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Imaginary-Time Path Integrals from Klein-Gordon Equation.

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Abstract. – The ordinary Feynman-Kac-Itô formula gives a path integral representation of the Schrödinger semi-groups. We discuss here an analogous probabilistic expression for the (positive energy) solutions of the imaginary-time Klein-Gordon equation in a static external electromagnetic field. When the external field is not purely magnetic, our result is different from the path integral associated to the semi-group $t \in [0, +\infty) \mapsto \exp[-t(H - mc^2)/\hbar]$ of a relativistic Schrödinger operator $H = \{c^2[-i\hbar\nabla - (e/c)\boldsymbol{A}]^2 + m^2c^4\}^{1/2} + V$.

What is the Hamiltonian of spin-zero elementary particle, with rest mass m and electric charge e, injected in an external (*i.e.* classical) static electromagnetic field $\boldsymbol{E} = -\nabla A^0 = -e^{-1}\nabla V$, $\boldsymbol{B} = \operatorname{rot}\boldsymbol{A}$? In the nonrelativistic approximation, the energy spectrum of the particle is adequately described by the Schrödinger operator $H = (1/2m)[-i\hbar\nabla - (e/c)\boldsymbol{A}]^2 + V$ and the corresponding Schrödinger semi-group $t \in [0, +\infty) \mapsto \exp[-tH/\hbar]$ admits a path integral representation which is the famous Feynman-Kac formula [1-3] for $\boldsymbol{A} = \boldsymbol{0}$ or, more generally, the Feynman-Kac-Itô formula [3] when the magnetic field is not trivial. The Schrödinger theory is not physically sound in the range of nonrelativistic energies with absolute value larger than mc^2, therefore we must turn to relativistic quantum mechanics and ask for a better quantum Hamiltonian H. One possible choice of H can be obtained from the classical relativistic energy $h(\boldsymbol{p},\boldsymbol{x}) = \{c^2[\boldsymbol{p} - (e/c)\boldsymbol{A}(\boldsymbol{x})]^2 + mc^2\}^{1/2} + V(\boldsymbol{x})$ according to the usual rule of quantization $\boldsymbol{p} \to -i\hbar\nabla$. The resulting Hamiltonians are the «relativistic Schrödinger operators» $H \doteq \{c^2[-i\hbar\nabla - (e/c)\boldsymbol{A}]^2 + m^2c^4\}^{1/2} + V$ which are extensively considered in all papers involved with problems of stability for the relativistic matter [4]. The relativistic Schrödinger semi-group $t \in [0, +\infty) \mapsto \exp[-t(H - mc^2)/\hbar]$ admits a path integral representation [5-7] which has useful consequences [6]. In spite of that, we choose to follow an alternative route by looking at the Klein-Gordon theory in an external (static) electromagnetic field. When the electric field is not zero, we get a different path integral representation for the semi-group $t \in [0, +\infty) \mapsto \exp[-t(H_{sp} - mc^2)/\hbar]$ of the single-particle quantum Hamiltonian H_{sp}. Our starting point is the free Klein-Gordon equation in the $(1 + d)$-dimensional space-time ($\mu = mc/\hbar$):

$$\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} - \Delta_d \varphi + \mu^2 \varphi = 0 . \tag{1}$$

The pseudodifferential operator $H_0 = \{-c^2\hbar^2\Delta_d + m^2c^4\}^{1/2}$ is closely linked with eq. (1). Among the solutions $\varphi(t,\boldsymbol{x})$ of the free Klein-Gordon equation, the positive-frequency ones, with initial data $\varphi_0(\boldsymbol{x})$ and $\dot\varphi_0(\boldsymbol{x})$ related by $\dot\varphi_0(\boldsymbol{x}) = -(i/\hbar)(H_0\varphi_0)(\boldsymbol{x})$, play a distinguished role. The linear space of positive-frequency solutions can be equipped with the Hilbert norm $\{N_0(\varphi,\dot\varphi)\}^{1/2}$, where $N_0(\varphi,\dot\varphi)$ is the charge corresponding to the conserved current $J^\mu(t,\boldsymbol{x}) = (i/2\hbar)(\overline{\varphi}\partial^\mu\varphi - \varphi\partial^\mu\overline{\varphi})(t,\boldsymbol{x})$ and the resulting complex Hilbert space carries an irreducible unitary representation of the Poincaré group. Despite the existence of such a Hilbert structure, the quantum meaning of the Klein-Gordon equation is subtle and it is safer to consider eq. (1) as a classical field equation. It acquires a particle interpretation only after the quantization of the field $\varphi(t,\boldsymbol{x})$. In such a quantum field theory there is a single-particle Hilbert space $\mathfrak{D}_1^0$, which, by restricting positive-frequency solutions of eq. (1) to the hyperplane $t = 0$, can be identified with the Sobolev space $H^{1/2}(\boldsymbol{R}^d)$ with the scalar product $\langle\varphi_0,\psi_0\rangle_{\mathfrak{D}_1^0} = (1/c\hbar^2)(H_0^{1/2}\varphi_0, H_0^{1/2}\psi_0)_{L^2(\boldsymbol{R}^d)}$ and there is a single-particle quantum Hamiltonian $H_{\rm sp}^0$ given by H_0 as self-adjoint operator in $H^{1/2}(\boldsymbol{R}^d)$. The one-particle free Klein-Gordon theory coincides with the quantum theory associated to the free relativistic Schrödinger operator H_0. The semi-group $t \mapsto \exp[-t(H_{\rm sp}^0 - mc^2)/\hbar] = \exp[-t(H_0 - mc^2)/\hbar$ admits a well-known probabilistic representation [5, 8, 9]. For pedagogical reasons, we review this result from a perspective which will be useful later when the previous considerations will be extended to the Klein-Gordon theory in an external electromagnetic field. In order to pick up the single-particle Hamiltonian $H_{\rm sp}^0$, it is expedient to enter into the Euclidean region where the (hyperbolic) free Klein-Gordon equation becomes the elliptic one:

$$-\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} - \Delta_d \varphi + \mu^2 \varphi = 0 . \tag{2}$$

People who look for (analytically continued) positive-frequency solutions, must read eq. (2) as a Dirichlet problem in the $(1 + d)$-dimensional half-space $t > 0$. The analogous problem for $t < 0$ is related to the single-antiparticle structure. Since eq. (2) is invariant under time translation, the Dirichlet problem defines a semi-group $t \mapsto P_0^t$ acting on the linear space $\mathfrak{D}_1^0$ of data on the boundary, in the sense that the solution $\varphi(t,\boldsymbol{x}) = (P_0^t\varphi_0)(\boldsymbol{x})$ of eq. (2) with boundary value $\varphi_0(\boldsymbol{x}) \in \mathfrak{D}_1^0$ and suitable properties of regularity to the infinity, is such that $\varphi(t,\cdot) \in \mathfrak{D}_1^0$ for all $t > 0$. The single-particle free Hamiltonian can be defined through $P_0^t = \exp[-tH_{\rm sp}^0/\hbar]$ and, of course, $H_{\rm sp}^0 = H_0$. There are standard probabilistic techniques [10] for solving Dirichlet problems. In our free case, the result looks

$$(\exp[-t(H_{\rm sp}^0 - mc^2)/\hbar]\varphi_0)(\boldsymbol{x}) = \boldsymbol{E}_{(ct,\boldsymbol{x})}(\varphi_0(\boldsymbol{Y}_{T_0})) . \tag{3}$$

Here $s \in [0, +\infty) \mapsto Y_s^\mu = (Y_s^0, \boldsymbol{Y}_s)$ is the $(1 + d)$-dimensional diffusion, starting at the point $(ct,\boldsymbol{x}) \in (0, +\infty) \times \boldsymbol{R}^d$ of the Euclidean space-time, given by the formula $Y_s^\mu = (Y_s^0, \boldsymbol{Y}_s) = (c(t-s) + \sqrt{\hbar/m}\,W_s^0, \boldsymbol{x} + \sqrt{\hbar/m}\,\boldsymbol{W}_s)$, where $s \mapsto W_s^\mu$ stands for a $(1 + d)$-dimensional Wiener process. The random variable T_0, appearing in eq. (6), is the first hitting time of the boundary $t = 0$ of $(0, +\infty) \times \boldsymbol{R}^d$ by $s \mapsto Y_s^\mu$. By changing the 1-dimensional Wiener process $s \mapsto W_s^0$ into the new one $s \mapsto -W_s^0$, the diffusion $s \mapsto Y_s^\mu$ can be replaced by $s \mapsto X_s^\mu = (X_s^0, \boldsymbol{X}_s) = (cs + \sqrt{\hbar/m}\,W_s^0, \boldsymbol{x} + \sqrt{\hbar/m}\,\boldsymbol{W}_s)$, starting at the space-time point $(0,\boldsymbol{x})$ of the boundary, provided that the Markov time T_0 gets substituted by

$$\tau_t = \inf\{s \geq 0:\ X_s^0 > ct\} = \inf\{s \geq 0:\ cs + \sqrt{\hbar/m}\,W_s^0 > ct\} . \tag{4}$$

Therefore, eq. (3) can be equivalently written as

$$(\exp[-t(H^0_{sp} - mc^2)/\hbar]\varphi_0)(\boldsymbol{x}) = \boldsymbol{E}_{(0,\boldsymbol{x})}(\varphi_0(\boldsymbol{X}_{\tau_t})) = \boldsymbol{E}_x(\varphi_0(\xi_t)), \tag{5}$$

where $t \mapsto \xi_t$ is the right continuous jump Markov process in $\boldsymbol{R}^d$ with generator $L_0 = mc^2/\hbar - \{-c^2\Delta_d + m^2c^4/\hbar^2\}^{1/2}$, given by the formula [8] $\xi_t = \boldsymbol{X}_{\tau_t}$. Most of the above discussion remains essentially unchanged when a purely magnetic external field is present. For instance, it is quite obvious that the operator H_0 must be replaced by the new one $H_0(\boldsymbol{A}) = \{c^2[-i\hbar\nabla - (e/c)\boldsymbol{A}]^2 + m^2c^4\}^{1/2}$ in the case of the Klein-Gordon equation in an external static magnetic field $\boldsymbol{B} = \operatorname{rot}\boldsymbol{A}$, with $\operatorname{div}\boldsymbol{A} = 0$:

$$\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} - \left(\nabla - \frac{ie}{c\hbar}\boldsymbol{A}\right)^2\varphi + \mu^2\varphi = 0. \tag{6}$$

The operator $H_0(\boldsymbol{A})$ makes sense, as a positive self-adjoint one in $L^2(\boldsymbol{R}^d)$, for vector potentials $\boldsymbol{A}(\cdot) \in L^2_{loc}(\boldsymbol{R}^d)$ and the single-particle Hilbert space $\mathfrak{D}^0_1$ can be identified with the domain of $H_0(\boldsymbol{A})^{1/2}$ equipped with the scalar product $\langle\varphi_0, \psi_0\rangle_{\mathfrak{D}^0_1} = (1/c\hbar^2)(H_0(\boldsymbol{A})^{1/2}\varphi_0, H_0(\boldsymbol{A})^{1/2}\psi_0)_{L^2(\boldsymbol{R}^d)}$. The generalization of eq. (5) to the purely magnetic case is not difficult and can be found in [7]. When a (static) electric field is also present, life is not so easy, nevertheless the field $\varphi(t, \boldsymbol{x})$ can be still quantized according to the canonical commutation relations and the Klein-Gordon theory provides a honest external field problem [11, 12]. If the external electric field is not too strong, in such a quantum field theory there will be still room for a single-particle Hilbert space $\mathfrak{D}_1$ and a single-particle quantum Hamiltonian H_{sp}. The operator H_{sp} is the Hamiltonian for a spin-zero high-energy particle (a charged pion, for instance) in the external e.m. field. We cannot exhibit this operator explicitly, nevertheless we are able to produce a quite elegant «Feynman-Kac-Itô» formula for the semi-group $t \mapsto \exp[-t(H_{sp} - mc^2)/\hbar]$.

In an external e.m. field A^μ, obeying the Lorentz condition $\partial_\mu A^\mu = 0$ (in particular, $\operatorname{div}\boldsymbol{A} \doteq 0$ in the static case), the Klein-Gordon equation in the Euclidean domain turns into

$$-\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} - \Delta_d\varphi - \frac{2e}{c\hbar}\mathcal{A}^\mu\partial_\mu\varphi + \frac{(m^2c^4 - e^2\mathcal{A}^\mu\mathcal{A}_\mu)}{\hbar^2c^2}\varphi = 0, \tag{7}$$

where $\mathcal{A}_\mu$ is related to the electromagnetic «four» potential A^μ by

$$\mathcal{A}_\mu = \mathcal{A}^\mu = (A^0, i\boldsymbol{A}). \tag{8}$$

If the field A^μ is a static one, the corresponding Dirichlet problem in the half-space $t > 0$ defines, again, a semi-group $t \in [0, +\infty) \mapsto P^t = \exp[-tH_{sp}/\hbar]$ on the linear space $\mathfrak{D}_1$ of the boundary data. The analogous Dirichlet problem in the lower half-space defines the single-antiparticle quantum Hamiltonian H_{sa}. Of course, by time reversal and charge conjugation, $H_{sa}(\mathcal{A}_\mu) = H_{sp}(-\mathcal{A}_\mu)$. One can easily obtain a path integral representation of the semi-group $t \in [0, +\infty) \mapsto P^t = \exp[-tH_{sp}/\hbar]$ by the following steps. The first one is the general probabilistic recipe [10] for solving Dirichlet problems. In this way one gets a probabilistic formula involving a suitable $(1 + d)$-dimensional diffusion and its first hitting time on the boundary of the upper half-space. The second step uses the well-known Girsanov formula [10] in order to transform the expectation involved in a new path integral taken with respect to the «free» stochastic process $s \mapsto Y^\mu_s = (Y^0_s, \boldsymbol{Y}_s) = (c(t-s) + \sqrt{\hbar/m}\,W^0_s, \boldsymbol{x} + \sqrt{\hbar/m}\,\boldsymbol{W}_s)$ and then one can proceed as in the free case by replacing $s \mapsto Y^\mu_s$ with $s \mapsto X^\mu_s$. The final result is

$$(\exp[-t(H_{sp} - mc^2)/\hbar]\varphi_0)(\boldsymbol{x}) = \boldsymbol{E}_{(0,\boldsymbol{x})}\left(\varphi_0(\xi_t)\exp\left[-\frac{e}{c\hbar}\int_0^{\tau_t}\mathcal{A}_\mu(\boldsymbol{X}_s)\,\mathrm{d}X^\mu_s\right]\right). \tag{9}$$

In particular, in a purely electric external field $\boldsymbol{E} = -\nabla A^0 = -e^{-1}\nabla V$:

$$(\exp[-t(H_{\rm sp} - mc^2)/\hbar]\,\varphi_0)(\boldsymbol{x}) = \boldsymbol{E}_{(0,\boldsymbol{x})}\left(\varphi_0(\xi_t)\exp\left[-\frac{1}{c\hbar}\int_0^{\tau_t} V(\boldsymbol{X}_s)\,\mathrm{d}X_s^0\right]\right). \tag{10}$$

Equation (9) holds true, as solution of the Dirichlet problem given by eq. (7), also in the case where $\mathcal{A}_\mu$ is time dependent but, of course, it does not define a semi-group and there is no quantum Hamiltonian. The worldline of a relativistic particle wanders up and down the space-time in a very intricate fashion. Consider, for instance, a free particle, of rest mass m, which, at the time $t = 0$ has been created at rest (*i.e.* with its minimal energy mc^2) at the point $\boldsymbol{x}$ of the space. Clasically, its worldline would be $s \in [0, +\infty) \mapsto X_s^\mu = (cs, \boldsymbol{x})$ but, owing to the quantum fluctuations [9], the actual space-time path of the particle will be the $(1 + d)$-dimensional diffusion $s \in [0, +\infty) \mapsto X_s^\mu = (cs + \sqrt{\hbar/m}\,W_s^0, \boldsymbol{x} + \sqrt{\hbar/m}\,\boldsymbol{W}_s)$ on which eq. (9) is based. This random process represents the «ground-state process» for a free, spin-zero, relativistic particle. We remark that also the d-dimensional Brownian motion $t \in [0, +\infty) \mapsto \boldsymbol{X}_t = x + \sqrt{\hbar/m}\,\boldsymbol{W}_t$, which appears in the ordinary Feynman-Kac-Itô formula, can be interpreted, in the spirit of Nelson's stochastic mechanics [13], as the «ground-state process» for a free nonrelativistic particle. It is clear, from eq. (4), that $\tau_t \to t$ as $c \uparrow +\infty$ (a precise result is that $t \mapsto \tau_t$ converges in probability to the deterministic time t, uniformly on compact intervals [9]). The Markov process $t \mapsto \xi_t = \boldsymbol{X}_{\tau_t}$ converges in probability to the Brownian motion $t \mapsto \boldsymbol{X}_t$ when $c \uparrow +\infty$ and the semi-group defined by eq. (9) has the right nonrelativistic limit. Equation (10) looks subtly different from the formula [5]

$$(\exp[-t(H - mc^2)/\hbar]\,\psi)(\boldsymbol{x}) = \boldsymbol{E}_{\boldsymbol{x}}\left(\psi(\xi_t)\exp\left[-\frac{1}{\hbar}\int_0^t V(\xi_s)\,\mathrm{d}s\right]\right) \tag{11}$$

for the relativistic Schrödinger operator $H = \{-c^2\hbar^2\Delta_d + m^2c^4\}^{1/2} + V$ but both formulae share the same nonrelativistic limit:

$$(\exp[-tH/\hbar]\,\psi)(\boldsymbol{x}) = \boldsymbol{E}_{\boldsymbol{x}}\left(\psi(X_t)\exp\left[-\frac{1}{\hbar}\int_0^t V(\boldsymbol{X}_s)\,\mathrm{d}s\right]\right), \tag{12}$$

namely the ordinary Feynman-Kac formula for the semi-group $t \mapsto \exp[-tH/\hbar]$ generated by the Schrödinger Hamiltonian $H = -(\hbar^2/2m)\Delta_d + V$.

It is important to notice that the single-particle quantum Hamiltonian $H_{\rm sp}$, implicitly defined by eq. (9), is not self-adjoint in the Hilbert space $L^2(\boldsymbol{R}^d)$ when the external electric field is not trivial. This means that the single-particle Hilbert space $\mathfrak{D}_1$ must be equipped with a scalar product $\langle\varphi_0, \psi_0\rangle_{\mathfrak{D}_1}$ different from $(\varphi_0, \psi_0)_{L^2(\boldsymbol{R}^d)} = \int_{\boldsymbol{R}^d}\overline{\varphi}_0(\boldsymbol{x})\,\psi_0(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$. Equation (9) is our main result, but we must still exhibit the right scalar product between single-particle wave functions.

In order to construct the correct single-particle Hilbert structure, for the sake of simplicity, we shall consider only the case in which the electric field alone is present as the effect of the magnetic field is quite trivial. Let $V(\boldsymbol{x})$ be defined as $V(\boldsymbol{x}) = eA^0(\boldsymbol{x})$, where e is the elementary electric charge. The resulting Klein-Gordon equation

$$\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} - \Delta_d\varphi + \frac{2iV}{c^2\hbar}\frac{\partial\varphi}{\partial t} + \frac{(m^2c^4 - V^2)}{c^2\hbar^2}\varphi = 0 \tag{13}$$

admits a conserved energy $E(\varphi, \dot{\varphi})$ given by the quadratic form

$$E(\varphi, \dot{\varphi}) = \frac{1}{2c} \int_{\boldsymbol{R}^d} |\dot{\varphi}|^2 \mathrm{d}\boldsymbol{x} + \frac{c}{2} \int_{\boldsymbol{R}^d} \left(|\nabla \varphi|^2 + \frac{(m^2 c^4 - V^2)}{c^2 \hbar^2} |\varphi|^2 \right) \mathrm{d}\boldsymbol{x} . \tag{14}$$

In the free case, the classical energy is positive definite and we shall consider only potentials V for which $E(\varphi, \dot{\varphi})$ has still such a property. For instance, in $d = 3$, if $V(\boldsymbol{x}) = -Ze^2/|\boldsymbol{x}|$ is the Coulomb potential of an atomic nucleus with electric charge, Ze, the positivity condition is verified provided that $Z < 1/2\alpha$, where $\alpha = e^2/\hbar c \approx 1/137$. When the energy $E(\varphi, \dot{\varphi})$ is bounded from below, eq. (13) defines a respectable classical field theory and $\varphi(t, \boldsymbol{x})$ can be safely quantized. As is well known, there is also a conserved current $J^{\mu}(t, \boldsymbol{x})$ with time-component given by the formula $J^0(t, \boldsymbol{x}) = (i/2c\hbar)(\bar{\varphi}\partial_t \varphi - \varphi \partial_t \bar{\varphi})(t, \boldsymbol{x}) - (1/c\hbar^2) \cdot \cdot V(\boldsymbol{x}) |\varphi(t, \boldsymbol{x})|^2$. It follows that the (dimensionless) quadratic form

$$N(\varphi, \dot{\varphi}) = \frac{i}{2c\hbar} \int_{\boldsymbol{R}^d} (\bar{\varphi}\partial_t \varphi - \varphi \partial_t \bar{\varphi}) \mathrm{d}\boldsymbol{x} - \frac{1}{c\hbar^2} \int_{\boldsymbol{R}^d} V |\varphi|^2 \mathrm{d}\boldsymbol{x} \tag{15}$$

is constant in time. It will be presently shown that $N(\varphi, \dot{\varphi})$ is positive on positive-frequency solutions $\varphi(t, \boldsymbol{x})$ of eq. (13), and, as is done [11] in the free case, the single-particle space can be identified with the set of such functions endowed with a Hilbert norm given precisely by $\{N(\varphi, \dot{\varphi})\}^{1/2}$. In an equivalent way, $\mathfrak{D}_1$ can be built up as a linear space of complex-valued functions $\varphi_0(\boldsymbol{x})$ on $\boldsymbol{R}^d$ obtained by restricting positive-frequency solutions of eq. (13) to the hyperplane $t = 0$. Positive-energy solutions $\varphi(t, \boldsymbol{x})$ of eq. (13) have initial data $\varphi_0(\boldsymbol{x})$ and $\dot{\varphi}_0(\boldsymbol{x})$ related by the formula $\dot{\varphi}_0(\boldsymbol{x}) = -(i/\hbar)(H_{\mathrm{sp}} \varphi_0)(\boldsymbol{x})$, where H_{sp} is the single-particle quantum Hamiltonian already defined through eq. (9), therefore

$$N(\varphi, \dot{\varphi}) = \frac{1}{2c\hbar^2} \int_{\boldsymbol{R}^d} (\bar{\varphi}_0 H_{\mathrm{sp}} \varphi_0 + \varphi_0 H_{\mathrm{sp}} \bar{\varphi}_0 - 2V |\varphi_0|^2) \mathrm{d}\boldsymbol{x} =$$

$$= \frac{1}{2c\hbar^2} \left((\varphi_0, (H_{\mathrm{sp}} - V)\varphi_0)_{L^2(\boldsymbol{R}^d)} + ((H_{\mathrm{sp}} - V)\varphi_0, \varphi_0)_{L^2(\boldsymbol{R}^d)} \right) = \langle \varphi_0, \varphi_0 \rangle_{\mathfrak{D}_1} . \tag{16}$$

Equation (16) defines a Hermitian bilinear form $\langle \cdot, \cdot \rangle_{\mathfrak{D}_1}$, with a suitable domain $\mathfrak{D}_1$. In order to conclude that this form is a true scalar product on its domain, we must show that $\langle \varphi_0, \varphi_0 \rangle_{\mathfrak{D}_1}$ is positive. This can be accomplished by the following observations:

i) As H_{sp} is reality preserving, we can restrain ourselves to the real wave function $\varphi_0(\boldsymbol{x})$. Let $\varphi(t, \boldsymbol{x})$ be, now, the solution, regular to the infinity as it is required in an unbounded domain, of the Dirichlet problem in the $(1 + d)$-dimensional half-space $t > 0$:

$$-\frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} - \Delta_d \varphi - \frac{2V}{c^2 \hbar} \frac{\partial \varphi}{\partial t} + \frac{(m^2 c^4 - V^2)}{c^2 \hbar^2} \varphi = 0 \tag{17}$$

with boundary value $\varphi_0(\boldsymbol{x})$ on the hyperplane $t = 0$. Since φ and its time derivative $\partial_t \varphi$ are vanishing for $t \uparrow +\infty$, $\lim_{t \uparrow +\infty} \int_{\boldsymbol{R}^d} (\varphi \partial_t \varphi + (V/\hbar) \varphi^2) \mathrm{d}\boldsymbol{x} = 0$.

ii) The function $t \in (0, +\infty) \mapsto \int_{\boldsymbol{R}^d} (\varphi \partial_t \varphi + (V/\hbar)\varphi^2) \mathrm{d}\boldsymbol{x}$ of the Euclidean time t is strictly increasing because $(\mathrm{d}/\mathrm{d}t) \int_{\boldsymbol{R}^d} (\varphi \partial_t \varphi + (V/\hbar)\varphi^2) \mathrm{d}\boldsymbol{x} = 2cE(\varphi, \dot{\varphi}) > 0$ (remember that we assumed $E(\varphi, \dot{\varphi}) > 0$). By exploiting the previous point i), it follows that $\lim_{t \downarrow 0} \int_{\boldsymbol{R}^d} (\varphi \partial_t \varphi + (V/\hbar)\varphi^2) \mathrm{d}\boldsymbol{x} < 0$.

iii) $\langle\varphi_0,\varphi_0\rangle_{\mathfrak{D}_1} = -(1/c\hbar)\lim_{t\downarrow 0}\int_{\mathbf{R}^d}(\varphi\partial_t\varphi + (V/\hbar)\varphi^2)\,\mathrm{d}\boldsymbol{x}.$

The operator H_{sp} is symmetric with respect to the scalar product $\langle\cdot,\cdot\rangle_{\mathfrak{D}_1}$, indeed, since $\varphi(t,\boldsymbol{x}) = (\exp[-tH_{\mathrm{sp}}/\hbar]\varphi_0)(\boldsymbol{x})$, eq. (17) provides the identity

$$H_{\mathrm{sp}}^2 - 2VH_{\mathrm{sp}} = -c^2\hbar^2\Delta_d + m^2c^4 - V^2 = H_0^2 - V^2. \tag{18}$$

Therefore, for all pairs of real wave functions $\varphi_0(\cdot)$, $\psi_0(\cdot)$ in the domain of H_{sp}:

$$\langle\varphi_0,H_{\mathrm{sp}}\psi_0\rangle_{\mathfrak{D}_1} = \frac{1}{2c\hbar^2}\big((\varphi_0,(H_{\mathrm{sp}}^2 - 2VH_{\mathrm{sp}})\psi_0)_{L^2(\mathbf{R}^d)} + (H_{\mathrm{sp}}\varphi_0,H_{\mathrm{sp}}\psi_0)_{L^2(\mathbf{R}^d)}\big) =$$

$$= \frac{1}{2c\hbar^2}\big((H_{\mathrm{sp}}\varphi_0,H_{\mathrm{sp}}\psi_0)_{L^2(\mathbf{R}^d)} + (H_0\varphi_0,H_0\psi_0)_{L^2(\mathbf{R}^d)} - (V\varphi_0,V\psi_0)_{L^2(\mathbf{R}^d)}\big) = \langle H_{\mathrm{sp}}\varphi_0,\psi_0\rangle_{\mathfrak{D}_1}. \tag{19}$$

The linear operator H_{sp} in the Hilbert space $\mathfrak{D}_1$ is not only Hermitian but also positive, in fact it follows from eq. (19) that $\langle\varphi_0,H_{\mathrm{sp}}\varphi_0\rangle_{\mathfrak{D}_1} = \lim_{t\downarrow 0}E(\varphi,\dot\varphi) \geq 0$, in this way, we are sure that H_{sp} has all the properties which a good quantum Hamiltonian should be possessed of.

In this paper we have shown how to give a path integral representation of the «Klein-Gordon» semi-group in the presence of a static external electromagnetic field. In a previous work [14] we found a similar result for the «Dirac» semi-group but only in the case of a purely magnetic external field. A natural extension of our research would be to find an analogous of eq. (9) for the «Dirac» semi-group when the external electric field is not trivial. It also would be interesting to study the relations between formulae (10) and (11) corresponding to Klein-Gordon and relativistic Schrödinger operators, respectively, because an open problem is the existence, for a nontrivial external electric field, of a unitary map $U:\mathfrak{D}_1 \mapsto L^2(\mathbf{R}^d)$ such that $HU = UH_{\mathrm{sp}}$ for some relativistic Schrödinger operator H.

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