum_{j=1}^{s} b_{ij}\ (c_j - 1) - \sum_{j=1}^{s} a_{ij}\ - \sum_{j=1}^{i-1} r_{ij} \right) \cdot \\ \cdot D(D^2 - \mu^2)^{s/2} y(t); \tag{2.41}$$

- if s is odd

$$(lte_{ef})_i = \frac{(-1)^{\frac{s+1}{2}}\ h^{s+1}}{Z^{\frac{s+1}{2}}} \left(1 - \sum_{j=1}^{s} b_{ij} \right) (D^2 - \mu^2)^{\frac{s+1}{2}} y(t). \tag{2.42}$$

3 Examples of methods

We now present two methods constructed as described in Section 2. For both methods we select as nodes

$$c_i = \frac{i-1}{s-1}, \quad i = 1, \ldots, s, \tag{3.43}$$

which clearly verify conditions (2.2) and $c_s = 1$.

3.1 Case 1: $s = 2$

We recall that in this case we have chosen $K = 0$ for simplicity. As a consequence, $P = 0$ due to self-consistency condition (2.20). Solving system (2.35) for $s = 2$, we compute the coefficients of a generic exponentially fitted 2−stage peer method:

$$\begin{aligned} b_{i1} &= 1 - b_{i2}, \quad i = 1, 2, \\ a_{11} &= \frac{(1 - b_{12})\,(\eta_{-1}\,(Z) - 1)}{Z\,\eta_0\,(Z)}, \\ a_{12} &= (1 - b_{12}) \left(\eta_0\,(Z) - \frac{(\eta_{-1}\,(Z) - 1)\,\eta_{-1}\,(Z)}{Z\,\eta_0\,(Z)} \right), \\ a_{21} &= \frac{b_{22}(\,1 - \eta_{-1}\,(Z)\,)}{Z\,\eta_0\,(Z)}, \\ a_{22} &= \frac{b_{22}\,\eta_{-1}\,(Z)\,(\eta_{-1}\,(Z) - 1)}{Z\,\eta_0\,(Z)} + \eta_0\,(Z)\,(2 - b_{22}) - r_{21}, \end{aligned} \tag{3.44}$$

which, for $Z \to 0$, tend to the coefficients of the corresponding classic peer method obtained by solving the system (2.7) with $\beta = 0$:

$$\begin{aligned} b_{i1} &= 1 - b_{i2}, \quad i = 1, 2, \\ a_{11} &= \frac{1 - b_{12}}{2}, \quad a_{12} = \frac{1 - b_{12}}{2}, \\ a_{21} &= -\frac{b_{22}}{2}, \quad a_{22} = \frac{4 - b_{22} - 2r_{21}}{2}. \end{aligned} \tag{3.45}$$

We observe that in the aforementioned expressions b_{12}, b_{22} and r_{21} are free parameters.

3.2 Case 2: $s = 3$

In this case, we have selected $K = -1$, for simplicity. Hence $P = 1$ due to self-consistency condition (2.20). Solving system (2.39), we compute the coefficients of a class of 3-stage exponentially fitted peer methods, as follows:

$$\begin{aligned} B &= \begin{bmatrix} 0 & 0 & 1 + Z\,\eta_0\,a_{11} + \frac{Z}{2}\,a_{12}\tilde{\eta}_0 \\ 0 & 0 & \tilde{\eta}_{-1} + Z\,\eta_0\,a_{21} + \frac{Z}{2}\,a_{22}\tilde{\eta}_0 \\ 0 & 0 & \eta_{-1} + Z\,\eta_0\,a_{31} + \frac{Z}{2}\,a_{32}\tilde{\eta}_0 - \frac{Z}{2}\,\tilde{\eta}_0 r_{32} \end{bmatrix}, \\ a_{1j} &= 0, \\ a_{2j} &= \frac{\varphi_j\,\tilde{\eta}_0 + \psi_j\,\tilde{\eta}_1}{4^{\delta_{1j}}\,\Phi} + \delta_{j3}\left(\frac{\tilde{\eta}_0}{2} - r_{21}\right), \\ a_{3j} &= \frac{\varphi_j(8\eta_0 - r_{32}\,(8\,\tilde{\eta}_0 + \tilde{\eta}_1\,Z)) + \psi_j\,(4\eta_1 - \tilde{\eta}_0\,r_{32})}{4^{\delta_{1j}}\cdot 2\,\Phi} \\ &\quad + \delta_{j3}\,(\eta_0 - r_{31} - \tilde{\eta}_{-1}\,r_{32}), \end{aligned} \tag{3.46}$$

for $j = 1, 2, 3$, where δ_{j3} is Kronecker delta and

$$\eta_\sigma = \eta_\sigma(Z), \quad \tilde{\eta}_\sigma = \eta_\sigma\left(\frac{Z}{4}\right), \text{ for } \sigma = -1, 0, 1,$$

$$\Phi = 2\eta_0\,\tilde{\eta}_0 - \tilde{\eta}_0\,\eta_1 Z + \frac{\eta_0\,\tilde{\eta}_1\,Z}{2},$$

$$\psi_1 = 2\,\tilde{\eta}_0 + \frac{\tilde{\eta}_1\,Z}{4}, \quad \psi_2 = -\eta_0 - \frac{\eta_1\,Z}{2},$$
$$\psi_3 = \eta_0\,\tilde{\eta}_{-1} - \frac{\tilde{\eta}_0\,\eta_{-1}}{2} - \frac{\tilde{\eta}_1\,\eta_{-1}\,Z}{16} + \frac{\tilde{\eta}_{-1}\,\eta_1\,Z}{2},$$

$$\varphi_1 = \tilde{\eta}_0, \quad \varphi_2 = -\eta_0, \quad \varphi_3 = \eta_0\,\tilde{\eta}_{-1} - \frac{\tilde{\eta}_0\,\eta_{-1}}{4}.$$

These coefficients for $Z \to 0$ tend to the coefficients of the classic 3-stage peer method obtained by solving system (2.7) with $\beta = 0$:

$$\begin{aligned}
b_{i3} &= 1 - b_{i1} - b_{i2}, \quad i = 1, 2, 3,\\
a_{11} &= \frac{4b_{11} - b_{12}}{24},\\
a_{12} &= \frac{2b_{11} + b_{12}}{3},\\
a_{13} &= \frac{4b_{11} + 5b_{12}}{24},\\
a_{21} &= \frac{5 + 4b_{21} - b_{22}}{24},\\
a_{22} &= \frac{2b_{21} + b_{22} - 2}{3},\\
a_{23} &= \frac{23 + 4b_{21} + 5b_{22} - 24r_{21}}{24},\\
a_{31} &= \frac{28 + 4b_{31} - b_{32} - 24r_{32}}{24},\\
a_{32} &= \frac{2b_{31} + b_{32} + 9r_{32} - 10}{3},\\
a_{33} &= \frac{76 + 4b_{31} + 5b_{32} - 24r_{31} - 72r_{32}}{24}.
\end{aligned} \tag{3.47}$$

We observe that in the afore-mentioned expressions b_{i1}, b_{i2} for $i = 1, 2, 3$ and r_{21}, r_{31} and r_{33} are free parameters.

4 Numerical experiments

In this section, we present some numerical tests in order to prove the effectiveness of the introduced exponentially fitted two-step peer methods. In the following, we compare the exponentially fitted peer methods with 2 and 3 stages characterized by the coefficients (3.44) and (3.46) to their classic counterparts (3.45) and (3.47). For both classic methods, we choose

$$B = \begin{bmatrix} 0 & \dots & 0 & 1 \\ \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & 1 \end{bmatrix}, \quad R = \begin{bmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{bmatrix} \in \mathcal{M}^{s\times s}. \tag{4.48}$$

For both exponentially fitted methods we select R as in (4.48), whereas we employ B as defined in (4.48) only for $s = 2$. In all the numerical tests the error is computed as the infinite norm of the difference between the numerical solution and the exact solution.

Example 1 We integrate in $[0, 10\,\pi]$ the following Kepler's problem [18, 25]

$$\begin{aligned} \frac{dp_i}{dt} &= -\frac{\partial H}{\partial q_i}, \quad i = 1,2, \\ \frac{dq_i}{dt} &= \frac{\partial H}{\partial p_i}, \quad i = 1,2, \end{aligned} \tag{4.49}$$

provided by the Hamiltonian function

$$H = \frac{1}{2}(p_1^2 + p_2^2) - \frac{1}{\sqrt{q_1^2 + q_2^2}} - \frac{2\delta + \delta^2}{3\sqrt{\left(q_1^2 + q_2^2\right)^3}}, \tag{4.50}$$

with δ small positive parameter, and initial conditions

$$q_1(0) = 1, \quad q_2(0) = 0, \quad p_1(0) = 0, \quad p_2(0) = 1 + \delta. \tag{4.51}$$

Its exact solution

$$q_1(t) = \cos(t + \delta\, t), \quad q_2(t) = \sin(t + \delta t), \quad p_i(t) = q_i'(t),\ i = 1,2 \tag{4.52}$$

has an oscillatory behaviour, so this system is a good candidate for the adoption of the exponentially fitted method (3.44) with a complex value for the parameter $\mu = \mathrm{i}\omega$. We first solve system (4.49) with $\delta = 0$. In this case, the exact solution

can be expressed as a linear combination of the basis functions belonging to the fitting space (2.8), so the problem is integrated exactly (within round-off error) by the exponentially fitted peer methods (3.44) and (3.46), where the parameter ω is chosen equal to the frequency of the exact solution, i. e. $\omega = 1$. Indeed, Table 1 shows that these methods are much more accurate than their classic counterparts (3.45) and (3.47) and produce an error almost equal to the round-off one. Moreover, we have performed a comparison between the exponentially fitted methods (3.44) and (3.46) derived in this paper with those obtained in [25]. We observe that, in case of 200 grid points, the global error associated to the whole integration interval related to exponentially fitted methods (3.44) and (3.46) is of the order of magnitude of 10^{-11}, whereas the global error computed in [25] is about 10^{-4}. The larger accuracy can be motivated by the fact that, in this paper, we have employed the regularizing six-step procedure that, as highlighted in [19, 21], better controls the propagation of round-off error.
We next integrate system (4.49) with $\delta = 10^{-2}$. In this case, the exact solution can be written in terms of the basis functions (2.8) with $\omega = 1$ but not as a linear combination. However, the exponentially fitted peer methods (3.44) and (3.46) integrate exactly (within an error almost equal to the round-off one) the considered problem also in this case, as reported in Table 1. Similarly to the first case, they exhibit an higher accuracy than the corresponding classic ones (3.45) and (3.47). Also for this problem, we have compared the exponentially fitted methods (3.44) and (3.46) derived in this work with those presented in [25]. We observe that, in case of 200 grid points, the global error associated to the whole integration interval obtained by exponentially fitted methods (3.44) and (3.46) is of the order of magnitude of 10^{-11}, whereas the global error computed in [25] is about 10^{-7}.

Example 2 We integrate the following Prothero-Robinson problem [18]

$$\begin{aligned} &y'(t) = \lambda\,(\,y(t) - \sin(\omega\,t + t)\,) + (\omega + 1)\,\cos(\omega\,t + t), \quad t \in \left[0, \frac{\pi}{2}\right], \\ &y(0) = 0, \end{aligned} \tag{4.53}$$

where $\lambda = -1$. The oscillating behaviour of its exact solution

$$y(t) = \sin(\omega\,t + t) = \sin(\omega\,t)\,\cos(t) + \cos(\omega\,t)\,\sin(t)$$

suggests to employ the exponentially fitted methods (3.44) and (3.46) with the parameter μ characterizing the functions belonged to the fitting space equal to $\mu = \mathrm{i}\,\omega$. Indeed, Table 2 shows that the exponentially fitted methods (3.44) and (3.46) are more accurate than their classic counterparts (3.45) and (3.47), which become totally inaccurate for highly oscillating problems. However, the exact solution is given by a nonlinear combination of these basis functions, so the exponentially

		N			
	δ	200	400	800	1600
CLASSIC $s = 2$	0	1.25	$2.59 \cdot 10^{-1}$	$2.43 \cdot 10^{-2}$	$5.50 \cdot 10^{-4}$
EXP. FITTED $s = 2$	0	$3.60 \cdot 10^{-13}$	$5.13 \cdot 10^{-13}$	$3.66 \cdot 10^{-12}$	$8.31 \cdot 10^{-13}$
CLASSIC PEER $s = 3$	0	1.91	$2.48 \cdot 10^{-1}$	$3.02 \cdot 10^{-2}$	$3.75 \cdot 10^{-3}$
EXP. FITTED $s = 3$	0	$8.67 \cdot 10^{-13}$	$2.49 \cdot 10^{-12}$	$4.29 \cdot 10^{-12}$	$1.24 \cdot 10^{-12}$
CLASSIC PEER $s = 2$	10^{-2}	1.40	$2.88 \cdot 10^{-1}$	$2.77 \cdot 10^{-2}$	$1.00 \cdot 10^{-3}$
EXP. FITTED $s = 2$	10^{-2}	$3.60 \cdot 10^{-13}$	$5.13 \cdot 10^{-13}$	$3.66 \cdot 10^{-12}$	$8.32 \cdot 10^{-13}$
CLASSIC $s = 3$	10^{-2}	1.91	$2.48 \cdot 10^{-1}$	$3.02 \cdot 10^{-2}$	$3.75 \cdot 10^{-3}$
EXP. FITTED $s = 3$	10^{-2}	$8.67 \cdot 10^{-13}$	$2.49 \cdot 10^{-12}$	$4.29 \cdot 10^{-12}$	$1.24 \cdot 10^{-12}$

Table 1. Accuracy of the classic peer methods (3.45) and (3.47) and the exponentially fitted ones (3.44) and (3.46) in the integration of problem (4.49) with N grid points, the parameter characterizing the fitting space (2.8) equal to $\mu = \mathrm{i}\,\omega$ and $\omega = 1$.

		N			
	ω	80	160	320	640
CLASSIC $s=2$	50	$3.79 \cdot 10^{-1}$	$1.05 \cdot 10^{-1}$	$2.65 \cdot 10^{-2}$	$6.60 \cdot 10^{-3}$
EXP. FITTED $s=2$	50	$1.53 \cdot 10^{-2}$	$4.1 \cdot 10^{-3}$	$1.00 \cdot 10^{-3}$	$2.57 \cdot 10^{-4}$
CLASSIC $s=3$	50	$1.39 \cdot 10^{-1}$	$1.10 \cdot 10^{-2}$	$9.42 \cdot 10^{-4}$	$8.98 \cdot 10^{-5}$
EXP. FITTED $s=3$	50	$1.23 \cdot 10^{-4}$	$1.07 \cdot 10^{-5}$	$1.26 \cdot 10^{-6}$	$1.33 \cdot 10^{-7}$
CLASSIC $s=2$	100	$3.59 \cdot 10^{-1}$	$3.02 \cdot 10^{-1}$	$9.33 \cdot 10^{-2}$	$2.47 \cdot 10^{-2}$
EXP. FITTED $s=2$	100	$3.33 \cdot 10^{-2}$	$5.3 \cdot 10^{-3}$	$1.8 \cdot 10^{-3}$	$4.86 \cdot 10^{-4}$
CLASSIC $s=3$	100	1.12	$6.92 \cdot 10^{-2}$	$2.40 \cdot 10^{-3}$	$1.22 \cdot 10^{-4}$
EXP. FITTED $s=3$	100	$2.87 \cdot 10^{-5}$	$3.08 \cdot 10^{-5}$	$2.30 \cdot 10^{-6}$	$1.58 \cdot 10^{-8}$

Table 2. Accuracy of the classic peer methods (3.45) and (3.47) and the exponentially fitted ones (3.44) and (3.46) in the integration of problem (4.53) with N grid points and different values for the frequency ω.

N	EXP. FITTED $s=2$	EXP. FITTED $s=3$
80	2.57	–
160	1.91	3.52
320	1.98	3.09
640	2.00	3.24

Table 3. Estimated order of the exponentially fitted peer methods (3.44) and (3.46) computed by Equation (4.54) within the integration of problem (4.53) with $\omega = 50$.

fitted peer methods (3.44) and (3.46) exhibit an error bigger than the round-off one, as reported in Table 2. Moreover, we estimate the order p of the exponentially fitted peer methods (3.44) and (3.46) employing the following relations

$$p = \lim_{h \to 0} p(h), \quad p(h) \approx \log_2 \left(\frac{E(h)}{E(h/2)} \right), \tag{4.54}$$

where $E(h)$ and $E(h/2)$ are the errors with a stepsize h and $h/2$, respectively. Table 3 shows that the estimated order $p(h)$ of the exponentially fitted peer methods (3.44) and (3.46) are equal to 2 and 3, respectively.

5 Conclusions

We have constructed a general class of exponentially fitted peer methods following the six-step procedure presented in [19,21]. These methods are widely suitable for the numerical integration of ordinary differential equations having a solution with a particular feature such as an oscillatory behaviour or an exponential decay. The adopted strategy is based on adapting already existing methods in order to be exact (within round-off error) on trigonometric or hyperbolic functions. In the sixth step of the procedure, we have computed the expression of the leading term of the local truncation error. It may lead to an estimate of the parameter characterizing the basis functions, which we aim to study as future work. Numerical experiments have confirmed the effectiveness of the approach. Further efforts of the research in the topic will include the parallel implementation of these methods on a selection of high dimensional problems arising from partial differential equations. In this way, it will be possible to exploit, in practice, the effectiveness of peer methods in a parallel implementation environment, properly coupled with the adaptation to the problem provided by the exponential fitting approach.

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