A-stability preserving perturbation of Runge-Kutta methods for stochastic differential equations

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Abstract

The paper is focused on analyzing the linear stability properties of stochastic Runge-Kutta (SRK) methods interpreted as a stochastic perturbation of the corresponding deterministic Runge-Kutta methods. In particular, we give a condition such that deterministic A-stability is automatically inherited by stochastic Runge-Kutta methods as mean-square A-stability. This issue provides classes of mean-square A-stable SRK methods straightforwardly.

Keywords: Mean-square A-stability, stochastic differential equations, stochastic Runge-Kutta methods.

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1. Introduction

We consider a system of stochastic differential equations (SDEs) of Itô type, assuming the form

$$\begin{cases} \mathrm{d}Y(t) = f(Y(t))\mathrm{d}t + g(Y(t))\mathrm{d}W(t), & t \in [0, T], \\ Y(0) = Y_0 \in \mathbb{R}^d, \end{cases} \tag{1.1}$$

where $f, g : \mathbb{R}^d \to \mathbb{R}^d$ and $W(t)$ is an m-dimensional Wiener process. For theoretical results on the existence and uniqueness of solutions to (1.1), we refer to the monographs [17, 20, 22]. Referring to the discretized domain

$$\mathcal{I}_{\Delta t} = \{t_n = n\Delta T,\ n = 0, 1, \ldots, N,\ N = T/\Delta t\},$$

we focus our attention on the following family of stochastic numerical methods

$$\begin{cases} \widehat{Y}_i = Y_n + \Delta t \displaystyle\sum_{j=1}^{s} a_{ij} f(\widehat{Y}_j) + g(Y_n)\Delta W_n, & i = 1, ..., s, \\ Y_{n+1} = Y_n + \Delta t \displaystyle\sum_{i=1}^{s} b_i f(\widehat{Y}_i) + g(Y_n)\Delta W_n, \end{cases} \tag{1.2}$$

where Y_n is an approximation to $Y(t_n)$ and $\widehat{Y}_i$ approximates $Y(t_n + c_i\Delta t)$, $i = 1, ..., s$, with $c_i \in [0, 1]$. The class of methods (1.2), hereinafter denoted as *stochastic Runge-Kutta methods* (SRK), arises from the stochastic perturbation of deterministic Runge-Kutta methods by adding an explicit additional diffusive term depending on the discretized Weiner increment ΔW_n, that is a Gaussian random variable with zero mean and variance Δt. SRK methods (1.2) are characterized by the weights b_i, the nodes c_i and the scalars a_{ij}, $i, j = 1, 2, \ldots, s$, collected as usual in the Butcher tableau

$$\begin{array}{c|c} c & A \\ \hline & b^{\intercal} \end{array} = \begin{array}{c|cccc} c_1 & a_{11} & a_{12} & \ldots & a_{1s} \\ c_2 & a_{21} & a_{22} & \ldots & a_{2s} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_s & a_{s1} & a_{s2} & \ldots & a_{ss} \\ \hline & b_1 & b_2 & \ldots & b_s \end{array} \tag{1.3}$$

that is equal to the Butcher tableau of the underlying deterministic Runge-Kutta method

$$\begin{cases} Z_i = z_n + \Delta t \sum_{j=1}^{s} a_{ij} f(Z_j), \qquad i = 1, ..., s, \\ z_{n+1} = z_n + \Delta t \sum_{i=1}^{s} b_i f(Z_i) \end{cases} \tag{1.4}$$

for the deterministic equation $z' = f(z)$.

The analysis of strong and weak accuracy properties of SRK methods of various types has extensively been addressed by the existing literature; see, for instance [1, 3, 4, 5, 6, 7, 13, 16, 21, 23, 24] and references therein. As regards, in particular, the family of SRK methods (1.2), they belong to the family of stochastic Runge-Kutta methods introduced by Gard in the monograph [17], so the analysis of their convergence comes straightforward from [17]. Hence, we can infer from [17], that the strong convergence of (1.2) is equivalent to the convergence of the underlying RK method (1.4), i.e., it occurs when $\sum_{i=1}^{s} b_i = 1$.

In this paper, we address our investigation to the analysis of the linear stability properties of SRK methods (1.2), by providing the conditions under which the stability properties of the underlying deterministic Runge-Kutta method (1.4) are automatically inherited by its stochastic perturbation (1.2). This issue would provide, for instance, classes of mean-square A-stable SRK methods (1.2) straightforwardly, which may also eventually be good candidates to numerically inherit the stability properties of nonlinear test problems, such as those in [2, 8, 9, 10, 11, 12, 14, 15].

The manuscript is organized as follows: Section 2 provides the main tools for the linear stability analysis of SRK methods; Section 3 contains the main result of the manuscript, dealing with conditions guaranteeing that the stability properties of the underlying RK methods are inherited by the corresponding SRK methods, also confirmed by the numerical evidence.

2. Mean-square stability of SRK methods

In order to analyze the linear stability properties of SRK methods, we consider the following linear scalar test SDE with multiplicative noise

$$\begin{cases} \mathrm{d}Y(t) = \lambda Y(t)\mathrm{d}t + \mu Y(t)\mathrm{d}W(t), \quad t \in [0, T], \\ Y(0) = Y_0, \end{cases} \tag{2.1}$$

where $\lambda, \mu \in \mathbb{C}$. The following definition occurs (see, for instance [18, 19]).

Definition 2.1. *The solution $Y(t)$ of (2.1) is mean-square stable if*

$$\lim_{t\to\infty} \mathbb{E}\,|Y(t)|^2 = 0.$$

As proved in [18, 19, 25], the solution $Y(t)$ to (2.1) is mean-square stable if and only if

$$\mathrm{Re}(\lambda) + \frac{1}{2}|\mu|^2 < 0 \tag{2.2}$$

and the set

$$\mathcal{S}_{SDE} = \left\{\lambda, \mu \in \mathbb{C} : \mathrm{Re}(\lambda) + \frac{1}{2}|\mu|^2 < 0\right\}$$

is the stability region of the problem (2.1). Let us provide the numerical counterpart of above arguments, described by the following result.

Theorem 2.1. *Let Y_n be the numerical solution computed by a SRK method* (1.2) *to the linear scalar SDE* (2.1)*. Then,*

$$\mathbb{E}|Y_{n+1}|^2 = R_s(\eta, \zeta)\mathbb{E}|Y_n|^2, \tag{2.3}$$

where

$$R_s(\eta, \zeta) = (1 + |\zeta|^2)|R_d(\eta)|^2, \tag{2.4}$$

being $\eta = \lambda\Delta t$, $\zeta = \mu\sqrt{\Delta t}$, $R_d(\eta) = 1 + \eta b^{\intercal} (I - \eta A)^{-1} e$, I identity matrix in $\mathbb{R}^{s\times s}$ and e unit vector in $\mathbb{R}^s$.

Proof: Applying (1.2) to problem (2.1) yields

$$\begin{cases} \widehat{Y} = Y_n e + \eta A \widehat{Y} + \zeta V_n Y_n e, \\ Y_{n+1} = Y_n + \eta b^{\intercal} \widehat{Y} + \zeta \, V_n Y_n, \end{cases} \tag{2.5}$$

where $\widehat{Y} = [\widehat{Y}_1 \ \ \widehat{Y}_2 \ \ \cdots \ \ \widehat{Y}_s]^{\intercal}$ and V_n is a standard Gaussian random variable. Then,

$$Y_{n+1} = (1 + \zeta V_n) R_d(\eta) Y_n,$$

and passing to the mean-square side by side leads to the thesis.

□

Definition 2.2. *The function $R_s(\eta, \zeta)$ in* (2.4) *is denoted as the stability function of the SRK method* (1.2).

Theorem 2.1 highlights a connection between the SRK method (1.2) and its underlying RK method (1.4) in terms of linear stability properties. Indeed, the stability function of (1.2) is related to that of (1.4) according to (2.4). In particular, the following stability definitions will be useful for the core result of this paper.

Definition 2.3. *A SRK method* (1.2) *is said to be mean-square stable for a fixed couple $(\eta, \zeta) \in \mathbb{C}^2$, if*

$$R_s(\eta, \zeta) < 1. \tag{2.6}$$

Moreover, the set

$$\mathcal{S}_{SRK} = \{\eta, \zeta \in \mathbb{C} : R_s(\eta, \zeta) < 1\}$$

is defined as the mean-square stability region of the stochastic Runge-Kutta method.

Definition 2.4. *A SRK method* (1.2) *is said to be mean-square A-stable if*

$$\mathcal{S}_{SRK} \supseteq \mathcal{S}_{SDE}.$$

3. Inheriting the A-stability of the underlying RK method

We now provide the link between mean-square A-stability of a SRK method (1.2) and the A-stability of its underlying RK method (1.4).

Theorem 3.1. *For a given A-stable deterministic Runge-Kutta method* (1.4)*, the corresponding stochastic perturbation* (1.2) *is mean-square A-stable if and only if*

$$|R_d(\eta)|^2 \leq \frac{1}{1 - 2\mathrm{Re}(\eta)}, \tag{3.1}$$

for any $\eta \in \mathbb{C}^-$.

Proof: For a stochastic Runge-Kutta method (1.2) whose underlying RK method (1.4) is A-stable, (2.6) is equivalent to

$$|\zeta|^2 < \frac{1}{|R_d(\eta)|^2} - 1, \tag{3.2}$$

for any $\eta \in \mathbb{C}^-$. The mean-square A-stability of (1.2) would require that $R_s(\eta, \zeta) < 1$, for any $\eta, \zeta \in \mathcal{S}_{SDE}$. By (2.2), $\eta, \zeta \in \mathcal{S}_{SDE}$ if and only if

$$|\zeta|^2 \leq -2\mathrm{Re}(\eta). \tag{3.3}$$

Equations (3.2) and (3.3) provides that the mean-square A-stability of (1.2) holds true if and only (3.1) is satisfied.

□

Let us apply this result to relevant cases of A-stable methods, i.e., Gauss and Radau Runge-Kutta methods.

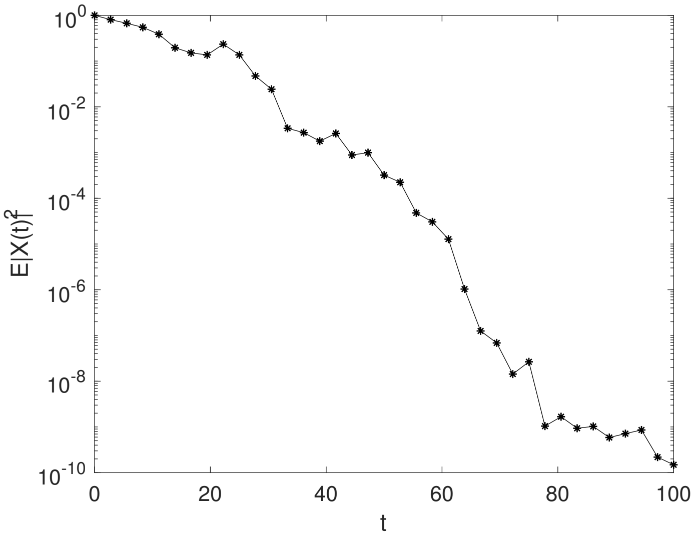

Figure 1: Mean-square over 1000 paths of the numerical solution to (2.1) with $\lambda = -2$, $\mu = 1$ and $X_0 = 1$, computed by the SRK method (1.2) having the midpoint rule as underlying RK method. The employed stepsize is $\Delta t = 2.73$.

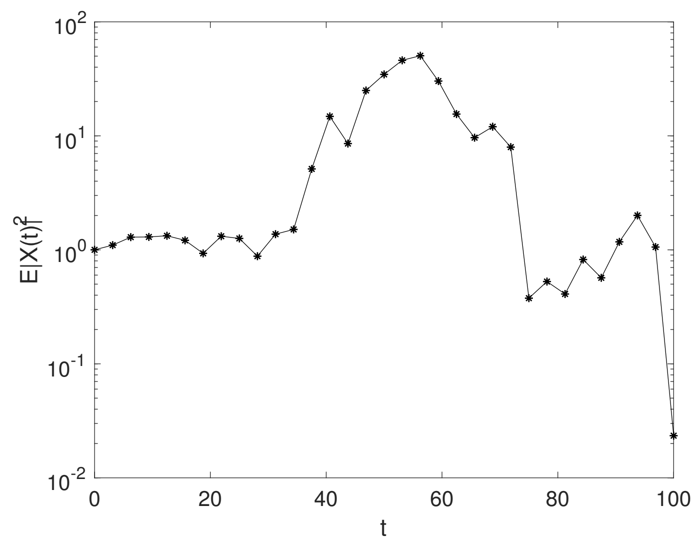

Figure 2: Mean-square over 1000 paths of the numerical solution to (2.1) with $\lambda = -2$, $\mu = 1$ and $X_0 = 1$, computed by the SRK method (1.2) having the midpoint rule as underlying RK method. The employed stepsize is $\Delta t = 3.12$.

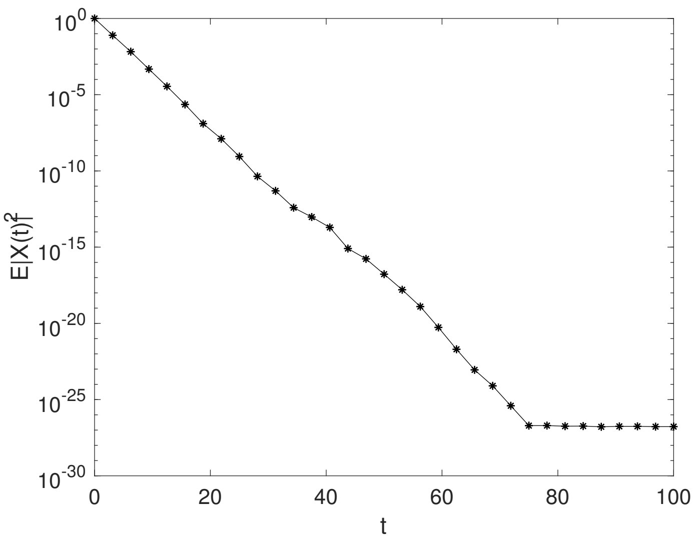

Figure 3: Mean-square over 1000 paths of the numerical solution to (2.1) with $\lambda = -2$, $\mu = 1$ and $X_0 = 1$, computed by the SRK method (1.2) having the Radau IA method (equal to the Radau IIA method, since the problem is autonomous) as underlying RK method. The employed stepsize is $\Delta t = 3.12$.

Example 3.1 (One-stage Gaussian methods). *The SRK method* (1.2) *having the one-stage Gaussian method (i.e., the midpoint rule) as underlying RK method, does not inherit the A-stability property. Indeed, we recall that*

$$R_d(\eta) = \frac{1 + \frac{1}{2}\eta}{1 - \frac{1}{2}\eta},$$

then (3.1) *holds true only for* $Re(\eta) \geq -\frac{1}{4}|\eta|^2$. *To give a numerical evidence of this issue, let us consider the problem* (2.1) *with* $\lambda = -2$, $\mu = 1$ *and* $X_0 = 1$, *in the time interval* $[0, 100]$*; then,* (2.6) *holds true if and only if* $\Delta t < 3$. *The confirmation of this bound is visible from Figures 1 and 2.*

Example 3.2 (One-stage Radau IA and IIA methods). *Both the SRK methods* (1.2) *having the one-stage Radau IA and IIA methods as underlying RK ones, are mean-square A-stable. Indeed, both deterministic methods share the same stability function*

$$R_d(\eta) = \frac{1}{1 - \eta}.$$

Then, condition (3.1) *is equivalent to*

$$\frac{1}{|1 - \eta|^2} \leq \frac{1}{1 - 2Re(\eta)},$$

that holds true for any $\eta \in \mathbb{C}$, *as required by Theorem 3.1. A numerical confirmation on the same problem as in Example 3.1 can be inferred by Figure 3.*

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