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Lack of self-averaging in weakly disordered one-dimensional systems

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Abstract . — We introduce a one-dimensional disordered Ising model which at zero temperature is characterized by a non-trivial, non-self-averaging, overlap probability distribution when the impurity concentration vanishes in the thermodynamic limit. The form of the distribution can be calculated analytically for any realization of disorder. For non-zero impurity concentration the distribution becomes a self-averaging delta function centered on a value which can be estimated by the product of appropriate transfer matrices.

1. Introduction.

Disordered systems attracted much work in the last years. One of the main features of these systems is the large number of locally stable states. As a consequence, a natural characterization of the equilibrium states is through the distribution of their mutual distance, that is through the probability distribution $P(q)$ of the overlap q between pairs of states [1]. In general, the organization of the equilibrium states depends on the realization of disorder, so that $P(q)$ depends on the disorder, even after the thermodynamic limit has been taken. For instance, this loss of self-averaging is observed in the Sherrington-Kirkpatrick model of spin glasses [2] where it can be related to a symmetry breaking in the replica space [1]. A big effort has been devoted to the search of simpler systems which share the main features of the Sherrington-Kirkpatrik model. The most celebrated among them is the random energy model [3] now used in very different contexts such as biological evolution, neural networks, polymers, and so on. On the other hand, it is commonly assumed that one-dimensional systems lack for such a rich behaviour. In this paper we introduce a disordered Ising model which, in spite of being a one-dimensional model with short range interactions, exhibits at zero temperature a non-trivial, non-self-averaging, $P(q)$ if the impurity concentration vanishes in the thermodynamic limit. This latter condition is crucial since, if the disorder concentration does not vanish

in the thermodynamic limit, $P(q)$ becomes a self-averaging delta function, even at zero temperature. This is the main result of the paper which allows one to get a deeper understanding of the phenomena which appear in non-trivial disordered systems: existence of many degenerate ground states, frustration, lack of self-averaging.

The paper is organized as follows. In section 2, we introduce the model and the thermodynamic quantities of interest. In section 3, we study the model in the weakly disorder limit, where it is possible to calculate the form of the overlap distribution $P(q)$ for any disorder realization. In section 4, the model is analysed for non-zero impurity concentration where $P(q)$ is shown to be a self-averaging delta function. The value of the overlap where the delta function is centered can be obtained by means of random transfer matrices.

In Appendix A, we discuss an extension of the Landau theorem on the absence of phase transitions in one dimension to the overlap probability distribution. In Appendix B, we give a brief description of the numerical algorithm used in section 4 to compute thermodynamical quantities *via* product of transfer matrices [4].

2. The system.

One of the main reasons for the existence of a large number of equilibrium states in statistical systems is the presence of "frustration". In general, a disordered system cannot satisfy all the constraints originated from the competing effects of disorder and interactions. We introduce a one-dimensional model which captures the main features of disordered systems but, at the same time, is simple enough for an analytical study. The model is a random field Ising chain of N spins described by the Hamiltonian

$$H = -\sum_{i=1}^{N} [\sigma_i \sigma_{i+1} + h_i \sigma_i] \tag{2.1}$$

with

$$h_i = \eta_i - \eta_{i+1} \tag{2.2}$$

where the η_i are random independent variables which take the value $+1$ and -1 with probability p and $(1-p)$, respectively. As a consequence, the magnetic field on a site assumes the value $h_i = \pm 2$ with probability $p(1-p)$, and $h_i = 0$ with probability $p^2 + (1-p)^2$. By changing the value of p, we can modulate the disorder in the model. The pure ferromagnetic system is obtained for $p = 1$ or 0, yielding $h_i = 0\ \forall i$. We can also consider a pure system with a finite number I of impurities, so that the impurity concentration vanishes in the thermodynamic limit, *weakly disordered system*. The integer number I is a random variable distributed according to

$$P(I) = c^I \exp(-c)/I!$$

where $c = \sum_I P(I) I$ is its mean value. The weakly disorder limit is achieved by taking $\eta_i = 1$ with probability $p = 1 - c/N$, and hence $p \to 1$ when the number of spins $N \to \infty$.

For any value of p the local magnetic field has a certain degree of spatial correlation, all field realizations are made by a sequence of strings

$$2,\ 0 \ldots\ 0, -2,\ 0 \ldots\ 0\ 2,$$

where the number of zeros between two successive 2 and -2 or -2 and 2 is a random non-negative variable.

The overlap between two spin configurations "α" and "β" for the same disorder realization is defined as

$$q_{\alpha\beta} = \frac{1}{N}\sum_{i=1}^{N} \sigma_i^{\alpha}\sigma_i^{\beta}.$$

This can be usefully computed by introducing two identical replicas of the system. The equilibrium probability distribution of $q_{\alpha\beta}$ is, therefore, given by

$$P(q;N) = Z_N^{-2}\sum_{\sigma^1}\sum_{\sigma^2}\delta\left(q - q_{12}\right)\prod_{\alpha=1,2}\exp\left(-\beta\, H^{\alpha}\right), \tag{2.3}$$

where H^1 and H^2 are the Hamiltonians (2.1) of the two replicas and $\beta = 1/T$. The normalization factor Z_N is the single replica partition function, which can be defined in terms of the product of random transfer matrices as

$$Z_N = \sum_{\sigma_i=\pm 1}\prod_{i=1}^{N}\exp\beta\left[\sigma_i\sigma_{i+1} + \eta_{i+1}\left(\sigma_{i+1} - \sigma_i\right)\right] = \mathrm{Tr}\prod_{i=1}^{N} A_i \tag{2.4}$$

where, for this model,

$$\begin{aligned} A_i &= \begin{pmatrix} e^{\beta} & e^{\beta} \\ e^{-3\beta} & e^{\beta} \end{pmatrix} \quad \text{with probability } p \\ A_i &= \begin{pmatrix} e^{\beta} & e^{-3\beta} \\ e^{\beta} & e^{\beta} \end{pmatrix} \quad \text{with probability } 1-p. \end{aligned} \tag{2.5}$$

In general $P(q;N)$ depends on the particular sequence of random fields, even in the thermodynamic limit $N \to \infty$. However, one can show that in one-dimensional systems with short range interactions for any finite temperature $P(q;N)$ is self-averaging, and hence $P(q;N)$ converges towards a well defined function $P(q)$ as $N \to \infty$. This is no more true at zero temperature, where the Landau theorem on the absence of phase transitions does not hold, and one can have a non-trivial $P(q)$. [see Appendix A]. The zero temperature limit of this model, with $p = 1/2$, was studied numerically in reference [5] by extracting the diverging part in (2.5), i.e. by writing $A_i = e^{\beta}R_i$ with

$$R_i = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \tag{2.6}$$

and using the matrices R_i in the product (2.4). The non-zero values of the zero temperature entropy of the model found in reference [5] indicates that frustration plays an important role.

3. The weakly disordered system.

The weakly disordered system has a great theoretical relevance because it exhibits a non-trivial lack of self-averaging at zero temperature. To our knowledge, this is the first disordered one-dimensional model where the overlap probability distribution $P(q)$ is a non-self-averaging smooth function of q.

It has to be noted that the pure system, with $h_i = 0\ \forall i$, also exhibits a lack of self-averaging. The overlap between two equilibrium configurations should be zero for any finite temperature, so that $P(q) = \delta(q)$. On the other hand, at zero temperature the overlap is $q = \pm 1$ and

$P(q) = [\delta(q-1) + \delta(q+1)]/2$. This is a quite trivial ergodicity breaking due to the zero temperature phase transition in the Ising model with zero field. It simply reflects the breaking of the "up-down" symmetry of the model.

The weakly disordered system is defined by assuming $\eta_i = 1$ with probability $p = 1 - c/N$, where N is the number of spins. As a consequence, there is a finite average number c of impurities in the thermodynamic limit, though $p = 1$.

Any field realization of the weakly disordered system obeys some simple rules. A sequence

$$\{\eta_{k-1} = 1, \ \eta_k = -1, \ \eta_{k+1} = 1\} \text{ has probability } p^2(1-p) \sim c/N$$

while a sequence

$$\{\eta_{k-1} = 1, \ \eta_k = -1, \ \eta_{k+1} = -1\} \text{ has probability } p(1-p)^2 \sim (c/N)^2,$$

and the latter event can be neglected in the limit of large N. It follows that if $h_i = 2$, then $h_{i+1} = -2$. Therefore in the thermodynamic limit, a field configuration is given, with probability one, by "islands" of zero h_i of arbitrary length separated by I interfaces made of the pair of random fields $h_j = 2$ and $h_{j+1} = -2$.

Within this picture, we can determine the overlap probability distribution by combinatorial considerations. Let us consider the case of $I = 1$ impurity, that is particularly simple since there is only one field interface. Assuming periodic boundary conditions, the interface can be moved to the boundary, so we can take $h_1 = -2$, $h_N = 2$ and $h_i = 0$ for $i = 2, .. N-1$. At zero temperature, the system has two possible choices: either σ_1 is aligned with h_1 and σ_N with h_N or all spins (σ_1 and σ_N included) are aligned in the same direction. The first choice has $N-1$ possibility states since one should have a spin flip in an arbitrary point $1 < L \leq N$. The ground state is then made of spin configurations with the first L spins down and the remaining $N-L$ spins up and its energy is $E_{\rm I} = [(N-2)-2] + 2h$ where $J = 1$ and $h = |h_1| = |h_N| = 2$ so that $E_{\rm I} = N$.

The second choice (all spins aligned) implies that either σ_1 or σ_2 have opposite direction with respect to the field. Therefore there are only 2 possible ground states for any value of N, with energy $E_{\rm II} = N$.

Although both choices lead to ground states with the same energy, the 2 spin configurations corresponding to the second choice can be neglected with respect to the $N-1$ configurations of the first choice in the thermodynamic limit.

It is worth stressing that if $h > 2$ the system becomes trivially equivalent to independent one-dimensional Ising models. On the contrary the value $h = 2$ maximizes the frustration since a single spin pays the same energy cost to align with the field or with its neighbours.

It follows that for $h = 2$ and zero temperature the weakly disorder system is not a pure Ising model without external field and with fixed boundary conditions $\sigma_1 = -1$ and $\sigma_N = +1$ but it is thermodynamically equivalent to it.

We have seen that most of the ground states are made of spin configurations with the first L spins down and the remaining $N-L$ spins up. The overlap between two of such spin configurations with spin-flip points L and L' is

$$q(L, L') = 1 - 2\frac{|L - L'|}{N} \tag{3.1}$$

The probability distribution is obtained by counting how many times the overlap is equal to q, i.e.

$$P_1(q) = \frac{1}{N^2} \sum_{L=1}^{N} \sum_{L'=1}^{N} \delta(q - q(L, L')) \tag{3.2}$$

since all values of L and L' are equiprobable. In the limit of large N the sum can be estimated, by defining the new variables $l = L/N$ and $x = |L - L'|/N$, as

$$\int_0^1 \mathrm{d}l \int_0^l \mathrm{d}x \; \delta(q - 1 + 2x). \tag{3.3}$$

The integration can be easily performed and yields

$$P_1(q) = \int_0^1 \mathrm{d}l \; \Theta(q - 1 + 2l) = \frac{1+q}{2} \tag{3.4}$$

It is worth stressing that $P_1(q)$ is the same for all disorder configurations, due to the cyclic property of the trace, though it is not a delta function. The average overlap $\bar{q} = \int P_1(q) q \, \mathrm{d}q = 1/3$ is different from the most probable value $q_{\mathrm{mp}} = 1$.

Consider now the case of $I = 2$ impurities, which is qualitatively different from $I = 1$. Each disorder realization has two islands of zero fields. Assuming periodic boundary conditions, one interface can still be moved to the boundaries, i.e. $h_1 = -2$, $h_N = 2$. The other must be placed on an arbitrary site $j = xN$, i.e. $h_j = 2$, $h_{j+1} = -2$. We can repeat the consideration made for $I = 1$, in order to neglect the $O(N)$ configurations which have at least one spin not aligned with h, in the thermodynamic limit. This is possible since there are $O(N^2)$ configurations where the spins σ_1, σ_j, σ_{j+1} and σ_N are aligned with the field.

In the limit of large N, the systems with $I = 1$ and $I = 2$ have the same energy but many significatively different realizations of the disorder. In fact, for $I = 2$ there are N allowed positions j for the second interface. A realization is therefore individuated by the position of the interface, i.e. by the particular value of x in the interval [0, 1].

In the thermodynamical limit, we can regard the system with $I = 2$ impurities as the superposition of two systems with $I = 1$ impurity, made of xN and of $(1 - x)N$ spins, respectively. The overlap between two ground states of the same disorder realization x is then given by the weighted sum of the overlaps of the two systems with $I = 1$,

$$q(x) = x \; q_1 + (1 - x) q_2 \tag{3.5}$$

where the q_i are distributed according to

$$P_1(q_i) = \frac{1 + q_i}{2} \tag{3.6}$$

Unlike the case $I = 1$, for $I = 2$ the overlap depends on the particular disorder realization, implying that the overlap probability distribution $P_2(q)$ is not self-averaging. For any value of x, $P_2(q)$ is given by

$$P_2(q; x) = \int_{-1}^1 \mathrm{d}q_1 \int_{-1}^1 \mathrm{d}q_2 \; P_1(q_1) P_1(q_2) \, \delta(q - q(x)). \tag{3.7}$$

By performing the first integral, we get

$$P_2(q; x) = \frac{1}{x^2} \int_a^b \mathrm{d}q_2 \left[(x + q) + q_2(q - 1 + 2x) - q_2^2(1 - x) \right] \tag{3.8}$$

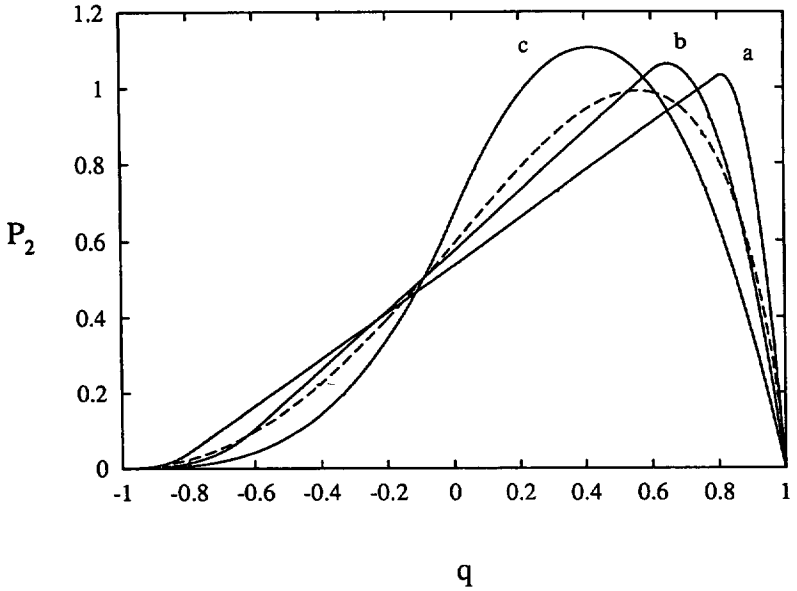

Fig. 1. — Overlap probability $P_2(q;x)$ for $x = 0.1$ (a), $x = 0.2$ (b) and $x = 0.5$ (c). The dashed line is the average overlap probability $\overline{P_2}(q) = \int_0^1 \mathrm{d}x\ P_2(q;x)$.

where

$$a = \max\left(-1,\ \frac{q-x}{1-x}\right), \quad b = \min\left(1,\ \frac{q+x}{1-x}\right) \tag{3.9}$$

The overlap probability (3.8) is shown in figure 1 for different positions of the second impurity ($x = 0.1$, $x = 0.2$ and $x = 0.5$). For comparison, we also report the average overlap probability distribution $\overline{P_2}(q)$.

For $I > 2$ impurities, it is immediate to repeat the above argument. Thus, the overlap probability distribution of the system with I field interfaces located on the sites $x_1\ N, (x_1 + x_2)N, ..\ N$ is given by

$$P_I(q; x_1, x_2, .., x_{I-1}) = \int_{-1}^{1} \mathrm{d}q_1 .. \int_{-1}^{1} \mathrm{d}q_I\ \delta\left(q - \sum_{j=1}^{I} x_j q_j\right) \prod_{j=1}^{I} P_1(q_j) \tag{3.10}$$

with $x_I = 1 - \sum_{1}^{I-1} x_j$. This expression takes a simple form in the Fourier space. Let us define the Fourier transform of the overlap probability distribution

$$P(\omega) = \int_{-\infty}^{\infty} \mathrm{e}^{i\omega q} P(q)\mathrm{d}q \tag{3.11}$$

where $P(q) = 0$ for $|q| > 1$. In the case $I = 1$, one can easily see that

$$P_1(\omega) = \int_{-1}^{1} \frac{(1+q)}{2} \mathrm{e}^{i\omega q}\ \mathrm{d}q = \left(1 - i\frac{\partial}{\partial\omega}\right) \frac{\sin(\omega)}{\omega} \tag{3.12}$$

Inserting into (3.10) the intergal representation of the delta function, we have for $I > 1$ impurities

$$P_I(q; x_1, x_2, .., x_{I-1}) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega q} \int_{-1}^{1} dq_1 .. \int_{-1}^{1} dq_I \prod_{i=1}^{I} P_1(q_i) e^{i\omega x_i q_i}$$

$$= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega q} \prod_{i=1}^{I} P_1(\omega\, x_i). \tag{3.13}$$

Therefore, the Fourier transform of the overlap probability distribution is given by the convolution

$$P_I(\omega; x_1, x_2, .., x_{I-1}) = \prod_{i=1}^{I} P_1(\omega x_i) \quad \text{with} \quad \sum_{i=1}^{I} x_i = 1 \tag{3.14}$$

where

$$P_1(\omega x_i) = \frac{\sin(\omega\, x_i)}{\omega\, x_i} - i\frac{\cos(\omega\, x_i)}{\omega\, x_i} + i\frac{\sin(\omega\, x_i)}{(\omega\, x_i)^2} \tag{3.15}$$

For any number $I > 1$ of impurities, the overlap probability distribution depends on the particular disorder configuration individuated by the position of the $I-1$ field interfaces, i.e. by the sequence $\{x_1, x_2, ...x_{I-1}\}$. The disorder average is, therefore, obtained by averaging over the probability of the sequence $\mathcal{P}(\{x_1, x_2, ...x_{I-1}\})$. In our model the x_i are independent random variables uniformly distributed in the interval [0, 1], so that

$$\mathcal{P}(\{x_1, x_2, .. x_{I-1}\}) = \prod_{i=1}^{I-1} \mathcal{P}(x_i) \tag{3.16}$$

where $\mathcal{P}(x) = 1$ for $x \in [0, 1]$ and $\mathcal{P}(x) = 0$ otherwise.

Since the x_i are independent, in the limit of large I the average overlap probability distribution tends to the most probable value $\tilde{P}_c(\omega)$ obtained for $x_i = 1/I$,

$$\tilde{P}_c(\omega) = [P_1(\omega)]^c \tag{3.17}$$

since for large I the fluctuations about the mean value c are negligible. For the central limit theorem, $\tilde{P}_c$ is a Gaussian with mean value $\bar{q} = 1/3$ and variance $\overline{(q-1/3)^2} = 2/(9c)$. The Gaussian form very quickly becomes a good approximation, as shown in figure 2 where $\tilde{P}_3(q)$ (the most probable overlap probability with impurities at $x_1 = 1/3$ and $x_2 = 1/3$) as well as the average $\overline{P_3}(q)$ are very close to the Gaussian with mean value 1/3 and variance 2/27.

The limit $c \to \infty$, but $c/N \to 0$, corresponds to vanishing impurity concentration with $p \to 1$. As a consequence, at zero temperature the overlap probability distribution is $P(q) = \delta(q-1/3)$ for $p \to 1$, while it is a double delta $P(q) = [\delta(q-1) + \delta(q+1)]/2$ for $p = 1$, i.e. the pure system.

The weakly disordered system exhibits a non-trivial new phase between these two extreme behaviours. For finite temperature, the overlap vanishes both in the pure and in the weakly disordered system leading to $P(q) = \delta(q)$.

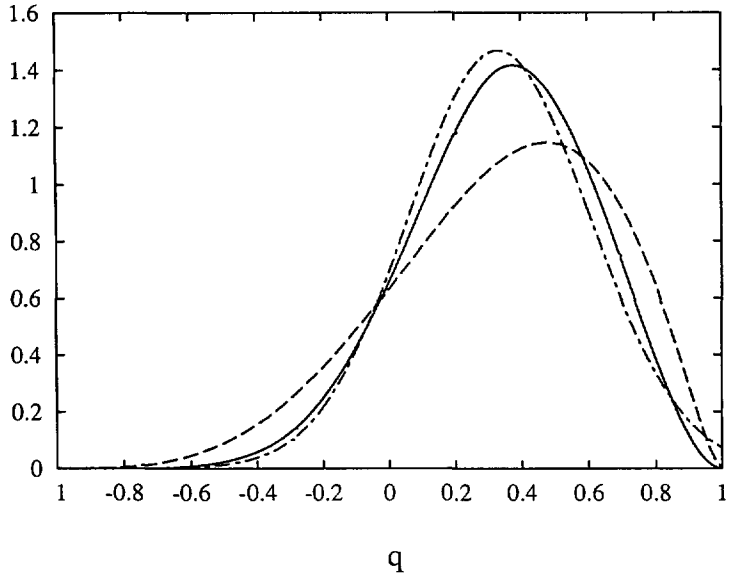

Fig. 2. — Overlap probability distributions: $P_3(q; x_1, x_2)$ with impurities at $x_1 = 1/3$ and $x_2 = 1/3$ (full line), average probability $\overline{P_3}(q)$ (dashed line) and Gaussian distribution with mean value 1/3 and variance 2/27 (dot dashed line).

4. The disordered system.

In this section we discuss the system with non-zero impurity concentration, i.e. $p \neq 1$ or 0. Unlike the weakly disordered case, the number of ground states with spin not aligned with the field becomes comparable to the total number of ground states. We cannot, therefore, perform simple analytical calculations. However, we can employ the transfer matrix approach for a numerical calculation [4].

The study of (2.3) is equivalent to that of [5, 6]

$$\mathcal{N}_N(\omega) = \sum_{\sigma^1} \sum_{\sigma^2} e^{N\omega q_{12}} \prod_{\alpha=1,2} e^{-\beta H^\alpha} \tag{4.1}$$

which can be regarded as the partition function of a system made of two interacting replicas, with Hamiltonian $H^1 + H^2 + \omega N q_{12}$. The term ωN is the macroscopic coupling between the two replicas. Only if $\omega = 0$, or if the overlap $q_{12} = 0$, $\mathcal{N}_N(\omega)$ becomes the square of the usual partition function (2.4).

The advantage of (4.1) stems from the possibility to write it as a product of random transfer matrices [4, 5]. The maximum Lyapunov exponent $\Gamma(\omega)$ of the product yields the average of the logarithm of $\mathcal{N}_N(\omega)$,

$$\mathcal{N}_N(\omega) \propto \exp[N\Gamma(\omega)] \quad \text{for} \quad N \gg 1. \tag{4.2}$$

There is no average over the disorder in the l.h.s. of (4.2) since the Oseledec theorem [7] ensures that the Lyapunov exponent is a non-random quantity for $N \to \infty$, i.e. it has the same value for almost all realizations of disorder, a part a set of zero probability measure. This

is not in contradiction with the fact that the overlap probability distribution could be not self-averaging. Indeed, it is possible to prove [5] that the self-averaging of the Lyapunov exponent only implies that the extrema of the support of $P(q)$ are the same for all configurations of disorder. Consider the quantity

$$\Gamma_N(\omega) = \ln \int_{-1}^{1} \mathrm{e}^{N\omega q} P(q;N)\mathrm{d}q \tag{4.3}$$

where $P(q;N)$ is the overlap probability distribution of a system of size N, and in general depends on the realization of disorder. The Lyapunov exponent is related to the thermodynamic limit of

$$\frac{1}{N}\Gamma_N(\omega) = \frac{1}{N}\ln \int_{-1}^{1} \mathrm{e}^{N\omega q} P(q;N)\mathrm{d}q \tag{4.4}$$

In the limit of large N, we can insert into (4.4) the saddle point estimate obtaining for small ω

$$\Gamma(\omega) = \begin{cases} \omega\, q_{\max} & \text{if } \omega > 0 \\ \omega\, q_{\min} & \text{if } \omega < 0 \end{cases} \tag{4.5}$$

In general, $\Gamma(\omega)$ is a non-linear function of ω.

A reasonable ansatz on the finite N corrections to the asymptotic form of $P(q)$ for a given disorder realization is

$$P(q;N) \sim P(q;\infty) + A\mathrm{e}^{-S(q)N}$$

In this case, the saddle point estimate of (4.4) gives the Lyapunov exponent as the Legendre transform of the convex envelope $s(q)$ of $S(q)$:

$$\Gamma(\omega) = \max_q [q\omega - s(q)] \tag{4.6}$$

As the Lyapunov exponent of a product of random matrices is a non-random quantity (Oseledec theorem), (4.6) shows that the envelope $s(q)$ is self-averaging in systems with short range interactions. By definition $s(q) = 0$ for $q \in [q_{\min}, q_{\max}]$ and $s(q) > 0$ for $q_{\max} < q \leq 1$ and $-1 \leq q < q_{\min}$. Moreover, for large $|\omega|$ the saddle point is given by $q = \pm 1$ so that the asymptotic behaviour is $\Gamma(\omega) \simeq C_{(\pm)} \pm \omega$ where $C_{(\pm)}$ are constants.

This result could appear of rather limited utility, but it assumes a great importance when considered as a mark of a replica symmetry breaking. Indeed, if $\lim_{N\to\infty} P(q,N) = \delta(q - \bar{q})$, the derivative $\mathrm{d}\Gamma(\omega)/\mathrm{d}\omega$ at $\omega = 0$ does exist and is equal to $\bar{q}$. This is the case in one-dimensional systems when $T \neq 0$. On the other hand, if $P(q)$ differs from a delta function it implies a non-differentiable $\Gamma(\omega)$ at $\omega = 0$.

No information is lost if $P(q;\infty)$ is a delta function, implying $q_{\min} = q_{\max} = \bar{q}$, and $s(q)$ has only one zero at $q = \bar{q}$.

The theoretical relevance of this result follows from the possibility of estimating the Lyapunov exponent either by a direct numerical calculation or by many analytic methods such as weak disorder expansions [8], cycle expansions of appropriate zeta functions [9], microcanonical tricks [10, 11], and so on.

This approach has been used in reference [5] to evaluate the average overlap $\bar{q}$ for the model with $p = 1/2$ for both zero and finite temperature. In this reference $\bar{q}$ was obtained from the numerical derivative of $\Gamma(\omega)$ at $\omega = 0$. Moreover, the zero temperature case was done by using an *ad hoc* trick to extract the diverging part. Here we report some results on the model with $0 < p \leq 1/2$ for both zero and finite temperature obtained by using a different method [4]

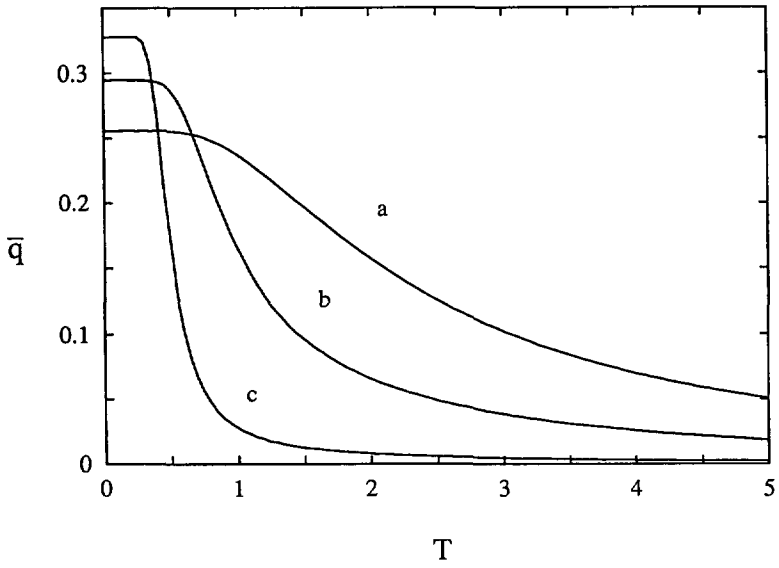

Fig. 3. — Numerical results for the average overlap $\bar{q}$ as function of the temperature $T = \beta^{-1}$ in the disordered system with $p = 0.5$ (a), $P = 0.1$ (b), $p = 0.01$ (c).

which avoids numerical estimates of derivatives. As we discuss in Appendix B, the entropy $s(q)$ defined in (4.6) is the Legendre transform of $(1/N)\overline{\ln \mathcal{N}_N(\omega)}$. The quantity $\mathcal{N}_N(\omega)$ can be written as the weighted sum of $\exp(N\omega q)$ over the states with a given value of q,

$$\mathcal{N}_N(\omega) = \sum_q \mathcal{N}_N(q) \mathrm{e}^{N\omega q} \tag{4.7}$$

For large N the saddle point estimate yields

$$-s(q) = \frac{1}{N}\overline{\ln \mathcal{N}_N(q)} = \frac{1}{N}\overline{\ln \mathcal{N}_N(\omega)} - \omega q \tag{4.8}$$

where $q \equiv q(\omega)$ is the overlap selected by the chosen value of ω. All the quantities on the r.h.s. of (4.8) can be computed directly by means of products of suitable transfer matrices. A brief description of the method is in Appendix B. We report the numerical results for the average overlap as function of the temperature $T = \beta^{-1}$ in figure 3 and as function of p at different temperatures in figure 4. We have also applied our method to the direct computation of second derivatives. Figure 5 gives the behaviour of the second derivative of $(1/N)\Gamma_N(\omega)$, that is

$$\chi_{\mathrm{SG}} \equiv \lim_{N\to\infty} N\overline{(q-\bar{q})^2} = \lim_{N\to\infty} \frac{1}{N} \left.\frac{\partial^2 \Gamma_N(\omega)}{\partial \omega^2}\right|_{\omega=0} \tag{4.9}$$

as function of temperature at $p = 1/2$ and $p = 0.1$. The spin glass susceptibility χ_{SG} measures the finite N fluctuations of the overlap. If $P(q)$ is a delta function χ_{SG}/N is the variance of the typical (Gaussian) distribution $P_N(q)$ when $N \to \infty$, The divergence of χ_{SG} denotes a non trivial $P(q)$.

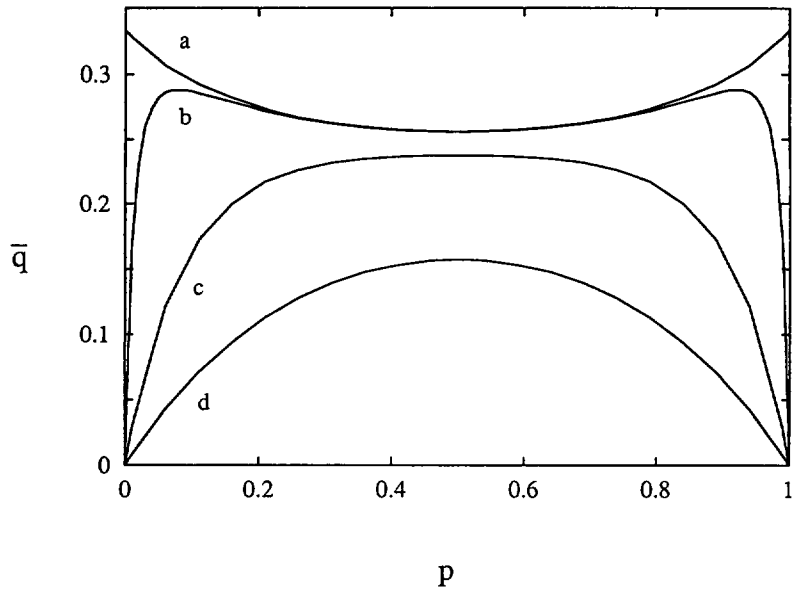

Fig. 4. — Numerical results for the average overlap $\bar{q}$ as function of the disorder concentration p at temperatures $T = 0$ (a), $T = 0.5$ (b), $T = 1$ (c), $T = 2$ (d).

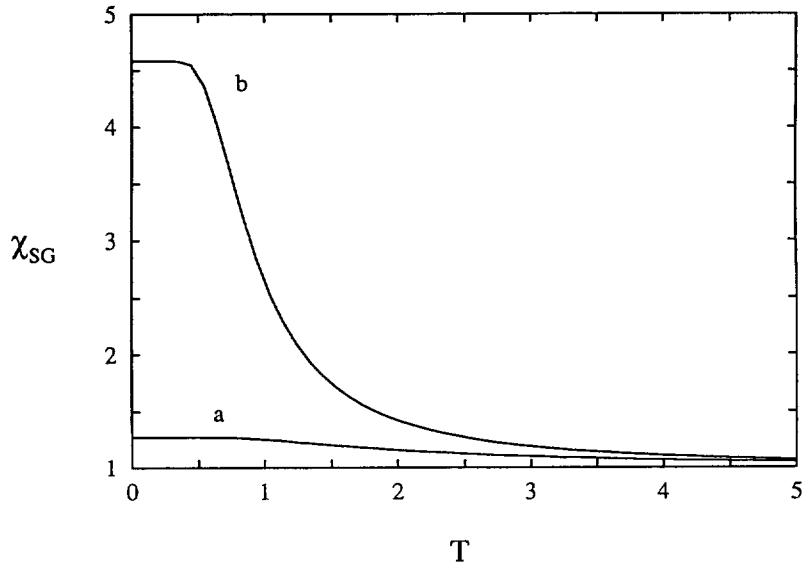

Fig. 5. — Spin glass susceptibility χ_{SG} as function of the temperature in the disordered system with $p = 0.5$ (a) and $p = 0.1$ (b).

For small $1 - p$ (or small p, due to the symmetry $p \leftrightarrow 1 - p$) and zero temperature we find that χ_{SG} diverges as $(1-p)^{-1}$ (or p^{-1}). This is a mark of the transition from the self-averaging

regime ($p < 1$) to the non-self-averaging regime, i.e. finite number of impurities with vanishing concentration. It is worth stressing the analogy between the weak disorder phase and the critical point in second order phase transition where the susceptibility diverges.

5. Conclusions.

We have shown that in one-dimensional systems with short range disorder, lack of self-averaging cannot appear at finite temperature. In generic situations the overlap between two equilibrium configurations does not vanish and has a self-averaging value which can be computed by products of random transfer matrices.

To describe the zero temperature phase transition, a new one-dimensional model has been introduced. It is complex enough to be a good candidate for testing ideas which can be useful in more realistic situations. In the weak disorder regime, our model exhibits an overlap probability $P(q)$ which is a smooth function of q. It does not self-average in the thermodynamic limit but depends on the disorder configuration. The $P(q)$ has an analytic form which can be computed in a transparent way without using the replica trick or other indirect methods. It is an open problem to find analogous models in two dimensions.

We conclude by noting that in long range models a non trivial $P(q)$ is associated to the breaking of the ergodic property in the thermodynamic limit. The $P(q)$ describes the overlap distribution between the different ergodic components present in the system.

In our case the non trivial $P(q)$ found at $T = 0$ reflects the high degeneracy of the ground state: the ground state is made of a macroscopic number of degenerate equilibrium states. These states, however, are not separated by high energy barriers, so we cannot interpret them as "ergodic components". The ergodic component is made by the ensemble of all these states. Consequently, at difference with the overlap probability distribution between the equilibrium states equation (2.3), the overlap distribution between ergodic components is a self-averaging delta function.

Appendix A.

In this appendix, we show that the overlap probability distribution of a one-dimensional random system with short range interactions is a self-averaging delta function at any temperature different from zero.

Consider a one-dimensional random Ising model with short range interactions. Because of the well-known Landau theorem on the absence of phase transitions in one dimension, in any realization of the random coupling and/or fields, the spin-spin correlation should have the exponential decay

$$\langle \sigma_i \sigma_{i+r} \rangle - \langle \sigma_i \rangle \langle \sigma_{i+r} \rangle \sim \mathrm{e}^{-r/\xi} \tag{A1}$$

where $\langle . \rangle$ denotes the thermal average with the Gibbs measure $\exp(-\beta H)$ for a given disorder realization. Since in a typical realization of disorder ξ^{-1} is the difference of the Lyapunov exponent of the product of random transfer matrices related to the model, (A1) also follows from the Perron theorem assuring that there is no degeneracy of the eigenvalues in the spectrum of matrices with positive non-zero elements. The value of ξ can depend on the sites i, $i + r$ and on the disorder realization. Nevertheless, (A1) implies that the σ_i are (exponentially) independent random variables with respect to the Gibbs measure.

The overlap between two spin configurations σ^1 and σ^2 is

$$q_{12} \equiv \frac{1}{N} \sum_{i=1}^{N} \langle \sigma_i^1 \rangle \langle \sigma_i^2 \rangle .$$

As the σ_i are independent, we can apply the law of large numbers so that in the thermodynamic limit with probability one all spin configurations related to a disorder realization have the same overlap, i.e. $\lim_{N\to\infty} q_{12} \to \bar{q}$ and the overlap probability distribution $P(q)$ converges towards a delta function.

We should still prove that $\bar{q}$ has the same value for all disorder realizations. This can be seen by means of a very general result: the support of the overlap probability distribution $P(q)$ is self-averaging. Consequently, if $P(q)$ is a delta function for a disorder realization, it should be a delta function centered on the same value $\bar{q}$, for all the other realizations. In conclusion, for one dimensional systems with short range interactions one has in the thermodynamic limit for any non-zero temperature

$$P(q) = \delta (q - \bar{q}) .$$

for almost all disorder realizations, apart from a set of zero probability measure.

Appendix B.

In this appendix we briefly review the numerical method used in section 4 to evaluate the average overlap probability distribution. We only describe the finite temperature method. We shall denote by $\mathbf{S}_i \equiv (\sigma_i^1, \sigma_i^2)$ the pair of spin at the same site in the two replicas. The average of the logarithm of $\mathcal{N}_N(\omega)$, equation (4.1), can be obtained from the vector $\mathbf{N}_i(\mathbf{S}_{i+1})$ which satisfies the recursion relation

$$\mathbf{N}_i(\mathbf{S}_{i+1}) = \sum_{\mathbf{S}_i} \exp\left(-\beta\epsilon_i^1 - \beta\epsilon_i^2 + \omega\sigma_i^1\sigma_i^2\right) \mathbf{N}'_{i-1}(\mathbf{S}_i) \tag{B1}$$

where $\epsilon_i^\alpha = -\left[\sigma_{i+1}^\alpha + h_i\right]\sigma_i^\alpha$ is the single spin energy in the site i of the replica α, and

$$\sum_{\mathbf{S}_{i+1}} \mathbf{N}'_i(\mathbf{S}_{i+1}) = 1. \tag{B2}$$

From (4.1) follows the initial condition

$$\mathbf{N}_1(\mathbf{S}_2) = \sum_{\mathbf{S}_1} \exp\left(-\beta\epsilon_1^1 - \beta\epsilon_1^2 + \omega\sigma_1^1\sigma_1^2\right). \tag{B3}$$

By denoting with n_i the sum of the elements of $\mathbf{N}_i(\mathbf{S}_{i+1})$, the logarithm of $\mathcal{N}_N(\omega)$ averaged over the disorder is

$$\frac{1}{N}\overline{\ln\mathcal{N}_N(\omega)} = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \ln n_i \tag{B4}$$

i.e. the Lyapunov exponent of the product.

The value of q can be obtained directly from the derivative of the vector $\mathbf{N}_i(\mathbf{S}_{i+1})$ with respect to ω. The derivative with respect to ω leads to the vector $\mathbf{O}_i(\mathbf{S}_{i+1})$ obeying the recursion relation

$$\mathbf{O}_i(\mathbf{S}_{i+1}) = \sum_{\mathbf{S}_i} \left[\mathbf{O}'_{i-1}(\mathbf{S}_i) + \sigma^1\sigma^2\mathbf{N}'_{i-1}(\mathbf{S}_i)\right] \mathrm{e}^{\omega\sigma_i^1\sigma_i^2} \prod_{\alpha=1,2} \mathrm{e}^{-\beta\epsilon_i^\alpha} \tag{B5}$$

with initial condition

$$\mathbf{O}_1(\mathbf{S}_2) = \sum_{\mathbf{S}_1} \sigma_1^1 \sigma_1^2 \exp\left(-\beta\epsilon_1^1 - \beta\epsilon_1^2 + \omega\sigma_1^1\sigma_1^2\right). \tag{B6}$$

At each step the vector $\mathbf{O}_i$ is rescaled as

$$\mathbf{O}_i'(\mathbf{S}_{i+1}) = \left[\mathbf{O}_i(\mathbf{S}_{i+1}) - o_i \mathbf{N}_i'(\mathbf{S}_{i+1})\right]/n_i \tag{B7}$$

where n_i and o_i are the sum of the elements of $\mathbf{N}_i(\mathbf{S}_{i+1})$ and $\mathbf{O}_i(\mathbf{S}_{i+1})$, respectively. The value of $q = q(\omega)$ is then obtained from the average of o_i

$$q = \lim_{N\to\infty} \frac{1}{N} \sum_i \frac{o_i}{n_i} \tag{B8}$$

In the limit of zero temperature one has to extract the diverging part in (4.8). This is achieved by means of a Legendre transform to express all the quantities in terms of the average energy per spin $\epsilon \equiv \epsilon(\omega, \beta)$. The value of ϵ for given β and ω can be obtained by introducing a vector of the derivative of $\mathbf{N}_i(\mathbf{S}_{i+1})$ with respect to ω, and proceeding as done for q.

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