THE MEASUREMENT PROBLEM IN THE STOCHASTIC FORMULATION OF QUANTUM MECHANICS

PHILIPPE BLANCHARD
Research Center BiBoS, Universität Bielefeld
4800 Bielefeld 1, Germany

MARCELLO CINI*
Laboratoire de Physique Théorique, Université de Paris XI
91405 Orsay Cedex, France

MAURIZIO SERVA
Dipartimento di Matematica, Università dell'Aquila
67010 Coppito, L'Aquila, Italy,
also INFN, sezione dell'Aquila

Abstract: The description of physical systems obtained in the classical limit of Quantum Mechanics is not consistent with the one given by Classical Statistical Mechanics. In Quantum Mechanics the complete description of a physical system is given by specifying the wave function, which evolves completely deterministically. Probability comes into play only when a measurement is made. Classical Statistical Mechanics, on the contrary, concerns systems whose description is not complete and where probability enters as a consequence of a lack of knowledge. In this paper we show that Nelson's Stochastic Mechanics, whose predictions are identical with those of Quantum Mechanics, provides nevertheless a unified description of the deterministic and random aspects of Quantum Mechanics which eliminates the inconsistency outlined above and gives a simple and satisfactory solution to the measurement problem.

* On leave from *Dipartimento di Fisica, Università di Roma "La Sapienza", piazzale A. Moro 2, Roma 00185 Italy*

1. Introduction

In Quantum Mechanics (Q.M) the complete description of the state of a physical system with configuration space $\mathbb{R}^n$ is given by specifyng the wave function ψ, which is a square integrable function. The dynamics is given by the Schrödinger equation and so the state evolves completely deterministically.

Randomness comes into play only when a measurement is made. The manner in which it is incorporated into the formalism of Q.M. appears for many people to be not very satisfactory. Indeed physical systems are described in Q.M. as evolving according to one or the other of two apparentely incompatible ways: completely deterministic Schrödinger equation and probabilistic state vector reduction.

On the other side it is almost impossible to be dissatisfied with the very brilliant successes obtained by using Schrödinger equation for predicting the outcomes of experiments.

In this paper we give an answer to this problem using Nelson's Stochastic Mechanics (S.M.). This theory gives a more detailed description of physical systems with respect to Q.M. but is completely equivalent to Q.M. for predictions of the outcomes of experiments. Nevertheless, we show that S.M. is able to describe in a unifed language the deterministic and random aspects of quantum evolution giving a simple and completely satisfactory solution to the measurement problem.

In section 2 we discuss the measurement process in Q.M. showing that the problems are essentially a consequence of a more fundamental problem connected with the classical limit of the theory.

In section 3 we give a short description of Stochastic Mechanics. We show that the theory is a kind of minimal randomization of Classical Mechanics which reconciles the individual particle random trajectories with the Schrödinger equation.

In section 4 we rediscuss the classical limit and the measurement process in the framework of S.M. showing that both have a simple and coherent description in this theory.

In section 5 we test our results against a simple model of Stern-Gerlach measuring apparatus.

Finally, in the appendix, we briefly recall how to extend S.M. in order to describe also discrete processes and we produce another example based on a simple model of spin-bosons interaction.

2. Quantum Mechanics: the problems

2.1. *The classical limit*

The classical limit of Q.M. is generally discussed by exhibiting the connection between quantum and classical equations of motion. It is shown for example, that the phase S of a wave function satisfying Schrödinger equation becomes the classical action function S_c, which satisfies the Hamilton-Jacobi equation. In this way one concentrates on the emergence of classical trajectories and remains satisfied with this result, without considering further the limiting behaviour of the statistical properties of quantum mechanical states.

Generally speaking, quantum probabilities (which refer to the output of a measurement) for a given state become, in the limit, the probabilities associated to a classical statistical ensemble. For example the quantum probabilities of an energy eigenstate become those of the classical microcanonical ensemble with the same energy.

Two main difficulties however arise if one looks carefully into this limit [1]. The first is a consequence of the superposition principle. Consider a state

$$|\psi> = \frac{1}{\sqrt{2}}\left(|\psi_r> + |\psi_l>\right). \tag{2.1}$$

and suppose that in configuration space the wave functions $\psi_r(x,t) = <x|\psi_r>$ and $\psi_l(x,t) = <x|\psi_l>$ are "classically" separated. As an example for a free particle ψ_r and ψ_l can be chosen as two gaussians centered in $x_r = p_0 t$ and $x_l = -p_0 t$ respectively. The variance of any single gaussian has the well known form $\sigma = [\frac{(\Delta p)^2 t^2}{m^2} + \frac{\hbar^2}{4(\Delta p)^2}]^{\frac{1}{2}}$ while the separation of the two peaks is $2p_0 t$. Therefore classically separated means $2p_0 t \gg 2\sigma$, which for large t reduces to $p_0 \gg \Delta p$. The probability density for the x variable from (2.1) is

$$r(x,t) = \frac{1}{2}[|\psi_r|^2 + |\psi_l|^2 + 2Re(\psi_r^* \psi_l)] \tag{2.2}$$

since ψ_r and ψ_l have a very small overlap one can write

$$r(x,t) \approx \tilde{r}(x,t) = \frac{1}{2}[|\psi_r|^2 + |\psi_l|^2] \tag{2.3}$$

The approximate equality $r \approx \tilde{r}$ becomes exact in the limit $p_0/\Delta p \to \infty$.

The probability $\tilde{r}$ (which corresponds to a mixture) coincides with a classical statistical description in which the particle is on the right of the origin with probability 1/2 and on the left with the same probability. One would be tempted to say that also the probability r, well approximated by $\tilde{r}$ describes the same situation at least when $p_0/\Delta p$ is very large. This is not possible because the standard interpretation of Q.M. denies that a particle has a definite position before interacting with a "classical instrument".

In other words, the classical probability $\tilde{r}$ is interpreted as follows: "It is meaningful to say that the particle is on the right or on the left of the origin, but since we don't know we assign probability 1/2 to both these possibilities". On the other hand we interpret r (even in the limit $r \to \tilde{r}$) by saying: "It is meaningless to say that the particle is on the left or on the right. *Only if we look with a classical instrument* (mind?) we find it on the right or on the left with equal probability".

In conclusion even if $r \to \tilde{r}$ *we do not* obtain a classical statistical interpretation. There is a fundamental gap which is not eliminated in this limit.

The second difficulty arises from the interference terms. If we consider the density matrix $\rho = |\psi><\psi|$ and the mixture density matrix $\tilde{\rho}$

$$\tilde{\rho} = \frac{1}{2}[|\psi_r><\psi_r| + |\psi_l><\psi_l|] \tag{2.4}$$

we can always find an operator A for which

$$Tr[A(\rho - \tilde{\rho})] \neq 0 \tag{2.5}$$

even when $p_0/\Delta p \gg 1$. An obvious example is the operator

$$A = \frac{1}{2}(|\psi_r><\psi_l| + |\psi_l><\psi_r|) \tag{2.6}$$

This is not astonishing. It only means that some "quantum behaviour" can always be detected, in principle, also for variables which refer to macroscopic systems.

In our example the macroscopic variable is the distance between the two gaussian peaks or, equivalently, the variable $p_0/\Delta p$. In this case the limit $Tr[(\rho - \tilde{\rho})A]$ can be different from 0 only if the operator A introduces a dependence on the variable $p_0/\Delta p$ which cancels the effect of the vanishing of $Tr(\rho - \tilde{\rho})$ in the limit $p_0/\Delta p \to \infty$.

For a system made of a very large number of particles the macroscopic properties are described by the collective variables, while the microscopic details are

represented by the variables of each single particle. However we will not discuss this question in the present work and we refer to [1,2] for more details.

An orthodox physicist is therefore confirmed in his idea that the situation in which interference terms are present, even if they are almost vanishing, is qualitatively different from the situation in which they are totally absent.

2.2. *The measurement*

The measurement in Q.M. proceeds as follows [3-8]. Before the measurement the micro+macro system is described by the state vector

$$\frac{1}{\sqrt{2}}|\psi_0>[|\phi_+> + |\phi_->] \tag{2.7}$$

where $|\phi_+>$ and $|\phi_->$ are two eigenvectors corresponding to the variable to be measured. The apparatus is described before the measurement at $t = 0$ by a state vector $|\psi_0>$ corresponding to a "neutral" position of the pointer.

Assume that the interaction produces a one-to-one correspondence between the states $|\psi_r>$ and $|\psi_l>$ of the apparatus introduced in the previous subsection and the states $|\phi_+>$ and $|\phi_->$ of the microscopic object. Then the total wave function $\chi(x,y,t) \equiv <x,y|\chi>$, at any time t sufficiently large after the measurement, will be given by

$$\chi(x,y,t) = \frac{1}{\sqrt{2}}[\psi_r(x,t)\phi_+(y,t) + \psi_l(x,t)\phi_-(y,t)] \tag{2.8}$$

where y is the configurational variable corresponding to the microsystem. The probability density of the micro+macro system will be correspondingly

$$r(x,y,t) = \tilde{r}(x,y,t) + Re[\psi_r\psi_l^*\phi_+\phi_-^*] \tag{2.9}$$

where

$$\tilde{r}(x,y,t) = \frac{1}{2}[|\psi_r|^2|\phi_+|^2 + |\psi_l|^2|\phi_-|^2] \tag{2.10}$$

Since the pointer interference terms $\psi_r\psi_l^*$ and $\psi_r^*\psi_l$ become negligible when $p_0/\Delta p \to \infty$ the probability density r reduces to the probability density matrix $\tilde{r}$ of the statistical mixture describing the microscopic object in the state $|\phi_+>$ when the pointer "is" at the right of the origin or in the state $|\phi_->$ when the pointer is at the left. Unfortunately our preceding discussion about the classical limit shows that: (a) one cannot say that the pointer a "is" in a given position before it is, at its

turn, measured by another apparatus (infinite regression); (b) had we considered, instead of the pointer position x, another variable A not commuting with x, we would have obtained a probability density $r(A,y,t) \equiv |<A,y|\chi>|^2$ for variables A and y in which appears the interference product $<A|\psi_r><\psi_l|A>$ which might have not been negligible. Of course this would be possible only if A is a non local operator and this would correspond to possible "quantum behaviour" of the apparatus.

If the interference terms $\psi_r\psi_l^*$ and conjugate are not negligible, the "macroscopic system" is not a good measuring apparatus, becouse the main requisite that a measuring apparatus must fulfil is that it should exhibit a one-to-one probability correlation betwwen the values of the pointer and the values of the quantity to be measured.

The difficulty however remains that the position x is no longer an "objective" property of the pointer as it is for the position of a classical object. It is therefore clear that the difficulties of the mesurement problem are simply a consequence of those encountered in the classical limit of Q.M..

3. Basic facts about Stochastic Mechanics

In order to show that S.M. [9-12] is a minimal randomization of Classical Mechanics (C.M.) it is useful to formulate the last in a convenient form.

C.M. consists essentially in two assumptions, a kinematical and a dynamical one. We can say that a particle moving in Euclidean space $\mathbb{R}^n$ follows *smooth* trajectories (the kinematical assumption) which satisfy the Newton equation (the dynamical assumption).

We can put this statement in the form of two equations. The first equation corresponds to a definition of the kinematics

$$dx(t) = b(x(t),t)dt \tag{3.1}$$

Where b is a velocity field. The second equation is simply the Newton equation

$$m\ddot{x}(t) = -\nabla V(x(t)) \tag{3.2}$$

This can be regarded as a constraint on the allowed velocity fields appearing in (3.1). In order to see this fact (and also for future comparisons) it is useful to recall that the total derivative of a smooth real valued function $f(x,t)$ is defined by

$$Df(x,t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t}[f(x(t+\Delta t), t+\Delta t) - f(x(t),t)] \tag{3.3}$$

together with the condition $x(t) = x$ which means that the particle is in x at time t. A simple calculation, which takes into account formula (3.1), leads to

$$Df(x,t) = b(x,t)\nabla f(x,t) + \frac{\partial f(x,t)}{\partial t} \tag{3.4}$$

this formula permits to calculate the acceleration field, infact, the i-th component of the acceleration is

$$a_i(x,t) \equiv DDx_i = Db_i(x,t) = b(x,t)\nabla b_i(x,t) + \frac{\partial b_i(x,t)}{\partial t}$$

Let us now assume that b is a gradient function

$$b(x,t) = \frac{1}{m}\nabla S_c(x,t) \tag{3.5}$$

(where S_c is the classical action), a short calculation shows that the acceleration takes the simpler form

$$a(x,t) = \frac{1}{2}\nabla b^2(x,t) + \frac{\partial b(x,t)}{\partial t} \tag{3.6}$$

and therefore the Newton equation becomes

$$m[\frac{1}{2}\nabla b^2(x,t) + \frac{\partial b(x,t)}{\partial t}] = -\nabla V(x) \tag{3.7}$$

which can be integrated in order to obtain the Hamilton-Jacobi equation

$$\frac{1}{2m}[\nabla S(x,t)]^2 + \frac{\partial S(x,t)}{\partial t} + V(x) = 0 \tag{3.8}$$

On the other hand, it is an obvious fact that equation (3.1) is equivalent to the continuity equation

$$\frac{\partial r(x,t)}{\partial t} = -\mathrm{div}[r(x,t)b(x,t)] \tag{3.9}$$

which gives the evolution of the probability density r once its initial value is known. The single trajectory can be eventually obtained simply by assuming that the initial r is a Dirac delta.

The three equations (3.5), (3.8) and (3.9) give a "hydrodynamical" formulation of C.M. completely equivalent to the standard one.

S.M. can be viewed as a mimimal randomization of C.M.. The kinematical law of S.M. asserts that the paths of the particle are the *continuous but not differentiable* realizations of a diffusion process. In other words, the trajectories are given by the solution of a stochastic differential equation of the type

$$dx(t) = b^+(x(t),t)dt + \sqrt{\sigma}dw(t) \tag{3.10}$$

This equation is the random counterpart of (3.1). The trajectories are not differentiable becouse $w(t)$ is the standard Brownian motion and therefore $dw(t)$, roughly speaking, is proportional to $(dt)^{\frac{1}{2}}$. σ is the diffusion constant whose value we do not specify at the moment. This stochastic equation corresponds to the Fokker-Planck equation

$$\frac{\partial r(x,t)}{\partial t} = -\mathrm{div}[r(x,t)b^+(x,t)] + \frac{\sigma}{2}\Delta r(x,t) \tag{3.11}$$

which reduces to the continuity equation when σ is put equal 0.

Since the trajectories $x(t)$ are not differentiable it is necessary to define a kind of mean derivative in order to extend the Newton equation (2.2). This can be done in the following way: given a smooth real-valued function $f(x,t)$, we define a forward mean derivative as

$$D^+f(x,t) = \lim_{\Delta t\to 0}\frac{1}{\Delta t}E[f(x(t+\Delta t),t+\Delta t) - f(x(t),t)|x(t) = x] \tag{3.12}$$

where $E[\cdot|x(t) = x]$ means conditional expectation with respect to $x(t) = x$. In other words the expectation is taken under the condition that the particle is in the position x at time t. Taking into account (2.4) we easily obtain

$$D^+f(x,t) = b^+(x,t)\nabla f(x,t) + \frac{\sigma}{2}\Delta f(x,t) + \frac{\partial f(x,t)}{\partial t} \tag{3.13}$$

We also give another definition which is time-symmetric with respect to the first one. We define a backward mean derivative as

$$D^-f(x,t) = \lim_{\Delta t\to 0}\frac{1}{\Delta t}E[f(x(t+\Delta t),t+\Delta t) - f(x(t),t)|x(t+\Delta t) = x] \tag{3.14}$$

where we have changed the condition in the past with the condition in the future that the particle is in x at time $t+\Delta t$. One can easily see that

$$D^-f(x,t) = b^-(x,t)\nabla f(x,t) - \frac{\sigma}{2}\Delta f(x,t) + \frac{\partial f(x,t)}{\partial t} \tag{3.15}$$

where $b^-(x,t) \equiv b^+(x,t) - \sigma\nabla \log r(x,t)$. Both these definitions reduce to the ordinary total derivative when the trajectory is classical ($\sigma = 0$).

We can now define a mean acceleration which is the counterpart of the classical one; for the i-th component we write

$$a_i(x,t) \equiv \frac{1}{2}[D^+D^-x_i + D^-D^+x_i] = \frac{1}{2}[D^+b_i^-(x,t) + D^-b_i^+(x,t)] \tag{3.16}$$

We assume, as we have done in C.M., that b^+ is a gradient function (in this case the definition of b^- implies that b^- is also a gradient function). We can therefore introduce a scalar function $S(x,t)$ (the quantum action) such that

$$b^+(x,t) = \frac{1}{m}\nabla S(x,t) + \frac{\sigma}{2}\nabla \log r(x,t) \tag{3.17}$$

and

$$b^-(x,t) = \frac{1}{m}\nabla S(x,t) - \frac{\sigma}{2}\nabla \log r(x,t) \tag{3.18}$$

equation (3.16) can be integrated and becomes

$$\frac{1}{2m}[\nabla S(x,t)]^2 + \frac{\partial S(x,t)}{\partial t} - \frac{\sigma^2}{2}\frac{\Delta r^{\frac{1}{2}}(x,t)}{r^{\frac{1}{2}}(x,t)} + V(x) = 0 \tag{3.19}$$

The equations (3.11), (3.17) and (3.19) are the counterpart of the classical equations (3.5), (3.8) and (3.9). It is also clear that assuming that σ vanishes we recover the last three from the first three.

Stochastic Mechanics is therefore simply a generalization of C.M. which is obtained with a mininimal randomization of the trajectories.

Now comes the question where Q.M. enters in all that since we have, apparently, somethting which looks much more similar to C.M. than Q.M.. In order to answer the question we construct the function ψ in the following way

$$\psi(x,t) \equiv r^{\frac{1}{2}}(x,t)\exp[\frac{iS(x,t)}{m\sigma}] \tag{3.20}$$

we easily check that it satisfies the equation

$$i\sigma m\frac{\partial \psi(x,t)}{\partial t} = -\frac{m\sigma^2}{2}\Delta\psi(x,t) + V(x)\psi(x,t) \tag{3.21}$$

it is therefore sufficient to put $\sigma = \frac{\hbar}{m}$ in order to obtain the Schrödinger equation which, in this context appears simply as a "trick" to linearize the equations of our perturbed version of C.M..

We can now reverse our construction and say: given a solution $\psi(x,t)$ of Schröedinger equation we can associate a diffusion whose probability density is $r(x,t) = |\psi(x,t)|^2$ and whose drift field is

$$b^+(x,t) = \frac{\hbar}{m}(Re + Im)[\nabla \log \psi(x,t)] \tag{3.22}$$

The conclusion is therefore that S.M. embeds the computational formalism of conventional Q.M. into a more detailed description of nature so that ψ no longer fully describes the state of the system. Nevertheless, since the probability densities of S.M. and Q.M. coincide, the two theories are equivalent for what concerns the preduction of outputs of measurements.

Before ending this section we make two remarks. The first concerns the behaviour of trajectories in the proximity of the the nodal set $N_\psi = \{x \in \mathbb{R}^n | \psi(x,t) = 0\}$. It has been shown [see 13,14] that a path never crosses a point of this set. This leads to a decomposition of $\mathbb{R}^n$ into invariant regions C_i separated by N_ψ. One can speak in this case of "trapping" or "confinement" in C_i since a process starting in C_i does not leaves this region with probability one. In other words N_ψ acts as an impenetrable barrier.

The second remark is that the dynamics of S.M. can be alternatively obtained from variational principles [15-18].

4. Stochastic Mechanics: solution of the problems

4.1. *The classical limit*

Consider again the case of the second section. In S.M. one associates to the wave function

$$<x|\psi> = \frac{1}{\sqrt{2}}[\psi_r(x,t) + \psi_l(x,t)] \tag{4.1}$$

a process with probability density

$$r(x,t) = \frac{1}{2}[|\psi_r|^2 + |\psi_l|^2 + 2Re(\psi_r \psi_l^*)] \tag{4.2}$$

and drift

$$b^+(x,t) = \frac{\hbar}{m}(Re + Im)\nabla \log(\psi_r + \psi_l) \tag{4.3}$$

In the limit $p_0/\Delta p \to \infty$ ψ_r and ψ_l tend to have disjoint supports and therefore the probability of crossing of the origin becomes very small, furthermore, the density and the drift at right of the origin are well approximated by

$$r_r(x,t) = |\psi_r|^2 \tag{4.4}$$

and

$$b_r^+(x,t) = \frac{\hbar}{m}(Re + Im)\nabla \log \psi_r \tag{4.5}$$

while on the left they are approximated by

$$r_l(x,t) = |\psi_l|^2 \tag{4.6}$$

and

$$b_l^+(x,t) = \hbar(Re + Im)\nabla \log \psi_l \tag{4.7}$$

As a consequence of this fact the process splits in two separate processes [see also 19,20], the first will live on the right of the origin and will have density $r_r(x,t)$ and drift $b_r^+(x,t)$ and the second will live on the left and will have density $r_l(x,t)$ and drift $b_l^+(x,t)$. The system belongs to the first process or to the second one with probability $\frac{1}{2}$.

It should be remarked that the presence of nodes around the origin would improve the mechanism because the supports of the two processes would be disjoint

even before the limit. Obviusly the limit $p_0/\Delta p \to \infty$ is still necessary in order to have convergence of densities and drifts.

We can compare these two processes originated from the pure state with the other two which originated from the mixture. By definition the first process of the mixture will have density $r_r(x,t)$ and drift $b_r^+(x,t)$, furthermore, in the limit $p_0/\Delta p \to \infty$ it will be confined on the right of the origin ($r_r(x,t)$ vanishes on the left). An analogous discussion holds for the second process of the mixture. We conclude that the couple originated from the pure state and the couple associated to the mixture coincide in the limit.

The problems of Q.M. have now disappeared because the statement that the system is on the right (left) is always fully meanigful, therefore in the limit *there is not* any conceptual difference between a pure state and a mixture.

Now comes the question: what happens to the interference terms when the limit is not exact? They simply modify the behaviour of the process in the interference region. In principle one can detect this difference which corresponds to a semiclassical behavior.

In S.M., as in Q.M., one can define variables which are particularly sensitive to the details of the wave function (and therefore of the drift) in the interference region. These variables show, therefore, a typical quantum behaviour (an example of this fact will be given in section 5).

There is, however, no one-to-one correspondence with the operators of Q.M., because all variables in S.M. are defined in terms of the stochastic variable x_t. This is not a disadvantage, in our opinion, for two reasons. Firstly because an operator such as A of eq. (2.6) does not correspond to any operational definition, on the contrary, operators which are operationally well defined can be constructed from x_t (for example the output momenta of particles which scatters on a target [21,22]). Secondly, because no problems of interpretation arise in S.M., since the position always has a physical meaning.

4.2. *The measurement*

From eq.(2.7)we have, denoting, as before, by x the configurational macroscopic variable, and by y the configurational microscopic variable, that the wave function before the measurement is

$$<x,y|\chi> = \frac{1}{\sqrt{2}}<x|\psi_0>(<y|\phi_+> + <y|\phi_->) \tag{4.8}$$

The corresponding process for the two stochastic variables $x(t)$, $y(t)$ has density

$$r(x,y,t) = \frac{1}{2}[|\psi_0(x,t)|^2|\phi_+(y,t) + \phi_-(y,t)|^2] \tag{4.9}$$

and drift

$$b^+(x,y,t) = \frac{\hbar}{m}(Re + Im)[\nabla_y \log(\phi_+(y,t) + \phi_-(y,t)) \ , \ \nabla_x \log \psi_0(x,t)] \tag{4.10}$$

The drift has two components, the first one for the microscopic variable, the second for the macroscopic one. Since the density for the two components is a product and the two components of the drift are uncorrelated, the stochastic variables $x(t)$ and $y(t)$ are statistically and dynamically independent.

After the measurement we have

$$<x,y|\chi> = \frac{1}{\sqrt{2}}[<x|\psi_r><y|\phi_+> + <x|\psi_l><y|\phi_->] \tag{4.11}$$

Since ψ_r and ψ_l have disjoint supports the process splits into two separated processes [19,20]. The first one has density

$$r_r(x,y,t) = |\psi_r(x,t)|^2|\phi_+(y,t)|^2 \tag{4.12}$$

and drift

$$b_r^+(x,y,t) = \frac{\hbar}{m}(Re + Im)[\nabla_y \log \phi_+(y,t) \ , \ \nabla_x \log \psi_r(x,t)] \tag{4.13}$$

The second has density

$$r_l(x,y,t) = |\psi_l(x,t)|^2|\phi_-(y,t)|^2 \tag{4.14}$$

and drift

$$b_l^+(x,y,t) = \frac{\hbar}{m}(Re + Im)[\nabla_y \log \phi_-(y,t) \ , \ \nabla_x \log \psi_l(x,t)] \tag{4.15}$$

The total system belongs to one of them with probability 1/2. The effect of a measurement on the microscopic variable alone is therefore the following [23]. Before the measurement we have the process whith density and drift

$$\frac{1}{2}|\phi_-(y,t) + \phi_+(y,t)|^2 \ , \ \frac{\hbar}{m}(Re + Im)\nabla_y \log(\phi_+(y,t) + \phi_-(y,t)) \tag{4.16}$$

while after the measurement we have two processes, the first with density and drift

$$|\phi_+(y,t)|^2 \quad , \quad \frac{\hbar}{m}(Re + Im)\nabla_y \log \phi_+(y,t) \tag{4.17}$$

the second with

$$|\phi_-(y,t)|^2 \quad , \quad \frac{\hbar}{m}(Re + Im)\nabla_y \log \phi_-(y,t) \tag{4.18}$$

The microsystem belongs to one of these two with probability 1/2. The measurement has the effect of modifying the process of the microscopic variable. This fact leads to the conclusion that the trajectories of S.M. are not observable [23] since any observation modifies the process.

There is only one case in which this modification does not occur namely the case in which the "microscopic" variable to be measured is indeed classical. This is the same as saying that ϕ_+ and ϕ_- have disjoint supports. Therefore one has that the ante-measurement process (4.16) coincides with the couple of post-measurement processes (4.17) and (4.18) In other words, if the microsystem is classical, we can look at the dichotomic variable corresponding to the states ϕ_{pm} without disturbing it. Looking has the only effect of increasing our knowledge. Classical trajectories are observable and have an objective meaning.

This kind of reasoning can also be applied to measurements of the total system (microscopic + macroscopic). Since it is globally classical, the associated process is not affected by any further measuring interaction. This means that we can look at the pointer without fear of changing its state or the state of the microsystem.

It is clear at this point that the measurement in S.M. proceeds as follows:

(a) We produce an interaction which links the microscopic variable to a macroscopic one

(b) Since the total system is now classical we can look at it (measure, interact) without disturbing the micro+macro system. Therefore we look at it with only result the of a gain of information.

5. A model: the Stern-Gerlach apparatus

We consider a Stern-Gerlach apparatus in which a beam of atoms is split in two separated beams by a suitable interaction between their internal angular momentum and an external magnetic field.

A simple model for this system can be constructed by coupling the internal angular momentum $J = (J_1, J_2, J_3)$ with an inhomogeneous magnetic field $B(x_1, x_2, x_3) = (0, x_1, 0)$ acting in the time interval between $t = 0$ and $t = T$.The coordinates of the center of mass are x_1, x_2 and x_3 but we ignore x_2 and x_3 since they are not involved in the interaction and we simply use $x \equiv x_1$.The internal configurational variable is y. This leads to the quantum mechanical Hamiltonian

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - \lambda J_2 x\theta(t)\theta(T-t) \tag{5.1}$$

We take the initial state to be a product of a gaussian

$$\psi_0(x) = (2/\pi\alpha)^{\frac{1}{4}} \exp\{-x^2/\alpha\} \tag{5.2}$$

times a superpostion of two eigenstates $\phi_+(y)$ and $\phi_-(y)$ corresponding to the eigenvalues $+n\hbar$ and $-n\hbar$ of J_2:

$$<x, y|\chi> = \psi_0(x)\frac{1}{\sqrt{2}}[\phi_+(y) + \phi_-(y)] \tag{5.3}$$

where $\alpha \equiv \hbar^2/(\Delta p_0)^2$ and Δp_0 is the indeterminacy of momentum. The total wave function at time t will be easily obtained once the following Schrödinger equation for the x variable only is solved

$$i\hbar\frac{d\psi}{dt} = \frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - Lx\psi \tag{5.4}$$

with $L = \pm\lambda\hbar n$ and initial condition ψ_0. The method is standard and one obtains a solution which for t larger than T is

$$\psi(x,t) = (\frac{2}{\pi\alpha})^{\frac{1}{4}} \frac{1}{1 + i\frac{2\hbar}{m\alpha}t} \exp\{-\frac{(x + x_0 - \frac{p_0 t}{m})^2}{\alpha + i\frac{2\hbar}{m}t}\} \exp\{i\frac{p_0 x}{\hbar}\} \tag{5.5}$$

which is a gaussian whose parameters are

$$<p>_t = LT \equiv p_0 \quad , \quad <x>_t = \frac{p_0}{m}(t - \frac{T}{2}) \equiv \frac{p_0}{m}t - x_0 \tag{5.6}$$

$$<\Delta p>_t = \Delta p_0 \quad , \quad <\Delta x>_t = [\frac{(\Delta p_0)^2 t^2}{m^2} + \frac{\hbar^2}{4(\Delta p_0)^2}]^{\frac{1}{2}} \tag{5.7}$$

The classical condition is, therefore, $LT \gg \Delta p_0$. Of course there will be two different wave functions ψ_r and ψ_l depending on the two values $p_0 = LT = \pm\lambda\hbar nT$ which have to be substituted in the above expression.

The total quantum mechanical wave function has the form (2.8). Now we associate to this wave function a stochastic process with two components $x(t)$ and $y(t)$ whose probability density at time $t > T$ will coincide with the quantum mechanical expression

$$r(x,y,t) = \tilde{r}(x,y,t) + Re\{\psi_r^*(x,t)\psi_l(x,t)\phi_+^*(y)\phi_-(y)\} \tag{5.8}$$

with

$$\tilde{r}(x,y,t) = \frac{1}{2}(|\psi_r(x,t)|^2|\phi_+(y)|^2 + |\psi_l(x,t)|^2|\phi_-(y)|^2) \tag{5.9}$$

We will now show that there is a vanishing probability for the macrosystem to cross the origin. In fact, if we consider the region around the origin for which $|x| < \epsilon t$, we have from (5.5) that for large t and small ϵ

$$|\psi_{r,l}(x,t)| < [\frac{m^2}{2\pi(\Delta p_0)^2 t^2}]^{\frac{1}{4}} \exp\{-\frac{(p_0)^2}{4(\Delta p_0)^2}\} \tag{5.10}$$

Therefore the total probability $P(\epsilon,x) = \int_{-\epsilon t}^{+\epsilon t} r(x,y,t)dx$ that the microsystem is in y and the macrosystem in the region $|x| < \epsilon t$ satisfies

$$P(\epsilon,x) < [|\phi_-(y)|^2 + |\phi_+(y)|^2 + 2|\phi_-(y)||\phi_+(y)|]\frac{\epsilon m}{\sqrt{2\pi}\Delta p_0} \exp\{-\frac{(p_0)^2}{2(\Delta p_0)^2}\} \tag{5.11}$$

this proves our statement for large $p_0/\Delta p_0$. The process therefore splits into two processes. Since a similar inequality holds for

$$\tilde{P}(\epsilon,x) = \int_{-\epsilon t}^{+\epsilon t} \tilde{r}(x,y,t)dx \tag{5.12}$$

we can say that also the right and left process associated with the wave functions

$$\chi_r(x,y,t) = \psi_r(x,t)\phi_+(y) \tag{5.13}$$

$$\chi_l(x,y,t) = \psi_l(x,t)\phi_-(y) \tag{5.14}$$

never cross the origin and remain on the right and on the left of the origin respectively.

One can easily prove that the pair of processes originating from the pure state and the pair originating from the mixture coincide. One has, in fact, for any x_0 and sufficiently large times

$$\int_{x_0-\epsilon t}^{x_0+\epsilon t} |r(x,y,t) - \tilde{r}(x,y,t)|dx < |\phi_-(y)||\phi_+(y)| \frac{2\epsilon m}{\sqrt{2\pi}\Delta p_0} \exp\{-\frac{(p_0)^2}{2(\Delta p_0)^2}\} \quad (5.15)$$

Furthermore, since there are no processes in the region $|x| < \epsilon t$, we can restrict the calculations of the drift to the regions $|x| > \epsilon t$. The right and left processes associated to the pure state have, in the classical limit, drifts

$$b^+_{r,l}(x,y) = [(Re + Im)\{\frac{2i(x + x_0 \mp |p_0|t/m)}{m(\alpha + i2\hbar t/m)} \pm |p_0|\} \ , \ \frac{\hbar}{m}(Re + Im)\nabla_y \log \phi_\pm(y)] \quad (5.16)$$

They both coincide with the drifts of the two processes associated to the mixture.

Finally, we wish to show that in spite of this conclusions, there is, at least in principle, a way to detect the difference between the processes associated to the pure state and those associated to the mixture. In fact, if we evaluate the conditional probability density to find the microsystem in y after having found the macrosystem in the region $|x| < \epsilon t$, we obtain

$$r(y|\epsilon) - \tilde{r}(y|\epsilon) = \frac{2Re \int_{-\epsilon t}^{+\epsilon t} \phi_-(y)\phi_+^*(y)\psi_r^*(x)\psi_l(x)dx}{\int_{-\epsilon t}^{+\epsilon t} |\psi_r(x)|^2 + |\psi_l(x)|^2)dx} \quad (5.17)$$

where, as usual, r and $\tilde{r}$ refer to the pure state and to the mixture respectively. Now both the numerator and the denominator are of order $\exp\{-(p_0)^2/(\Delta p_0)^2\}$ and therefore the difference of the conditional probabilities has the form

$$r(y|\epsilon) - \tilde{r}(y|\epsilon) = a(\epsilon)Re[\phi_-(y)\phi_+^*(y)] + b(\epsilon)Im[\phi_-(y)\phi_+^*(y)] \quad (5.18)$$

with $a(\epsilon)$ and $b(\epsilon)$ of order 1 whith respect to $\exp\{-(p_0)^2/(\Delta p_0)^2\}$. The difference, which may be experimentally detected, does not vanish in the classical limit. Nevertheless these "non classical" events are extremely rare and one has to wait a time which grows as $\exp\{(p_0)^2/(\Delta p_0)^2\}$ to see them.

6. Discussion

The basic inconsistency between the properties attributed to physical systems by the classical limit of Quantum Mechanics and those described by Classical Statistical Mechanics is eliminated if we adopt the point of view that a representation of their state more complete than the one provided by the quantum mechanical wave function is possible. In spite of the widespread belief that this possibility does not exist, it is however a fact that Nelson's Stochastic Mechanics accomplishes this task by embedding the computational formalism of Quantum Mechanics into a more detailed description of physical systems. We have seen that the additional stochastic variable, far from being a meaningless complication of the picture, is instead necessary if a smooth and internally consistent transition from the quantum to the classical domain is required. As an example, we have shown that, according to this picture, in a Stern-Gerlach apparatus a given particle is always located in one of the two beams irrespectively of whether its position is measured or not. This localization occurs, of course, only when there is a clearcut separation in space between the two beams. As soon as they begin to overlap the usual interference phenomena start to show up.

This solution eliminates one of the most striking paradoxes of the conventional interpretation of Q.M., according to which each particle materializes instantaneously into one of the two beams, no matter how far away they may be, only when the corresponding counter is triggered at random.

Last, but not least, this picture of reality supports our cherished belief that we can look at macroscopic object without modifying its state. Classical trajectories have an objective meaning: looking, far from killing cats, only increases the knowledge of the observer.

Appendix
Quantum Mechanics and Discrete Stochastc Processes

Nelson's S.M. represents the position of a quantum mechanical system in terms of a diffusion associated to its position but it is unable to describe discrete variables like spin or photon number [24-26]. In order to extend the theory to this more general case many people have proposed to introduce also jump processes.

We briefly recall here one of these extensions, which is particularly suitable for treating the evolution of systems which provide models of macroscopic measuring instruments such as counters.

The attempt is to construct a stochastic field theory based on the assumption that for any normal mode of a given classical field with amplitude $q_k(t)$ the corresponding action variable J_k becomes a stochastic variable assuming only positive integer values n_k in units of $\hbar$.

The dynamical evolution of the field is described by means of the infinite set of discrete stochastic processes $n_k(t)$ representing the occupation number of each mode k. The state of the field at time t will be therefore specified by the probability $r(n_1, n_2, ...n_k, ..., t)$ that the stochastic variables $n_1(t)$, $n_2(t)$, ..., $n_k(t)$,..have the values $n_1, n_2, ...n_k, ...$ at time t. To obtain the time evolution of r the jump probability rates are needed. These quantities are defined, as in the case for the drift in Nelson's S.M., in terms of the quantum mechanical wave function of the state considered.

In order to show how the method works it is sufficient to treat a very simple model, namely the case of a single field oscillator in interaction with a fixed source [see 25]. The Hamiltonian is

$$H = \frac{1}{2}(p^2 + \omega^2 q^2) + \lambda\sqrt{2\omega}q \tag{A.1}$$

which reduces to

$$H = \omega a^+ a + \lambda(a + a^+) \tag{A.2}$$

when the Dirac oscillator variables

$$a = \frac{1}{\sqrt{2\omega}}(\omega q + ip) \quad a^+ = \frac{1}{\sqrt{2\omega}}(\omega q - ip) \tag{A.3}$$

are introduced. The Schrödinger equation in the occupation number representation reads ($\hbar = 1$)

$$i\frac{d}{dt}<n|\psi> = n\omega<n|\psi> + \lambda\sqrt{n+1}<n+1|\psi> + \lambda\sqrt{n}<n-1|\psi> \tag{A.4}$$

From (A.4) one obtains the time derivative of the probability

$$r(n,t) = |<n|\psi>|^2 \tag{A.5}$$

in the form

$$\frac{dr(n,t))}{dt} = -2[Im\Delta_-(n,t) + Im\Delta_+(n,t)]r(n,t) \tag{A.6}$$

with

$$\Delta_\pm(n,t) = \lambda\sqrt{n+(1\pm 1)/2}\frac{<\psi|n\pm 1>}{<\psi|n>} \tag{A.7}$$

Eq. (A.6) does not yet have the form of a Fokker-Planck equation for the jump process $n(t)$. It is however possible to define positive jump probability rates as follows

$$p_\pm(n,t) = |\Delta_\pm(n,t)| + Im\Delta_\pm(n,t) \tag{A.8}$$

In terms of which the equation (A.6) reads

$$\frac{dr(n,t)}{dt} = p_-(n+1,t)r(n+1,t) + p_+(n-1,t)r(n-1,t) - [p_-(n,t)+p_+(n,t)]r(n,t) \tag{A.9}$$

which is now a Fokker-Planck equation for the probability density $r(n,t)$. The transition probability density $p(n,t;n_0,t_o)$ to find the system in n at time t if it was in n_0 at time t_0 satisfies the same equation. These are all the ingredients we need to discuss from the point of view of discrete stochastic processes a simple model of measuring apparatus which was proposed in reference [3].

In this model the microsystem is represented by a dichotomic variable (spin) and the macrosystem by an ensemble of N particles (bosons) which may be in one of two states, the state 0 or the state 1. Introducing creation and destruction operators a_0^+, a_0 and a_1^+, a_1 respectively, a generic state of the counter with n particles in the state 0 and $N-n$ particles in the state 1 is given by

$$|n,N-n> = \frac{1}{\sqrt{n!(N-n)!}}(a_0^+)^n(a_1^+)^{N-n}|0,0> \tag{A.10}$$

The generic spin state of the microsystem is represented by

$$|w> = c_+|u_+> + c_-|u_-> \tag{A.11}$$

where $|u_{\pm}>$ are the two eigenstates of the spin corresponding to the eigenvalues ± 1.

The hamiltonian H is constructed in such a way that the counter interacts only with the spin state $|u_+>$, while the spin state $|u_->$ leaves the counter state unperturbed

$$H = \frac{\lambda}{2}(1+\sigma_z)[a_0^+ a_1 + a_1^+ a_0] \tag{A.12}$$

If the initial state of the total system is the product

$$\frac{1}{2}(|u_+> + |u_->)|N,0> \tag{A.13}$$

after a time t the total system will be in a state

$$|\chi> = \frac{1}{2}(\sum_{n=0}^{N} f_n(t)|u_+>|n, N-n> + |u_->|N,0>) \tag{A.14}$$

with [see 3]

$$f_n(t) = \frac{\sqrt{N!}}{\sqrt{n!(N-n)!}}(-i)^{N-n}(\cos\alpha(t))^n(\sin\alpha(t))^{N-n} \tag{A.15}$$

where $\alpha(t) = \lambda t/\sqrt{N}$ and $\tau \equiv 1/\lambda$ is the typical one-particle discharge time.

The state of the counter corresponding to $|u_+>$ is characterized by an average number $\bar{n}(t)$ of particles in the state 0 (and correspondingly $N - \bar{n}(t)$ particles in the state 1) while the counter's state corresponding to $|u_->$ has not changed from its initial state with N particles in the state 0 and no particles in the state 1.

The average number $\bar{n}(t)$ is easily obtained from the probability distribution $P(n,t)$ obtained from (A.14) and (A.15)

$$P(n,t) = \frac{N!}{(N-n)!n!}(\cos^2\alpha(t))^n(\sin^2\alpha(t))^{N-n} \tag{A.16}$$

This distribution is strongly peaked around the value

$$\bar{n}(t) = N\cos^2\alpha(t) \tag{A.17}$$

with a relative width $\Delta n/N$ which tends to 0 as $1/\sqrt{N}$.

The two states, the first corresponding to N particles in the state 0 and the second to $\bar{n}(t)$, becomes macroscopically different as $\bar{n}(t)/N$ become different from 1. The whole discussion made in Sections 2 can be completely repeated in this case.

We now introduce the stochastic interpretation sketched above. The bosons' state corresponding to $|u_->$ remains undischarged and therefore no jumps occur in this case. The discharged boson's state, on the contrary, has a density $P(n,t)$ given by (A.16). The jump probabilities $p_\pm(n,t)$ are easily calculated by using the formulas (A.7) and (A.8). One gets

$$p_+(n,t) = (N-n)\{|\mathrm{ctg}\alpha(t)| - \mathrm{ctg}\alpha(t)\} \tag{A.18}$$

$$p_-(n,t) = (N-n)\{|\mathrm{tg}\alpha(t)| + \mathrm{tg}\alpha(t)\} \tag{A.19}$$

For $t < \frac{\sqrt{N}}{\lambda}\frac{\pi}{2}$, one has

$$p_+(n,t) = 0 \quad p_-(n,t) = 2n\mathrm{tg}\alpha(t) \tag{A.20}$$

The solution of the Fokker-Planck equation, given that the counter has initially N bosons in the state ϕ_0, coincides, of course, with the probability $P(n,t)$ of equation (A.16). In other words, one has

$$P(n,t) \equiv P(n,t;N,0) \tag{A.21}$$

We can now speak of trajectories in the discrete n space. The probability that a trajectory of the discharged counter starts at N at time zero and ends in N at time t is

$$P(N,t;N,0) = (\cos^2\alpha(t))^N \simeq \exp -\frac{t^2}{\tau^2} \tag{A.22}$$

the second approximate equality holds for $t \ll \tau\sqrt{N}$. Therefore, looking at the counter at a time t which satisfies $\tau\sqrt{N} \gg t \gg \tau$ the probability to find all the N bosons in the state 0 becomes negligible. On the contrary, the counter remains in N all the time in correspondence of $|u_->$. Trajectories which tend to go in the "wrong" beam are therefore negligible. This shows that also in this case, if we ascribe a meaning to the process $n(t)$, all interpretational problems disappear. The counter *is* either discharged or not even *before* looking at it.

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