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STOCHASTIC INTERPRETATION OF EMISSION AND ABSORPTION OF THE QUANTUM OF ACTION

M.Cini and M.Serva

Dipartimento di Fisica
Università "La Sapienza"
Roma

INTRODUCTION

The possibility of reformulating quantum theory in the form of a theory of stochastic processes has been explored in recent times with some success[1]. The most elaborated attempt up to now is the theory known as Stochastic Mechanics developed by Nelson[2]. This theory describes the behaviour of a non relativistic particle in configuration space under the influence of a random disturbance of unspecified origin. Its motion is therefore the result of the joint action of the classical and the stochastic forces, leading to a continuous but non differentiable chaotic trajectory, typical of a Markov diffusion process. Quite recently a reformulation of Stochastic Mechanics has been proposed which lends itself to interesting generalizations[3,4].

With this method the spin can be treated as a discrete random variable and a probabilistic version of the non relativistic Pauli equation is obtained[5]. Likewise the Dirac equation in two dimensions can be formulated as a stochastic process in which the velocity assumes at random the values $\pm c$.[6]

In this talk I wish to illustrate the results obtained[7] by extending the approach discussed above to give a stochastic description of a harmonic oscillator in interaction with a source, in view of a possible generalization to the formulation of a stochastic theory of fields.

Before going into the details of our work, however, I will briefly sketch a simplified version of the treatment given in[3] for a spin 1/2 in a constant magnetic field $\vec{H}$ (for simplicity H_y=0). The Pauli equation reads in this case:

$$i \partial_t \chi(\sigma,t) = \frac{1}{2} \left[\sigma H_z \chi(\sigma,t) + H_x \chi(-\sigma,t) \right] \qquad \sigma = \pm 1 \qquad (1)$$

where $\chi(\sigma,t)$ is the σ component of a one column Pauli spinor. From (1) the continuity equation follows:

$$\partial_t |\chi(\sigma,t)|^2 = \frac{1}{2} H_x \, \mathrm{Im}[\chi^+(\sigma,t) \; \chi(-\sigma,t)] \tag{2}$$

Eq.(2) can be interpreted as a Kolmogorov equation for the probability density $\rho(\sigma,t) = |\chi(\sigma,t)|^2$ of a discrete Markov process $\tilde{\sigma}(t)$ taking values ± 1. Such an equation, of the form

$$\partial_t \rho(\sigma,t) = - p(\sigma,t) \; \rho(\sigma,t) + p(-\sigma,t) \; \rho(-\sigma,t) \tag{3}$$

reduces in fact to (2) provided the jump probability per unit time $p(\sigma,t)$ is given by

$$p(\sigma,t) = \frac{H_x}{2} \; \frac{\rho^{\frac{1}{2}}(-\sigma,t)}{\rho^{\frac{1}{2}}(\sigma,t)} \; [1 + \sin(S(\sigma) - S(-\sigma))] \tag{4}$$

with the phase $S(\sigma,t)$ defined through

$$\chi(\sigma,t) = \rho^{\frac{1}{2}}(\sigma,t) \exp[\, i \, S(\sigma,t)] \tag{5}$$

It is also useful to define the jump probability $p^*(\sigma,t)$ for the time reversed process ($t' = -t$; $\sigma' = \sigma$; $\rho' = \rho$; $S'(\sigma',t') = - S(\sigma,t)$) which satisfies

$$p^*(\sigma,t) \; \rho(\sigma,t) = p(-\sigma,t) \; \rho(-\sigma,t) \tag{6}$$

With these notations the Schrodinger equation (1) takes the simple form:

$$- \partial_t S(\sigma,t) = \frac{1}{2} H_z \sigma + \sqrt{p^*(\sigma,t) \, p(\sigma,t)} \tag{7}$$

Eqs.(3)(4)(7) provide therefore the required stochastic description of the quantum behaviour of a spin 1/2 in a constant magnetic field.

THE DISPLACED HARMONIC OSCILLATOR

We start with the Hamiltonian

$$H = \frac{1}{2} (p^2 + \omega^2 q^2) + vq = \omega a^+ a + \frac{v}{\sqrt{2\omega}} (a + a^+) \tag{8}$$

In the representation of the eigenstates of the free Hamiltonian $|n\rangle$

$$\omega a^+ a \, |n\rangle = n\omega \, |n\rangle$$

the Schrödinger equation reads:

$$i\partial_t \langle n|\psi(t)\rangle = n\omega \langle n|\psi(t)\rangle + \frac{v}{\sqrt{2\omega}} \sqrt{n+1} \, \langle n+1|\psi(t)\rangle + \frac{v}{\sqrt{2\omega}} \sqrt{n} \, \langle n-1|\psi(t)\rangle \tag{9}$$

We derive from (9) as before, the continuity equation:

$$\partial_t |\langle n|\psi(t)\rangle|^2 = \frac{2v}{\sqrt{2\omega}} \sqrt{n+1}\ \mathrm{Im}[\langle\psi(t)|n\rangle\langle n+1|\psi(t)\rangle] + \frac{2v}{\sqrt{2\omega}} \sqrt{n}\ \mathrm{Im}[\langle\psi(t)|n\rangle\langle n-1|\psi(t)\rangle] \tag{10}$$

The transformation of eq.(10) into a Kolmogorov equation for the probability density $\rho(n,t) = |\langle n|\psi(t)\rangle|^2$ of a discrete Markov process $n(t)$ taking all the integer values $0 < n < \infty$ is again straightforward:

$$\partial_t \rho(n,t) = -(p_+(n,t) + p_-(n,t))\rho(n,t) + p_+(n-1,t)\rho(n-1,t) + p_-(n+1,t)\rho(n+1,t) \tag{11}$$

where now

$$p_\pm(n,t) = \frac{V}{\sqrt{2\omega}} \sqrt{n + \frac{1}{2} \pm \frac{1}{2}}\ \frac{\rho^{\frac{1}{2}}(n\pm1,t)}{\rho^{\frac{1}{2}}(n,t)} \times \{1 - \sin[S(n\pm1,t) - S(n,t)]\} \tag{12}$$

with the phase $S(n,t)$ given by the relation analogous to (5):

$$\langle n|\psi(t)\rangle = \rho^{\frac{1}{2}}(n,t) \exp[i S(n,t)] \tag{13}$$

Again we define $p^*(n,t)$ as the probabilities per unit time for the time reversed process, which satisfy the relations

$$\begin{aligned} p_-(n+1,t)\rho(n+1,t) &= p_+^*(n,t)\rho(n,t) \\ p_+(n-1,t)\rho(n-1,t) &= p_-^*(n,t)\rho(n,t) \end{aligned} \tag{14}$$

We can now define the forward and backward time derivatives of any function $F(\bar{n}(t),t)$ of the stochastic process $n(t)$ as

$$(D_+F)(n,t) = \lim_{\Delta t\to 0} \Delta t^{-1} \mathbf{E}[F(\bar{n}(t+\Delta t),t+\Delta t) - F(\bar{n}(t),t) | \bar{n}(t) = n] \tag{15}$$

$$(D_-F)(n,t) = \lim_{\Delta t\to 0} \Delta t^{-1} \mathbf{E}[F(\bar{n}(t),t) - F(\bar{n}(t-\Delta t), t-\Delta t) | \bar{n}(t) = n] \tag{16}$$

By making use of the transition probability from n' at time t' to n at time t, $P(n,t;n',t')$ which satisfies forward and backward Kolmogorov equations derived from (11) one finds

$$(D_+n)(n,t) = p_+(n,t) - p_-(n,t) \tag{17}$$

$$(D_- n)(n,t) = p_-^*(n,t) - p_+^*(n,t) \quad (18)$$

The dynamical equation, equivalent to Schrödinger's equation, which specifies the process in addition to eqs.(11) and (12) can now be written in the form:

$$-\partial_t S(n,t) = n\omega + \sqrt{p_+(n,t)\, p_+^*(n,t)} + \sqrt{p_-(n,t)\, p_-^*(n,t)} \quad (19)$$

THE CLASSICAL LIMIT

In order to understand better the meaning of eqs.(11) (12)(17)(18)(19), it is useful to perform the limit $n \to \infty$, which is nothing else than the application of Bohr's correspondence principle. Since

$$\lim_{n\to\infty} S(n+1) - S(n) = \lim_{n\to\infty} S(n) - S(n-1) = \frac{\partial S(n)}{\partial n} \quad (20)$$

eq.(19) becomes

$$-\partial_t S(n,t) = n\omega + v \frac{\sqrt{2n}}{\omega} \cos\frac{\partial S}{\partial n} \quad (21)$$

This is just the Hamilton-Jacobi equation derived from a classical Hamiltonian of the form (7) in terms of the action variable n and the angle variable θ defined bu the canonical transformation

$$q = \sqrt{\frac{2n}{\omega}} \cos\theta \quad (22)$$

$$p = -\sqrt{2\omega n} \sin\theta \quad (23)$$

In fact the generating function $S_o(n,q,t)$ satisfies

$$\theta = \frac{\partial S_o}{\partial n} \quad (24)$$

The corresponding HJ equation is therefore

$$-\partial_t S_o = H = n\omega + v\sqrt{\frac{2n}{\omega}} \cos\frac{\partial S_o}{\partial n} \quad (25)$$

which coincides with (21).

Furthermore one has

$$\dot{n} = -\frac{\partial H}{\partial\theta} = v \frac{\sqrt{2n}}{\omega} \sin\theta \quad (26)$$

which is the limit of the current velocity $V(n,t)$ defined as

$$V(n,t) = \frac{1}{2}(D_+ n + D_- n)(n,t) \quad (27)$$

as one sees immediately by using (17) and (18):

$$\lim_{n\to\infty} V(n,t) = v \frac{\sqrt{2n}}{\omega} \sin \frac{\partial S}{\partial n} \qquad (28)$$

DISCUSSION

Eqs.(11)(12)(19) provide a stochastic description of the quantum mechanical behaviour of an oscillator whose state vector obeys the Schrodinger equation (9). To each quantum mechanical state $\psi(t)$ corresponds a discrete stochastic process $\bar{n}(t)$ with jump probabilities per unit time $p_{\pm}(n,t)$ and probability density $\rho(n,t)$. Under the effect of the interaction with the source the oscillator, which in its absence would be in a state with a fixed value of n, makes transitions between different values of n by successively emitting or absorbing one quantum of action at a time.

One might ask whether the dynamical equation (19) (or its equivalent one (7) for the spin case) could not be derived from first principles independently of the Schrödinger equation. This does not seem to be easy to do. A plausibility argument, however, can be found by considering that, under time reversal, both the classical eq.(25) and the corresponding stochastic equation must be invariant. It turns out then that the simplest invariant combination of $p_{\pm}(n,t)$ and $p^*_{\pm}(n,t)$ with the correct classical limit is just the expression

$$\sqrt{p^*_+(n,t)\; p_+(n,t)} + \sqrt{p^*_-(n,t)\; p_-(n,t)}$$

appearing in eq.(19).

Clearly the same argument holds for eq.(7). It seems therefore that, at least for the very simple cases discussed here, the Schrödinger equation is indeed the simplest way of describing the stochastic process of emission and absorption of one quantum of action at a time between a source and a system in periodic motion interacting with it.

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