J. Phys. A: Math. Gen. 19 (1986) 1163-1177. Printed in Great Britain

Stochastic theory of emission and absorption of quanta

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Received 1 July 1985

Abstract. The attempt to reformulate quantum theory in the form of a theory of stochastic processes, which has recently met with considerable success, has failed up to now to describe the most typical quantum effect: the emission and absoption of field quanta. The purpose of the present paper is to explore the possibility of constructing a stochastic field theory based on the assumption that for any normal mode of a given classical field, the corresponding action variable becomes a stochastic variable assuming only positive integer values in units of h.

The proposed method reproduces the known properties of simple quantum field theoretical models without recourse to probability amplitudes, which are quantities extraneous to classical probability theory. The probability transition rates, for simple states of the field, turn out to be those of simple Poisson processes.

1. Introduction

The possibility of reformulating quantum theory in the form of a theory of stochastic processes has been explored in recent times with some success (Guerra 1981). The most elaborate attempt is the theory known as stochastic mechanics developed by Nelson (1966). This theory describes the behaviour of a non-relativistic system in configuration space under the influence of a random disturbance of unspecified origin. Its motion is therefore the result of the joint action of the classical and the stochastic forces, leading to a continuous but non-differentiable chaotic trajectory, typical of a Markov diffusion process. From this point of view the wave-like behaviour of a single particle predicted by quantum mechanics is explained by showing that the time evolution of its position's probability density is the same as the one derived from the corresponding Schrödinger equation.

In spite of recent substantial progress on various aspects of the theory (Dohrn *et al* 1979, Jona-Lasinio *et al* 1981, Schucker 1980, Guerra and Morato 1983, De Falco *et al* 1982, Serva 1984), however, conceptual and computational difficulties arise, as Nelson himself has shown (Nelson 1983) already for a two-particle system. These difficulties are in fact of the same kind as those faced by Schrödinger in the early days of wave mechanics, when he tried, unsuccessfully, to give a physical meaning in configuration space to the wavefunction of an N-particle system, by connecting it with the actual motion of the particles in the physical three-dimensional space.

For the same reason the formalism of stochastic mechanics, which stresses so heavily the role of the particle trajectories, cannot readily be extended to the treatment of the quantum properties of fields.

In fact, attempts in this direction (Guerra and Loffredo 1980), realised by replacing the classical oscillators of the field's normal modes by Nelson's oscillators, are unable to explain the field's corpuscular properties, because in this theory oscillators do not have a discrete energy spectrum. It is therefore no longer possible to interpret, as one does in quantum field theory, the nth excited state of a given oscillator as a state of the field with n quanta of the corresponding mode. In this framework it seems therefore impossible to describe the most typical (and indeed the first one discovered historically) quantum effect: the emission and the absorption of light quanta. This obstacle shows, in our opinion, that the programme of reformulating quantum theory in terms of stochastic processes should take as its point of departure the established fact that the electromagnetic field, whose wave-like behaviour is guaranteed by classical physics, also shows, under suitable conditions, particle-like properties. In other words we believe that a stochastic theory should give priority to the task of reproducing the particle-like properties of fields predicted by quantum theory rather than aiming at explaining the wave-like behaviour of particles to begin with.

In a certain sense the question is not new. Historically the two fathers of quantum electrodynamics, P A M Dirac (Dirac 1927) and P Jordan (Jordan 1927) held widely different opinions about the nature of particles. Dirac believed that particles and fields are two essentially different entities, as shown by their respective classical limits. Correspondingly for him the quantisation procedure, which transforms into q numbers the c-number classical variables of any dynamical system maintaining their mutual relations (Poisson brackets and equations of motion), provides the (classical) particles with wave-like properties and the (classical) fields with corpuscular properties. Jordan, on the contrary, was convinced that the existence of particles is always a consequence of quantisation, both in the case of bosons (photons) and of fermions (electrons and protons, in his time). For him, in the classical limit only fields exist and therefore Schrödinger's equation (as well as its relativistic Dirac generalisation) has conceptually the same physical meaning as Maxwell's equations.

Of course, since the two ways of looking at physical reality led ultimately to the same theory (even if the Feynman picture can be seen as a direct offspring of the Dirac viewpoint and the Tomonaga–Schwinger formulation as a development of Jordan's) the choice between them can be considered a metaphysical question devoid of physical meaning. This is no longer true, however, when one is looking for a new theory. It may well be, in fact, that the monistic 'metaphysical core' of Jordan's programme is more fruitful as a guiding line for the formulation of a stochastic theory than the dualistic approach of Dirac.

Quite recently a formulation of stochastic mechanics has been advanced in which has been added to the diffusion process of Nelson a discrete stochastic process (De Angelis and Jona-Lasinio 1982). This work treats the spin component as a discrete random variable and gives a probabilistic version of the non-relativistic Pauli equation. Successively (Guerra and Marra 1984) a stochastic version of quantum mechanics in terms of Markov random processes taking values on the eigenvalues of a complete set of orthonormal functions in the system's Hilbert space has been proposed.

The present paper has been inspired by the first one. Only later have we realised that our approach fits the scheme proposed in the second one.

One purpose is to explore the possibility of constructing a stochastic field theory based on the assumption that for any normal mode of a given classical field with amplitude $q_k(t)$, the corresponding action variable J_k becomes a stochastic variable assuming only positive integer values in units of h. This of course means that the energy E_k of any normal mode is also a stochastic variable assuming only positive integer values in units $\hbar\omega_k$. In other words, we assume as a starting point of our theory what is usually considered to be a consequence of quantisation, namely of defining the amplitude q_k and its conjugate momentum p_k to be operators satisfying the commutator rule

$$[q_k, p_{k'}] = \mathrm{i}\hbar\delta_{kk'}. \tag{1.1}$$

This does not imply, however, that our theory has a weaker explanatory power than quantum theory. It simply means that the mathematical entities introduced in the two theories to start with are different: the assumption (1.1) is in fact equivalent to the assumption that $n_k = J_k/h$ can take only integer values.

The dynamical evolution of the field is described in our theory by means of the infinite set of discrete stochastic processes $\tilde{n}_k(t)$. In principle the state of the field at time t will therefore be defined by the probability distribution $\rho(n_1, n_2, \ldots, n_k, \ldots; t)$ that $\tilde{n}_1(t) = n_1$, $\tilde{n}_2(t) = n_2, \ldots, \tilde{n}_k(t) = n_K$. To obtain the time evolution of ρ the jump probabilities per unit time will be needed. We will show that these quantities, for simple states, are indeed very simple and have straightforward physical interpretations. Of course in the case of free fields without sources the number of quanta in each mode, if initially given, remains constant.

The purpose of the present paper is to discuss two simple field models and the equations that govern their stochastic time evolution. It will be found that the method proposed reproduces the known properties of the corresponding models in quantum field theory, without recourse to probability amplitudes, which are quantities extraneous to classical probability theory. The formalism developed here seems sufficiently general to allow generalisations to more realistic field theories. Attempts in this direction are in progress.

2. The fixed source model: stochastic equations

Assume a classical field $u(x, t)$ in one dimension with Lagrangian

$$L = \tfrac{1}{2}\int_0^l \left[\left(\frac{\partial u}{\partial t}\right)^2 - \left(\frac{\partial u}{\partial x}\right)^2\right] \mathrm{d}x + \int_0^l v(x)u(x, t)\,\mathrm{d}x. \tag{2.1}$$

The usual expansion in standing waves

$$u(x, t) = \sum_{k=1}^{\infty} q_k(t) \sin \omega_k x, \qquad \omega_k = \pi k/l, \tag{2.2}$$

together with the introduction of canonical variables

$$p_k(t) = \dot{q}_k(t) \tag{2.3}$$

yields the Hamiltonian

$$H = \tfrac{1}{2}\sum_{k=1}^{\infty} (p_k^2 + \omega_k^2 q_k^2) + \sum_{k=1}^{\infty} v_k q_k, \qquad v_k = \int_0^l v(x) \sin \omega_k x\,\mathrm{d}x. \tag{2.4}$$

In this simple case therefore the field decouples into a sum of independent harmonic oscillators. It will therefore be sufficient to treat each of them separately. We will drop the subscript k and deal, from now onwards, with the Hamiltonian

$$H = \tfrac{1}{2}(p^2 + \omega^2 q^2) + vq \tag{2.5}$$

which reduces to

$$H = H_0 + H_i, \tag{2.6}$$

$$H_0 = \omega a^+ a, \tag{2.7}$$

$$H_i = (v/\sqrt{2\omega})(a + a^+), \tag{2.8}$$

when the Dirac oscillator variables

$$a = \frac{1}{\sqrt{2\omega}}(\omega q + \mathrm{i}p), \qquad a^+ = \frac{1}{\sqrt{2\omega}}(\omega q - \mathrm{i}p), \tag{2.9}$$

are introduced.

The standard quantisation procedure transforms a, a^+ into operators satisfying

$$[a, a^+] = 1 \tag{2.10}$$

and expresses any state vector in terms of eigenstates of the free Hamiltonian

$$H_0|n\rangle = n\omega|n\rangle \tag{2.11}$$

with

$$|n\rangle = (1/\sqrt{n!})(a^+)^n|0\rangle, \qquad a|0\rangle = 0. \tag{2.12}$$

The Schrödinger equation

$$\mathrm{i}(\mathrm{d}/\mathrm{d}t)|\psi(t)\rangle = H|\psi(t)\rangle \tag{2.13}$$

gives therefore

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\langle n|\psi(t)\rangle = n\omega\langle n|\psi(t)\rangle + \frac{v}{\sqrt{2\omega}}(n+1)^{1/2}\langle n+1|\psi(t)\rangle + \frac{v}{\sqrt{2\omega}}\sqrt{n}\langle n-1|\psi(t)\rangle. \tag{2.14}$$

The Hamiltonian (2.6) can be diagonalised by means of the transformation

$$a = b - \lambda, \qquad \lambda = v/\omega\sqrt{2\omega}. \tag{2.15}$$

Equation (2.6) then becomes

$$H = \omega b^+ b - \lambda^2 \omega. \tag{2.16}$$

The lowest eigenstate $|0)$ of H therefore has eigenvalue $-\lambda^2\omega$. It can be expressed in terms of the eigenstates of H_0 as

$$|0) = \exp(-\tfrac{1}{2}\lambda^2) \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{\sqrt{n!}}|n\rangle = \exp(-\tfrac{1}{2}\lambda^2 - \lambda a^+)|0\rangle. \tag{2.17}$$

The ground state $|0)$ is therefore a coherent oscillator state of strength λ. It should be kept in mind that the complete set of the Hamiltonian eigenstates

$$H|n) = (n\omega - \lambda^2\omega)|n), \tag{2.18}$$

$$|n) = (1/\sqrt{n}!)(b^+)^n|0), \tag{2.19}$$

can be generated from the ground state (2.17) by means of the recurrence relation

$$(\mathrm{d}/\mathrm{d}\lambda)|n) = \sqrt{n}|n-1) - (n+1)^{1/2}|n+1). \tag{2.20}$$

Let us now use the Schrödinger equation (2.14) to derive the time derivative of the density

$$\rho(n, t) = |\langle n|\psi(t)\rangle|^2. \tag{2.21}$$

We have

$$\begin{aligned}\frac{\mathrm{d}\rho(n,t)}{\mathrm{d}t} &= \frac{v}{\sqrt{2\omega}}(n+1)^{1/2}\,\mathrm{Im}\left(\frac{\langle\psi(t)|n\rangle}{\langle\psi(t)|n+1\rangle}\right)\rho(n+1,t)\\ &\quad+\frac{v}{\sqrt{2\omega}}\sqrt{n}\,\mathrm{Im}\left(\frac{\langle\psi(t)|n\rangle}{\langle\psi(t)|n-1\rangle}\right)\rho(n-1,t)\\ &\quad-\left[\frac{v}{\sqrt{2\omega}}(n+1)^{1/2}\,\mathrm{Im}\left(\frac{\langle\psi(t)|n+1\rangle}{\langle\psi(t)|n\rangle}\right)\right.\\ &\quad\left.+\frac{v}{\sqrt{2\omega}}\sqrt{n}\,\mathrm{Im}\left(\frac{\langle\psi(t)|n-1\rangle}{\langle\psi(t)|n\rangle}\right)\right]\rho(n,t).\end{aligned} \tag{2.22}$$

Equation (2.22) is the point of departure for the stochastic interpretation of our field model. In fact one may try to interpret equation (2.22), following the method suggested by De Angelis and Jona-Lasinio (1982), as the forward Kolmogorov equation for a discontinuous Markov process in the state space of positive integers:

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(n,t) = p_-(n+1,t)\rho(n+1,t)+p_+(n-1,t)\rho(n-1,t)-[p_+(n,t)+p_-(n,t)]\rho(n,t) \tag{2.23}$$

where $p_\pm(u, t)$ are the jump probabilities per unit time from n to $n\pm1$ respectively. Because of the positivity condition on these probabilities the identification of the coefficients of the densities in (2.22) and (2.23) is not immediate. By setting

$$p_\pm(n,t) = |\Delta_\pm(n,t)| + \mathrm{Im}\,\Delta_\pm(n,t) \tag{2.24}$$

with

$$\Delta_+(n,t) = \lambda\omega(n+1)^{1/2}\langle\psi(t)|n+1\rangle/\langle\psi(t)|n\rangle, \tag{2.25}$$

$$\Delta_-(n,t) = \lambda\omega\sqrt{n}\,\langle\psi(t)|n-1\rangle/\langle\psi(t)|n\rangle, \tag{2.26}$$

one verifies that, since

$$\Delta_+(n,t)\rho(n,t) = \Delta_-^*(n+1,t)\rho(n+1,t), \tag{2.27}$$

equations (2.22) and (2.23) are identical and the condition of positivity is satisfied.

The quantities $\Delta_\pm(n, t)$ are not independent. In fact they are connected by the relation

$$\Delta_+(n,t)\Delta_-(n+1,t) = \lambda^2\omega^2(n+1). \tag{2.28}$$

In order to define completely the stochastic process one needs, in addition to the Kolmogorov equation (2.23), another equation for the jump probabilities (2.24). By analogy with Nelson's assumption, we may define the phase $S(n, t)$ as

$$\langle n|\psi|(t)\rangle = \rho^{1/2}(n,t)\exp[\mathrm{i}S(n,t)]. \tag{2.29}$$

Then one easily verifies that (2.24)-(2.26) can be written in the form

$$\Delta_+(n,t)=(n+1)^{1/2}\left(\frac{\rho(n+1,t)}{\rho(n,t)}\right)^{1/2}\exp\{\mathrm{i}[S(n,t)-S(n+1,t)]\},\qquad(2.30)$$

$$\Delta_-(n,t)=\sqrt{n}\left(\frac{\rho(n-1,t)}{\rho(n,t)}\right)^{1/2}\exp\{\mathrm{i}[S(n,t)-S(n-1,t)]\}.\qquad(2.31)$$

In order to reproduce the Schrödinger equation (2.14), $S(n,t)$ must satisfy the equation

$$-\frac{\mathrm{d}}{\mathrm{d}t}S(n,t)=n\omega+\mathrm{Re}[\Delta_+(n,t)+\Delta_-(n,t)].\qquad(2.32)$$

An equation of this form has been derived (Guerra and Marra 1984) by means of a variational principle on a suitable stochastic action. Within the framework of that theory, therefore, the stochastic process $\bar{n}(t)$ can be defined, independently from quantum theory, by equations (2.32), (2.23), (2.24), (2.30) and (2.31).

Instead of introducing the phases $S(n,t)$ we can derive the equations for the time evolution of $\Delta_\pm(n,t)$. One finds

$$-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\Delta_\pm(n,t)=\lambda^2\omega^2(n+\tfrac{1}{2}\pm\tfrac{1}{2})+\Delta_\pm(n,t)[\pm\omega+\Delta_\pm(n\pm1,t)-\Delta_\pm(n,t)-\Delta_\mp(n,t)].\qquad(2.33)$$

Of course the relation (2.28) connects Δ_+ and Δ_-, so that only one of the two equations (2.33) is really independent.

One may introduce explicitly the real and imaginary parts of $\Delta_\pm$ and write equations for them. We shall see, however, that it is better to work with $\Delta_\pm$ directly.

3. The ground state process

The most interesting physical properties of the model can be derived from the stochastic process describing the ground state. It is important therefore to discuss it in detail. When

$$|\psi(t)\rangle=|0\rangle\qquad(3.1)$$

one has

$$\mathrm{d}\rho(n,t)/\mathrm{d}t=0\qquad(3.2)$$

and $\Delta_\pm$ are real and time independent. From (2.17) it is easy to obtain

$$p_-(n)=n\omega,\qquad(3.3)$$

$$p_+(n)=\lambda^2\omega.\qquad(3.4)$$

Equations (3.3) and (3.4) are simple and very interesting. They show that in the ground state the emission of 'quanta' is a Poisson process with constant probability rate while the absorption is a decay process with probability rate proportional to the number of 'quanta'. It seems therefore quite reasonable at this stage to forget quantum theory and assume equations (3.3) and (3.4) to be a physically meaningful postulational basis for the ground state process. In fact one easily verifies that (3.3) and (3.4) are the simplest solutions of equations (2.34).

Equations (3.3) and (3.4) are sufficient to derive the transition probability $P(n, t; n_0, t_0)$ for finding in the field n 'quanta' at time t if their number was n_0 at time t_0. This conditional probability, which has the Kolmogorov property

$$P(n, t; n_0 t_0) = \sum_{n'} P(n, t; n', t')P(n', t'; n_0, t_0), \qquad t > t' > t_0, \tag{3.5}$$

satisfies the Kolmogorov equation

$$\begin{aligned}\frac{1}{\omega}\frac{\mathrm{d}}{\mathrm{d}t} P(n, t; n_0, t_0) &= -(n+\lambda^2)P(n, t; n_0 t_0) + \lambda^2 P(n-1, t; n_0 t_0)\\ &\quad + (n+1)P(n+1, t; n_0 t_0)\end{aligned} \tag{3.6}$$

which can be rewritten

$$\frac{1}{\omega}\frac{\mathrm{d}}{\mathrm{d}t} P(n, t; n_0 t_0) = \sum_{n'} (K)_{n'n} P(n', t; n_0 t_0) \tag{3.7}$$

in terms of the matrix

$$(K)_{n'n} = -(\lambda^2 + n)\delta_{nn'} + \lambda^2 \delta_{n',n-1} + n'\delta_{n',n+1}. \tag{3.8}$$

It satisfies also the initial condition

$$\lim_{t \to t_0} P(n, t; n_0, t_0) = \delta_{nn_0}. \tag{3.9}$$

Following the standard methods of probability theory we define a Kolmogorov generator L which is the transposed matrix of K:

$$(L)_{n'n} = (K^{\mathrm{T}})_{n'n} = (K)_{nn'} = -(\lambda^2 + n)\delta_{nn'} + \lambda^2 \delta_{n',n+1} + n\delta_{n',n-1}. \tag{3.10}$$

Then one has

$$P(n, t; n_0 t_0) = \{\exp[\omega(t - t_0)L]\}_{nn_0}. \tag{3.11}$$

That (3.11) satisfies (3.5) and (3.9) is evident. It is also easy to see that it satisfies (3.7) because

$$\begin{aligned}\frac{1}{\omega}\frac{\mathrm{d}}{\mathrm{d}t} P(n, t; n_0 t_0) &= \{L \exp[\omega(t - t_0)L\}_{nn_0}\\ &= \sum_{n'} L_{nn'}\{\exp[\omega(t - t_0)L]\}_{n'n_0} = \sum_{n'} K_{n'n} P(n', t; n_0 t_0).\end{aligned} \tag{3.12}$$

The expression (3.11) is however purely formal because L is the sum of three non-commuting matrices

$$(L_0)_{n'n} = -(\lambda^2 + n)\delta_{nn'}, \tag{3.13}$$

$$(L_+)_{n'n} = \lambda^2 \delta_{n',n+1}, \tag{3.14}$$

$$(L_-)_{n'n} = n\delta_{n',n-1}. \tag{3.15}$$

The exponential can however be 'disentangled' following the time ordering procedure invented many years ago by Feynman (1951) because the three matrices (3.13), (3.14) and (3.15) have the simple commutation properties:

$$[L_-, L_+] = \lambda^2 \mathbb{1}, \tag{3.16}$$

$$[L_-, L_0] = -L_-. \tag{3.17}$$

$$[L_+, L_0] = L_+. \tag{3.18}$$

One defines

$$\exp[\omega(t-t_0)L]=\exp\left(\omega\int_{t_0}^{t}L(s)\,\mathrm{d}s\right) \tag{3.19}$$

where the convention is that

$$L_i(s)L_j(s')=\begin{cases}L_iL_j, & s>s',\\ L_jL_i, & s<s',\end{cases}\qquad i,j=0,\pm. \tag{3.20}$$

The procedure is as follows. Define

$$S(t)=\exp(\omega L_0 t). \tag{3.21}$$

Then, because of (3.17) and (3.18) one has

$$S^{-1}(t)L_+S(t)=\mathrm{e}^{\omega t}L_+, \tag{3.22}$$

$$S^{-1}(t)L_-S(t)=\mathrm{e}^{-\omega t}L_-. \tag{3.23}$$

Then

$$\exp[\omega(t-t_0)L]=S(t)\exp\left(\omega\int_{t_0}^{t}[\mathrm{e}^{\omega s}L_+(s)+\mathrm{e}^{-\omega s}L_-(s)]\,\mathrm{d}s\right)S^{-1}(t_0). \tag{3.24}$$

Furthermore, defining

$$A(t)=\omega\int_{t_0}^{t}\mathrm{e}^{\omega s}\,\mathrm{d}s=\exp[\omega(t-t_0)]-1 \tag{3.25}$$

one has

$$\exp(-A(t)L_+)L_-\exp(A(t)L_+)=\lambda^2A(t)+L_-. \tag{3.26}$$

Then one obtains from (3.24)

$$\exp[\omega(t-t_0)L]=S(t)\exp(A(t)L_+)\exp(\bar{A}(t)L_-)S^{-1}(t_0)G(t,t_0) \tag{3.27}$$

where

$$\bar{A}(t)=\omega\int_{t_0}^{t}\mathrm{e}^{-\omega s}\,\mathrm{d}s=1-\exp[-\omega(t-t_0)], \tag{3.28}$$

$$G(t,t_0)=\exp\left(\lambda^2\omega\int_{t_0}^{t}\mathrm{e}^{-\omega s}A(s)\,\mathrm{d}s\right). \tag{3.29}$$

Now, since

$$\{\exp[A(t)L_+]\}_{nm}=(\lambda^2A(t))^{n-m}/(n-m)!,\qquad n\geqslant m, \tag{3.30}$$

$$\{\exp[\bar{A}(t)L_-]\}_{nm}=\binom{m}{n}(\bar{A}(t))^{n-m},\qquad n\leqslant m, \tag{3.31}$$

one finds immediately (we take $t_0=0$)

$$P(n,t;n_0,0)=\exp[-(n+\lambda^2)t]G(t)\sum_{k\leqslant n,n_0}\binom{n}{k}\binom{n_0}{k}\frac{k!}{n!}(A\lambda^2)^{n-k}(\bar{A})^{n_0-k} \tag{3.32}$$

which can be rewritten as

$$P(n,t;n_0,0)=[(\mathrm{e}^{\omega t}-1)^{-1}\;\mathrm{d}/\mathrm{d}\lambda^2+1]^{n_0}P(n,t;0,0) \tag{3.33}$$

where

$$P(n,t;0,0)=\mathrm{e}^{-n\omega t}\exp[\lambda^2(\mathrm{e}^{-\omega t}-1)](\mathrm{e}^{\omega t}-1)^n(\lambda^2)^n/n!. \tag{3.34}$$

4. Time dependent source: emission and absorption probability rates

It is interesting to point out that the results obtained thus far can be easily extended to the case of a time dependent source. This is important because, as is well known, it is precisely the solution of this model which is the point of departure for the Feynman reformulation of quantum electrodynamics.

By replacing equations (2.6) and (2.7) with

$$H = \omega a^+ a + (1/\sqrt{2\omega})(v(t)a + v^*(t)a^+) \tag{4.1}$$

one arrives at the same Kolmogorov equation (2.23) with

$$\Delta_+(n, t) = \frac{v^*(t)}{\sqrt{2\omega}}(n+1)^{1/2}\frac{\langle\psi(t)|n+1\rangle}{\langle\psi(t)|n\rangle}, \tag{4.2}$$

$$\Delta_-(n, t) = \frac{v(t)}{\sqrt{2\omega}}\sqrt{n}\frac{\langle\psi(t)|n-1\rangle}{\langle\psi(t)|n\rangle}. \tag{4.3}$$

Equation (2.27) still holds and in equation (2.28), λ^2 is replaced by $|v(t)|^2/2\omega^3$. Equations (2.34) now become

$$-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\Delta_+(n, t) = \frac{-\mathrm{i}}{v^*(t)}\frac{\mathrm{d}v^*(t)}{\mathrm{d}t}\Delta_+(n, t) + \frac{|v|^2}{2\omega}(n+1)$$
$$+\Delta_+(n, t)[\omega + \Delta_+(n+1, t) - \Delta_+(n, t) - \Delta_-(n, t)], \tag{4.4}$$

$$-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\Delta_-(n, t) = \frac{-\mathrm{i}}{v(t)}\frac{\mathrm{d}v(t)}{\mathrm{d}t}\Delta_-(n, t) + \frac{|v|^2}{2\omega}n$$
$$+\Delta_-(n, t)[-\omega + \Delta_-(n-1, t) - \Delta_-(n, t) - \Delta_+(n, t)]. \tag{4.5}$$

It is interesting to show how one obtains from the lowest-order approximation in $|v|^2$ from (4.4) and (4.5) and the Kolmogorov equation (2.23) the usual expressions for the probability rates of emission and absorption when the field is initially in a state with n_0 quanta:

$$\rho(n, 0) = \delta_{nn_0}. \tag{4.6}$$

Assume for the source a single frequency time dependence:

$$v(t) = v_0 \exp(\mathrm{i}\omega_0 t). \tag{4.7}$$

Then equation (4.4) becomes, neglecting terms of order v_0^4,

$$-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\Delta_+(n_0, t) = \frac{v_0^2}{2\omega}(n_0+1) + (\omega - \omega_0)\Delta_+(n_0, t). \tag{4.8}$$

Its solution is

$$\Delta_+(n_0, t) = \frac{v_0^2}{2\omega}(n_0+1)\frac{\exp[\mathrm{i}(\omega - v_0)t] - 1}{\omega - \omega_0} \tag{4.9}$$

which gives for the jump probability $p_+(n_0, t)$

$$p_+(n_0, t) = \frac{v_0^2}{2\omega}\frac{n_0+1}{\omega - \omega_0}\left(2\sin\frac{\omega - \omega_0}{2}t + \sin(\omega - \omega_0)t\right). \tag{4.10}$$

From (2.23) one obtains, inserting (4.6) in place of $\rho(n, t)$,

$$(\mathrm{d}/\mathrm{d}t)\rho(n_0+1, t) = p_+(n_0, t). \tag{4.11}$$

This expression gives a finite contribution, as usual, only when $\omega_0 \simeq \omega$. In fact, as first shown by Dirac in his historic paper of 1927, one integrates the probability rate in a small range around this value. Then, for t sufficiently large, one has

$$\int_{\omega-\varepsilon}^{\omega+\varepsilon} \mathrm{d}\omega_0\, p_+(n_0, t) = \frac{v_0^2}{\omega}(n_0+1) \int_{-\infty}^{+\infty} \frac{\sin x}{x}\,\mathrm{d}x = \pi\frac{v_0^2}{\omega}(n_0+1). \tag{4.12}$$

Similarly from equation (4.5) one finds

$$\Delta_-(n_0, t) = \frac{v_0^2}{2\omega} n_0 \frac{1-\exp[-\mathrm{i}(\omega-\omega_0)t]}{\omega-\omega_0} \tag{4.13}$$

which gives

$$\int_{\omega-\varepsilon}^{\omega+\varepsilon} \mathrm{d}\omega_0\, p_-(n_0, t) = \pi\frac{v_0^2}{\omega} n_0. \tag{4.14}$$

Equations (4.13) and (4.14) are the standard results for the Einstein probability rates of photon emission and absorption from an electric dipole source in the presence of radiation.

5. Time dependent source: transition probability from the vacuum state

In a simple case equations (4.4) and (4.5) can be solved exactly. In fact, by using (2.28) they can be rewritten as

$$-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\Delta_+(n, t) = \frac{-\mathrm{i}}{v^*}\frac{\mathrm{d}v^*}{\mathrm{d}t}\Delta_+(n, t) + \frac{|v|^2}{2\omega}(n+1)$$
$$+\Delta_+(n, t)\left(\omega + \Delta_+(n+1, t) - \Delta_+(n, t) - \frac{|v|^2}{2\omega}\frac{n}{\Delta_+(n-1, t)}\right), \tag{5.1}$$

$$-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\Delta_-(n, t) = \frac{-\mathrm{i}}{v}\frac{\mathrm{d}v}{\mathrm{d}t}\Delta_-(n, t) + n\frac{|v|^2}{2\omega}$$
$$+\Delta_-(n, t)\left(-\omega + \Delta_-(n-1, t) - \Delta_-(n, t) - \frac{|v|^2}{2\omega}\frac{n+1}{\Delta_-(n+1, t)}\right). \tag{5.2}$$

One sees immediately that these equations are satisfied, as was the case for the ground state process for a time independent source, by a Δ_+ independent of n and a Δ_- proportional to n. In fact with this assumption equations (5.1) and (5.2) reduce to

$$-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\Delta_+(t) = \frac{-\mathrm{i}}{v^*}\frac{\mathrm{d}v^*}{\mathrm{d}t}\Delta_+(t) + \frac{|v|^2}{2\omega} + \omega\Delta_+(t), \tag{5.3}$$

$$-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\Delta_-(n, t) = \frac{-\mathrm{i}}{v}\frac{\mathrm{d}v}{\mathrm{d}t}\Delta_-(n, t) - \omega\Delta_-(n, t) + \Delta_-(n, t)[\Delta_-(n-1, t) - \Delta_-(n, t)]. \tag{5.4}$$

The first one is easily solved because it is linear. Calling

$$\beta(t) = \frac{1}{\sqrt{2\omega}}\int_0^t v(s)\,\mathrm{e}^{-\mathrm{i}\omega s}\,\mathrm{d}s \tag{5.5}$$

one checks immediately that

$$\Delta_+ = \frac{\mathrm{i}v^*(t)}{\sqrt{2\omega}} \mathrm{e}^{\mathrm{i}\omega t}\beta(t) = \mathrm{i}\beta(t)\frac{\mathrm{d}}{\mathrm{d}t}\beta^*(t) \tag{5.6}$$

is the solution of (5.3). Furthermore, because of (2.28) one has

$$\Delta_-(n, t) = n(|v|^2/2\omega)(\Delta_+(t))^{-1}. \tag{5.7}$$

Equation (5.7) provides indeed the solution of (5.4). By inserting the former into the latter one verifies that (5.4) is satisfied as a consequence of (5.3). The solutions give the jump probabilities for the stochastic process corresponding to the initial state

$$\rho(n, 0) = \delta_{n0}. \tag{5.8}$$

In fact equation (2.27) entails, for $t \to 0$, taking into account (5.7),

$$(|v(0)|^2/2\omega)\rho(0, 0)t = t^{-1}\rho(1, 0) \tag{5.9}$$

which clearly implies

$$\rho(1, 0) = 0 \tag{5.10}$$

and, by induction, equation (5.8).

This result is important because it shows that for the time dependent source the vacuum has the same physical property as the ground state with constant source, namely that emission is a Poisson process independent of n and absorption is a decay process, proportional to n.

It is therefore possible, in this case, to evaluate the explicit expression for the transition probability, following the method developed in § 3. We define

$$p_+(n, t) = \zeta_+(t), \tag{5.11}$$

$$p_-(n, t) = n\zeta_-(t). \tag{5.12}$$

We have again

$$P(n, t; 0, 0) = \exp\left(\int_0^t L(s)\, \mathrm{d}s\right)_{n,0} \tag{5.13}$$

where now the three matrices $L_0(t)$, $L_\pm(t)$ depend on t not only as an ordering parameter but also through the explicit time dependence of the functions $\zeta_\pm(t)$. Equations (3.13), (3.14) and (3.15) are now replaced by

$$(L_0(t))_{n'n} = -[\zeta_+(t) + n\zeta_-(t)]\delta_{nn'}, \tag{5.14}$$

$$(L_+(t))_{n'n} = \zeta_+(t)\delta_{n',n+1}, \tag{5.15}$$

$$(L_-(t))_{n'n} = n\zeta_-(t)\delta_{n',n-1}. \tag{5.16}$$

The commutators (3.16), (3.17) and (3.18) are replaced by

$$[L_-(t), L_+(t)] = \zeta_+(t)\zeta_-(t)\mathbb{1}, \tag{5.17}$$

$$[L_-(t), L_0(t)] = -\zeta_-(t)L_-(t), \tag{5.18}$$

$$[L_+(t), L_0(t)] = \zeta_-(t)L_+(t). \tag{5.19}$$

The disentangling procedure is carried out by using the same method as in § 3. Defining

$$\chi_\pm(t) = \int_0^t \zeta_\pm(s)\,\mathrm{d}s, \tag{5.20}$$

$$\varphi_\pm(t) = \int_0^t \zeta_\pm(s) \exp[\pm\chi_-(s)]\,\mathrm{d}s, \tag{5.21}$$

$$\psi(t) = \int_0^t \zeta_-(s)\varphi_+(s) \exp[-\chi_-(s)]\,\mathrm{d}s, \tag{5.22}$$

one obtains

$$P(n, t; 0, 0) = \exp[-\chi_+(t) + \psi(t) - n\chi_-(t)](\varphi_+(t))^n/n!. \tag{5.23}$$

Explicit evaluation with the simple form of time dependence (4.7) gives

$$\zeta_+(t) = \frac{v_0^2}{2\omega}\frac{4}{\omega-\omega_0}\sin\frac{\omega-\omega_0}{2}t\cos^2\frac{\omega-\omega_0}{4}t, \tag{5.24}$$

$$\zeta_-(t) = \frac{\omega-\omega_0}{2}\sin\frac{\omega-\omega_0}{4}t\left(\cos\frac{\omega-\omega_0}{4}t\right)^{-1}. \tag{5.25}$$

For the simple case $n=1$ one easily recovers the result (4.12) in the lowest order of v_0^2 by taking

$$P(1, t; 0, 0) \simeq \int_0^t \zeta_+(s)\exp\left(-\int_s^t \zeta_-(s')\,\mathrm{d}s'\right)\mathrm{d}s = \frac{2v_0^2}{\omega}\frac{\sin^2[\frac{1}{2}(\omega-\omega_0)t]}{(\omega-\omega_0)^2} \tag{5.26}$$

which gives

$$\int_{\omega-\varepsilon}^{\omega+\varepsilon} \frac{\mathrm{d}}{\mathrm{d}t} P(1, t; 0, 0) = \pi\frac{v_0^2}{\omega}. \tag{5.27}$$

Equation (5.23) is the main result of this section. The transition *probability* from the vacuum to the state with n quanta corresponds to the expression given by Feynman for the transition *amplitude* of the same process in the form

$$G_{n0}(t) = \frac{1}{(n!)^{1/2}}(\mathrm{i}\beta(t))^n G_{00}(t) \tag{5.28}$$

with

$$G_{00}(t) = \exp\left(-\frac{1}{4\omega}\int_0^t \mathrm{d}s\int_0^t \mathrm{d}s'\, v(s)v^*(s')\exp[-\mathrm{i}\omega|s-s'|]\right). \tag{5.29}$$

A consistency check with quantum mechanics is given by the relation

$$P(0, t; 0, 0) = \exp\left[-\int_0^t \zeta_+(s)\exp\left(-\int_s^t \zeta_-(s')\,\mathrm{d}s'\right)\right] = \exp[-|\beta(t)|^2] = |G_{00}(t)|^2 \tag{5.30}$$

which can be proved by taking into account that, since

$$\zeta_+(s) - |\beta(s)|^2\zeta_-(s) = (\mathrm{d}/\mathrm{d}s)|\beta(s)|^2, \tag{5.31}$$

one has

$$\int_0^t \zeta_+(s)\exp\left(-\int_s^t \zeta_-(s')\,\mathrm{d}s'\right) = \int_0^t \frac{\mathrm{d}}{\mathrm{d}s}\left[|\beta(s)|^2 \exp\left(-\int_s^t \zeta_-(s')\,\mathrm{d}s'\right)\right]\mathrm{d}s = |\beta(t)|^2. \tag{5.32}$$

It is worthwhile mentioning that, since equation (5.1) becomes linear when $\Delta_+(n, t)$ is independent of n, one can add to the solution (5.6) a term

$$\Delta'_+(t) = (\mathrm{i}v^*(t)/\sqrt{2\omega})\,\mathrm{e}^{\mathrm{i}\omega t}\lambda \tag{5.33}$$

with λ time independent, which is a solution of the homogeneous equation. It is interesting that, when the term (5.33) is present, the initial state is no longer the vacuum, because equation (5.9) is replaced by

$$\rho(n, 0) = [(n+1)/\lambda^2]\rho(n+1, 0). \tag{5.34}$$

This is the same relation that holds for the ground state, or more generally for any coherent state of strength λ when the source is time independent.

6. An iterative method for the determination of the transition probability

We will develop in this section a method which allows the determination of the solution of the Kolmogorov equation (2.23) for the transition amplitude by means of successive approximations. The function $P(n, t; n', t')$ is defined as the probability that the field changes from n' to n in the time interval $t-t'$. This may happen through any number of jumps greater than or equal to $n-n'$. Let us therefore define the function $P_k(n, t; n', t')$ as the transition probability from n' to n with k jumps. Obviously it will be

$$P(n, t; n', t') = \sum_{k=0}^{\infty} P_k(n, t; n't'). \tag{6.1}$$

When $k < n-n'$, $P_k(nt; n't')$ vanishes. Then $P_0(nt, n't')$ is different from zero only when $n = n'$. This quantity is therefore the probability that the field does not make transitions in the interval $t-t'$. Therefore we will have

$$P_0(n, t; n', t') = \exp\left(-\int_{t'}^t [p_+(n, s) + p_-(n, s)]\,\mathrm{d}s\right). \tag{6.2}$$

The meaning of the following equality will then by now be clear:

$$\begin{aligned} P_k(n, t; n't') = &\int_{t'}^t P_{k-1}(n+1, s; n't')p_-(n+1, s)P_0(n, t; n, s)\,\mathrm{d}s \\ &+ \int_{t'}^t P_{k-1}(n-1, s; n't')p_+(n-1, s)P_0(nt; n, s)\,\mathrm{d}s. \end{aligned} \tag{6.3}$$

We have therefore the possibility of calculating the transition probability by an iteration procedure with increasing numbers of jumps.

The function $P_k(nt; n't')$ satisfies the differential equation

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}P_k(n, t; n't') = &-[p_+(n, t) + p_-(n, t)]P_k(n, t; n't') + p_+(n-1, t)P_{k-1}(n-1, t; n't') \\ &+ p_-(n+1, t)P_{k-1}(n+1, t; n't'). \end{aligned} \tag{6.4}$$

As a simple example we refer to the ground state process with jump probabilities given by (3.3) and (3.4). We will have

$$P_0(n, t; n't') = \exp[-(v^2/2\omega + n\omega)(t-t')] \tag{6.5}$$

which gives, as introduced in (6.3),

$$P_1(n+1, t; n't') = \frac{v^2}{2\omega}\int_{t'}^{t} P_0(n, s; n't')P_0(n+1, t; n+1, s)\,ds$$

$$= \frac{v^2}{2\omega^3}\exp\left[-\left(\frac{v^2}{2\omega}+n\omega\right)(t-t')\right]\{1-\exp[-\omega(t-t')]\}. \tag{6.6}$$

Higher-order terms can be computed similarly.

From equation (6.6) it further follows that

$$\lim_{\Delta t\to 0}\frac{P_1(n+1, t; n, t-\Delta t)}{\Delta t} = \frac{v^2}{2\omega^2}. \tag{6.7}$$

This equality is a consequence of the more general relations

$$\lim_{\Delta t\to 0}\frac{P_1(n+1, t; n, t-\Delta t)}{\Delta t} = p_+(n, t), \tag{6.8}$$

$$\lim_{\Delta t\to 0}\frac{P_1(n-1, t; n, t-\Delta t)}{\Delta t} = p_-(n, t), \tag{6.9}$$

that can be proved starting from (6.3).

7. Conclusions

In spite of the simplicity of the model the results obtained are, we believe, of some interest, both on their own merit, and in view of possible generalisations. We have in fact shown that equations (4.4) and (4.5) have simple solutions whose physical meaning can be directly understood in terms of probability theory by stating that in the coherent states of the field the emission probability of a quantum is independent of n and the absorption probability is proportional to n. This property is clearly shared by the vacuum, and by the ground state with time independent source.

Future investigations will show whether this simple property is also valid for more realistic field theories. The hope that the method developed here may also lead to a simple alternative formulation in these cases is, however, not completely unreasonable.

Acknowledgments

It is a pleasure to acknowledge stimulating discussions with G Jona-Lasinio. One of us (MS) would like to thank S Albeverio, Ph Blanchard and L Streit for the kind invitation to the BiBoS Center at the University of Bielefeld, and the ZiF for the warm hospitality.

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