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# Elastic Scattering in Stochastic Mechanics.

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*Summary.* – Shucker has shown how to define momentum in stochastic mechanics for a free particle. The purpose of this note is to generalize his result to the case in which a nonvanishing potential is present.

SHUCKER has defined [1] the stochastic momentum as the limit for $t \to \pm\infty$ of $\boldsymbol{\alpha}(t)/t$ (where $\boldsymbol{\alpha}(t)$ is the stochastic position variable). For any free particle such a limit has a finite definite value and its probability coincides with the one given by quantum mechanics for the momentum.

The advantage of this definition consists in providing an operational meaning to the corresponding physical quantity.

In scattering problems the experimentally measured quantities are the initial and final momenta. Therefore, the extension of Shucker's result to this case is obtained by suitably defining the corresponding stochastic variables.

It will be shown that also in this case the limit for $t \to \pm\infty$ of $\boldsymbol{\alpha}(t)/t$ is finite. Furthermore, it turns out that the probabi ity density of $\lim\limits_{t\to-\infty} \boldsymbol{\alpha}(t)/t$ is equal to the quantum probability density for the initial momentum, and $\lim\limits_{t\to+\infty} \boldsymbol{\alpha}(t)/t$ has the same probability density of the final momentum. We will start with a statement of a general nature.

In stochastic mechanics the function $p(\boldsymbol{x}, t; \boldsymbol{x}', t')$ represents the probability density for a particle to be found at point $\boldsymbol{x}$ at time $t$ if at time $t' = t - \Delta t$ its position was in $\boldsymbol{x}'$.

For small $\Delta t$ one has

$$p(\boldsymbol{x}, t; \boldsymbol{x}', t') = (4\pi\nu\,\Delta t)^{-\frac{3}{2}} \exp\left\{\frac{-(\boldsymbol{x} - \boldsymbol{x}' - \boldsymbol{b}^{+}(\boldsymbol{x}', t')\,\Delta t)^2}{4\nu\,\Delta t}\right\}. \tag{1}$$

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(1) D. S. SHUCKER: *J. Funct. Anal.*, **38**, 146 (1980).

By defining $\boldsymbol{y} = \boldsymbol{x}/t$ one obtains from (1) the corresponding transition probability for the $\boldsymbol{y}$:

$$p'(\boldsymbol{y}, t; \boldsymbol{y}', t') = (4\pi\nu\,\Delta t)^{-\frac{3}{2}} t^3 \exp\left\{\frac{-(\boldsymbol{y}-\boldsymbol{y}' + [\boldsymbol{y}' - \boldsymbol{b}^+(\boldsymbol{y}'t', t')]\,\Delta t/t)^2 t^2}{4\nu\,\Delta t}\right\}. \tag{2}$$

From this expression it is clear that, provided

$$\lim_{t\to\pm\infty} \boldsymbol{b}^+(\boldsymbol{y}t, t)/t = 0 \tag{3}$$

the transition probability vanishes for $\boldsymbol{y} \neq \boldsymbol{y}'$ when $t \to \pm\infty$. It will be therefore sufficient to show that (3) holds in order to conclude that $\boldsymbol{\alpha}(t)/t$ reaches definite (different) limits at $t \to \pm\infty$.

We will consider in this note the elastic scattering of a particle interacting with a spherically symmetric potential vanishing sufficiently rapidly. In fact we will assume that the potential vanishes at distances larger than an arbitrary but finite length $a$.

Let us define, as usually in quantum mechanics, $\varrho(\boldsymbol{x}, t) = |\psi(\boldsymbol{x}, t)|^2$ as the position probability density, and $\tilde{\varrho}(\boldsymbol{p}, t) = |\tilde{\psi}(\boldsymbol{p}, t)|^2$ as the momentum probability density, where $\tilde{\psi}(\boldsymbol{p}, t)$ is the Fourier transform of the wave function $\psi(\boldsymbol{x}, t)$.

We want to show that the two following equalities hold:

$$\lim_{t\to\pm\infty} t^3 \varrho(\boldsymbol{p}t, t) = \lim_{t\to\pm\infty} \tilde{\varrho}(\boldsymbol{p}, t)\,, \tag{4a}$$

$$\lim_{t\to\pm\infty} \boldsymbol{b}^+(\boldsymbol{p}t, t)/t = 0\,. \tag{4b}$$

The wave function describing the scattered particle is a superposition of eigenfunctions belonging to the continuum energy spectral range, which can be written in the form $R_{kl}(r)Y_{lm}(\theta, \varphi)$; $Y_{lm}(\theta, \varphi)$ are the angular-momentum eigenfunctions, $R_{kl}(r)$ are defined (for $r > a$) by

$$R_{kl}(r) = (-1)^l \frac{r^l}{k^l}\left(\frac{\mathrm{d}}{r\,\mathrm{d}r}\right)^l \left(\frac{A(k, l)\exp[ikr] + A^*(k, l)\exp[-ikr]}{r}\right) \tag{5}$$

with $|A(k, l)| = 1$.

The corresponding free-particle eigenfunctions are

$$R^0_{kl}(r) = (-1)^l\, 2\, \frac{r^l}{k^l}\left(\frac{\mathrm{d}}{r\,\mathrm{d}r}\right)^l \frac{\sin kr}{r}\,. \tag{6}$$

The wave function $\psi(\boldsymbol{x}, t)$ at time $t$ can be written:

$$\psi(r, \theta, \varphi, t) = \frac{1}{2\pi}\int_0^{+\infty}\int_0^{\pi}\int_0^{2\pi} \psi(r', \theta', \varphi', 0)\left(\sum_{l=0}^{+\infty}\sum_{m=-l}^{+l} Y_{lm}(\theta, \varphi) Y^*_{lm}(\theta', \varphi')\cdot\right.$$
$$\left.\cdot\int_0^{+\infty} R_{kl}(r) R_{kl}(r') \exp\left[-i\frac{k^2}{2}t\right]\mathrm{d}k\right)\sin\theta'\, r'^2\,\mathrm{d}\varphi'\,\mathrm{d}\theta'\,\mathrm{d}r'\,. \tag{7}$$

Its Fourier transform is

$$(8)\qquad \tilde{\psi}(p,\alpha,\beta,t)=\frac{1}{(2\pi)^{\frac{3}{2}}}\int\limits_0^{+\infty}\int\limits_0^{\pi}\int\limits_0^{2\pi}\psi(r',\theta',\varphi',0)\left(\sum_{l=0}^{+\infty}\sum_{m=-l}^{+l}Y_{lm}(\alpha,\beta)Y^*_{lm}(\theta',\varphi')\cdot\right.$$
$$\left.\cdot\int\limits_0^{+\infty}R_{kl}(r')(-i)^l\int\limits_0^{+\infty}R_{kl}(r)R^0_{pl}(r)\frac{r^2}{p}\,\mathrm{d}r\exp\left[-i\frac{k^2}{2}t\right]\mathrm{d}k\right)\sin\theta' r'^2\,\mathrm{d}\varphi'\,\mathrm{d}\theta'\,\mathrm{d}r'\,.$$

Let us perform the substitution $\boldsymbol{x}\to\boldsymbol{p}t$. This means the replacement (for $t>0$) $r\to pt$, $\varphi\to\beta$, $\theta\to\alpha$ in (7).

We obtain

$$(9)\qquad \psi(pt,\alpha,\beta,t)=\frac{1}{2\pi}\int\limits_0^{+\infty}\int\limits_0^{\pi}\int\limits_0^{2\pi}\psi(r',\theta',\varphi',0)\left(\sum_{l=0}^{+\infty}\sum_{m=-l}^{+l}Y_{lm}(\alpha,\beta)Y^*_{lm}(\theta',\varphi')\cdot\right.$$
$$\left.\cdot\int\limits_0^{+\infty}R_{kl}(pt)R_{kl}(r')\exp\left[-i\frac{k^2}{2}t\right]\mathrm{d}k\right)\sin\theta' r'^2\,\mathrm{d}\varphi'\,\mathrm{d}\theta'\,\mathrm{d}r'\,.$$

Let us assume that the wave function (7) contains only states with energy in the range between $k^2_{\min}/2$ and $k^2_{\max}/2$ . In this case we can replace in (9)

$$(10a)\qquad \int\limits_0^{+\infty}R_{kl}(pt)R_{kl}(r')\exp\left[-i\frac{k^2}{2}t\right]\mathrm{d}k$$

with

$$(10b)\qquad \int\limits_{k_{\min}}^{k_{\max}}R_{kl}(pt)R_{kl}(r')\exp\left[-i\frac{k^2}{2}t\right]\mathrm{d}k\,.$$

Now when $t\to+\infty$ we can change $R_{kl}(pt)$ with its asymptotic value:

$$(11)\qquad R'_{kl}(pt)=(-i)^l\frac{A(k,l)}{pt}\exp[ikpt]+(i)^l\frac{A^*(k,l)}{pt}\exp[-ikpt]$$

obtained from (5) in the limit of large $r$.

Therefore, one gets

$$(12)\qquad \lim_{t\to+\infty}\int\limits_{k_{\min}}^{k_{\max}}R_{kl}(pt)R_{kl}(r')\exp\left[-i\frac{k^2}{2}t\right]\mathrm{d}k\,t^{\frac{3}{2}}\exp\left[-i\frac{p^2}{2}t\right]=$$
$$=\lim_{t\to+\infty}\int\limits_{k_{\min}}^{k_{\max}}(-i)^lA(k,l)R_{kl}(r')\exp\left[ikpt-i\frac{k^2}{2}\right]\frac{\mathrm{d}k\,t^{\frac{3}{2}}}{pt}\exp\left[-i\frac{p^2}{2}t\right]+$$
$$+\lim_{t\to+\infty}\int\limits_{k_{\min}}^{k_{\max}}(i)^lA^*(k,l)R_{kl}(r')\exp\left[-ikpt-i\frac{k^2}{2}t\right]\frac{\mathrm{d}k\,t^{\frac{3}{2}}}{pt}\exp\left[-i\frac{p^2}{2}t\right].$$

By replacing $k$ with $\varepsilon/t^{\frac{1}{2}}+p$ in the first term and with $\varepsilon/t^{\frac{1}{2}}-p$ in the second term the limit (12) becomes

$$\frac{1}{p}(-i)^l R_{pl}(r')A(p,l)(-2\pi i)^{\frac{1}{2}} \quad (\text{for } k_{\min}<p<k_{\max}), \tag{13a}$$

$$0 \qquad (\text{for } p<k_{\min} \text{ or } p>k_{\max}). \tag{13b}$$

Consider now the limit

$$\lim_{t\to+\infty}\int_0^{+\infty} R_{kl}(r')\frac{(-i)^l}{(2\pi)^{\frac{1}{2}}}\int_0^{+\infty} R_{kl}(r)R^0_{pl}(r)\frac{r^2}{p}\,\mathrm{d}r\exp\left[-i\frac{k^2}{2}t\right]\mathrm{d}k\exp\left[i\frac{p^2}{2}t\right]. \tag{14}$$

It can be shown that, when integration on $p$ is limited to the range $k_{\min}\leqslant p\leqslant k_{\max}$, one can replace $R_{kl}(r)$, $R^0_{pl}(r)$ with their asymptotic values $R'_{kl}(r)$, $R^{0\prime}_{pl}(r)$, respecitvely, ($R^{0\prime}_{pl}(r)$ is obtained from (6) in the limit of large $r$).

With a suitable change of variable we obtain (for $k_{\min}<p<k_{\max}$):

$$\lim_{t\to+\infty}\int_{k_{\min}}^{k_{\max}} R_{kl}(r')\frac{(-i)^l}{(2\pi)^{\frac{1}{2}}}\int_0^{+\infty} R'_{kl}(r)R^{0\prime}_{pl}(r)\frac{r^2}{p}\,\mathrm{d}r\exp\left[-i\frac{k^2}{2}t\right]\mathrm{d}k\exp\left[i\frac{p^2}{2}t\right]= \tag{15}$$

$$=\frac{1}{p}(-i)^l R_{pl}(r')A(p,l)(-2\pi i)^{\frac{1}{2}}(i)^{\frac{3}{2}}.$$

When $p<k_{\min}$ or $p<k_{\max}$ the limit (15) vanishes.

In order to prove (4$a$), we now must compare the limits

$$\lim_{t\to+\infty} t^{\frac{3}{2}}\exp\left[-i\frac{p^2}{2}t\right](i)^{\frac{3}{2}}\psi(pt,\alpha,\beta,t), \tag{16a}$$

$$\lim_{t\to+\infty}\exp\left[i\frac{p^2}{2}t\right]\tilde{\psi}(p,\alpha,\beta,t). \tag{16b}$$

If we suppose that the wave function $\psi(r,\theta,\varphi,t)$ does not contain values of the angular momentum $l$ larger than $l_{\max}$, it is easy to see that the sums appearing in (8) and (9) are limited to $0\leqslant l\leqslant l_{\max}$. In this case the two limits (16$a$) and (16$b$) are equal. This means that (4$a$) holds.

In order to prove (4$b$) we start from the definition

$$\boldsymbol{b}^+(\boldsymbol{p}t,t)=\operatorname{Re}\left\{\frac{1}{t}\frac{\partial}{\partial\boldsymbol{p}}\psi(\boldsymbol{p}t,t)/\psi(\boldsymbol{p}t,t)\right\}+\operatorname{Im}\left\{\frac{1}{t}\frac{\partial}{\partial\boldsymbol{p}}\psi(\boldsymbol{p}t,t)/(\psi\boldsymbol{p}t,t)\right\}, \tag{17}$$

where $(\partial/\partial\boldsymbol{p})\psi(\boldsymbol{p}t,t)$ can be written in the form

$$\frac{\partial}{\partial\boldsymbol{p}}\psi(\boldsymbol{p}t,t)=\frac{1}{2\pi}\int_0^{+\infty}\int_0^{\pi}\int_0^{2\pi}\psi(r',\theta',\varphi',0)\left(\sum_{l=0}^{+\infty}\sum_{m=-l}^{+l}Y^*_{lm}(\theta',\varphi')\cdot\right. \tag{18}$$

$$\left.\cdot\frac{\partial}{\partial\boldsymbol{p}}Y_{lm}(\alpha,\beta)\int_{k_{\min}}^{k_{\max}}R_{kl}(pt)R_{kl}(r')\exp\left[-i\frac{k^2}{2}t\right]\mathrm{d}k\right)\sin\theta'\,r'^2\,\mathrm{d}\varphi'\,\mathrm{d}\theta'\,\mathrm{d}r'.$$

By expressing $\partial/\partial \boldsymbol{p}$ in polar co-ordinates, and taking into account that

$$\lim_{t\to+\infty}\frac{1}{t}\frac{\partial}{\partial p}\int\limits_{k_{\min}}^{k_{\max}} R_{kl}(pt)R_{kl}(r')\exp\left[-i\frac{k^2}{2}t\right]\mathrm{d}k\,t^{\frac{3}{2}}\exp\left[-i\frac{p^2}{2}t\right]= \tag{19}$$

$$= iR_{pl}(r')(-i)^l A(p, l)(-2\pi i)^{\frac{1}{2}}\,,$$

we obtain

$$\lim_{t\to+\infty}\frac{1}{t}\frac{\partial}{\partial \boldsymbol{p}}\,Y_{lm}(\alpha,\beta)\int\limits_{k_{\min}}^{k_{\max}} R_{kl}(pt)R_{kl}(r')\exp\left[-i\frac{k^2}{2}t\right]\mathrm{d}k\,t^{\frac{3}{2}}\exp\left[-i\frac{p^2}{2}t\right]= \tag{20}$$

$$= ip(\sin\alpha\cos\beta,\,\sin\alpha\sin\beta,\,\cos\alpha)Y_{lm}(\alpha,\beta)\,iR_{pl}(r')(-i)^l\frac{A(p,l)}{p}(-2\pi i)^{\frac{1}{2}}$$

and, therefore,

$$\lim_{t\to+\infty} b^+(\boldsymbol{p}t, t) = \boldsymbol{p}\,. \tag{21}$$

This implies the validity of (4*b*).

The same result can be obtained when $t\to-\infty$.

This complete the proof of our initial statements.

Its validity is limited by the assumptions made, namely: 1) the potential vanishes for $r > a$; 2) the wave function $\psi(r, \theta, \varphi, t)$ contains only states with $l \leqslant l_{\max}$; 3) the momentum range is between $k_{\min}$ and $k_{\max}$ From a physical point of view they are all reasonable and experimentally justified; from a mathematical point of view their replacement with weaker conditions seems feasible, but would require a more sophisticated formalism.

The extension of this method to the treatment of scattering of particles by targets with internal degrees of freedom is possible and it is presently under investigation.

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